A New Method for Computing the Stability Margin of Two-Dimensional Systems

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Abstract

The stability margin of 2-D (two-dimensional) Linear Shift Invariant causal single-input single-output discrete systems is investigated. A new method to compute the stability margin of 2-D continuous systems is considered. Illustrative examples are also included.

1 Introduction

A linear shift-invariant causal single-input single-output 2-D system is described by the following transfer function:

$$G(z_1,z_2)=\frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{1}$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are coprime polynomials in the independent complex variables z_1 and z_2. It is assumed that there are no nonessential singularities of the second kind on the closed unit bidisk, i.e. there are no points (z_1,z_2) with $|z_1|\le 1$ and $|z_2|\le 1$ such that $A(z_1,z_2)=B(z_1,z_2)=0$.
The system (1) is (Hurwitz) stable if and only if

$$B(0,z_2)\neq 0, \qquad \text{for} \quad |z_2|\le 1 \tag{2.1}$$

$$B(z_1,z_2)\neq 0, \qquad \text{for} \quad |z_1|\le 1, \quad |z_2|=1 \tag{2.2}$$

One should note that condition (2.1) is relatively easy to check using any 1-D stability test. However, condition (2.2) is more difficult since it includes two variables. We denote:

$$B(z_1,z_2)=\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2} b_{i_1,i_2} z_1^{i_1} z_2^{i_2}$$

The polynomial $B(z_1,z_2)$ is said to be *(Hurwitz) Stable* if and only if (2.1) and (2.2) are fulfilled.

Several algebraic methods for testing the stability of 2-D discrete systems or, equivalently, checking the Hurwitz character of 2-D polynomials already exist[1].

However, we are interested not only in whether the system is stable but also whether the system will remain stable in the presence of system parameter deviations.

So, for a stable 2-D (discrete) system, the following definitions have been introduced [3]:

***Definition* 1:** Given a 2-D discrete system described by the transfer function (1), we call stability margin σ_1 the supremum (i.e. the lower upper bound) of the positive real numbers for which $B((1+\sigma_1)\cdot z_1, z_2)$ is a *(Hurwitz) Stable Polynomial.*

***Definition* 2:** Given a 2-D discrete system described by the transfer function (1), we call stability margin σ_2 the supremum of the positive real numbers for which $B(z_1,(1+\sigma_2)\cdot z_2)$ is a *(Hurwitz) Stable Polynomial.*

***Definition* 3:** Given a 2-D discrete system described by the transfer function (1), we call stability margin σ the supremum of the positive real numbers for which $B((1+\sigma)\cdot z_1,(1+\sigma)\cdot z_2)$ is a *(Hurwitz) Stable Polynomial.*

Note that the special case where the stable system has nonessential singularities of the second kind on the closed unit bidisk is excluded, since all three stability margins will be zero. For the evaluation of the stability margin several methods already exist [3÷8]. In this paper, a new method is proposed. It is based on a recently proposed method for checking the stability of a 2-D system via inners determinants [9].

2 Computation of the stability margins for a 2-D (discrete) system

In this session, a method of computing the stability margins of 2-D systems is presented. We introduce the notation

$$k_1 = 1+\sigma_1 \tag{3}$$

The method is based on checking the inners matrix of the characteristic polynomial $B(z_1,z_2)$ of a stable system described by (1). For a stable 2-D discrete system, we recall that the polynomial $B(z_1,z_2)$ is a *(Hurwitz) Stable Polynomial* if and only if: (2.1) holds and the inners matrix $\Delta_{2N_1}(z_2)$ associated with $z_1^{N_1}B(z_1^{-1},z_2)$ is positive innerwise for all z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0,2\pi]$, [9]. Therefore $B(k_1 z_1, z_2)$ remains *(Hurwitz) Stable Polynomial* if and only if (2.1) holds and the inners matrix $\Delta_{2N_1}(k_1,z_2)$ associated with $z_1^{N_1}B(k_1 z_1^{-1},z_2)$ remains positive innerwise for all z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0,2\pi]$. However, because of the assumed stability of the considered system, (2.1) holds independent of k_1. (Note that (2.1) does not contain z_1, consequently it does not contain k_1). Thus, $B(k_1 z_1, z_2)$ remains *(Hurwitz) Stable Polynomial* if and only if the inners matrix $\Delta_{2N_1}(k_1,z_2)$ associated with $z_1^{N_1}B(k_1 z_1^{-1},z_2)$ remains positive innerwise for all z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0,2\pi]$.

Considering the inners matrix $\Delta_{2N_1}(k_1,z_2)$ associated with $z_1^{N_1}B(k_1 z_1^{-1},z_2)$, we obtain that for the supremum of k_1 for which $B(k_1 z_1,z_2)$ is *(Hurwitz) Stable* the inners matrix $\Delta_{2N_1}(k_1,z_2)$ will be singular i.e. $det\Delta_{2N_1}(k_1,z_2)=0$ (for some z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0,2\pi]$). For a detailed justification see Appendix. Therefore, the supremum of k_1 for which $B(k_1 z_1,z_2)$ is *(Hurwitz) Stable* is simultaneously the *minimum* of all k_1 with $det\Delta_{2N_1}(k_1,z_2)=0$ (for some z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0,2\pi]$). This implies that the computation of k_1 can be achieved by solving the following minimization problem

$$min\ k_1 \tag{4.1}$$

under the constraint

$$det\Delta_{2N_1}(k_1,z_2)=0 \tag{4.2}$$

where $\Delta_{2N_1}(k_1, z_2)$ is the inners matrix associated with $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$. In the sequel, we easily obtain σ_1 from Equation (3).

By interchanging the roles of the variables z_1 and z_2, a completely analogous method for the computation of σ_2 is obtained.

Analogously, for the computation of σ we denote

$$k=1+\sigma \tag{5}$$

Here, instead of (2.1) and (2.2), we use the equivalent condition $B(z_1, z_2) \neq 0$, for $|z_1| \leq 1$, $|z_2| \leq 1$, [1]. So, k is the supremum of the real numbers (≥ 1) for which $B(kz_1, kz_2) \neq 0$, for $|z_1| \leq 1$, $|z_2| \leq 1$. Varying only z_2, one can obtain that this condition is equivalent to $B(kz_1, kz_2) \neq 0$, for $|z_1| \leq 1$, $|z_2| = 1$. This latter equation is analogous to Equation (2.2). Therefore, following exactly the same steps as in the case of the stability margin σ_1, we formulate the following method for the stability margin σ.

$$min\ k \tag{6.1}$$

under the constraint

$$det\Delta_{2N_1}(k, z_2)=0 \tag{6.2}$$

where $\Delta_{2N_1}(k, z_2)$ is the inners matrix associated with $z_1^{N_1} B(kz_1^{-1}, kz_2)$. The following example illustrates the implementation of this method.

***Example* 1** [3÷8]: Consider the general first order characteristic polynomial of a stable system

$$B(z_1, z_2) = 1 + az_1 + bz_2 + cz_1z_2 \tag{7}$$

where *a, b, c* are real numbers. It is always assumed that the corresponding 2-D system has no nonessential singularities of the second kind. For the computation of the stability margin σ_1, one forms the inners matrix of $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$ (here $N_1 = 1$). This is

$$\Delta_{2N_1}(k_1, z_2)=$$

$$= \begin{bmatrix} (a + cz_2)k_1 & 1 + bz_2 \\ 1 + b\bar{z}_2 & (a + c\bar{z}_2)k_1 \end{bmatrix} \tag{8}$$

where $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$ and the overbar denotes complex conjugate. Then

$$det\Delta_{2N_1}(k_1, z_2) =$$

$$=(a^2 + c^2 + 2acx)k_1^2 - (1 + b^2 + 2bx) \tag{9}$$

where $x = cos\,\phi_2$ ($x \in [-1, 1]$). One obtains that $det\Delta_{2N_1}(k_1, z_2)$ is linear in x. So, for a certain k_1 the minimum value of $det\Delta_{2N_1}(k_1, z_2)$ is obtained for $x = -1$ (if $ack_1^2 - b \geq 0$) or for $x = +1$ (if $ack_1^2 - b < 0$). Thus, for the *minimum* k_1 with $det\Delta_{2N_1}(k_1, z_2)=0$ the determinant $det\Delta_{2N_1}(k_1, z_2)$ will be zero for $x = \pm 1$. Therefore, for the minimum k_1, we obtain:

$$(a^2 + c^2 + 2ac)k_1^2 - (1 + b^2 + 2b)=0 \tag{10.1}$$

or

$$(a^2 + c^2 - 2ac)k_1^2 - (1 + b^2 - 2b)=0 \tag{10.2}$$

Solving (10.1) and (10.2), we find $k_1 = min\left[\dfrac{|1+b|}{|a+c|}, \dfrac{|1-b|}{|a-c|}\right]$. From which

$$\sigma_1 = min\left[\frac{|1+b|}{|a+c|}, \frac{|1-b|}{|a-c|}\right] - 1 \quad (11)$$

Consequently, interchanging the variables z_1 and z_2, one evaluates

$$\sigma_2 = min\left[\frac{|1+a|}{|b+c|}, \frac{|1-a|}{|b-c|}\right] - 1 \quad (12)$$

The results agree with those of [3]÷[8]. Note that here they derived in a very simple manner.
Let us also compute σ. We form the inners matrix of $z_1^{N_1}B\left(kz_1^{-1}, kz_2\right)$. This is

$$\Delta_{2N_1}(k, z_2) =$$

$$= \begin{bmatrix} ak + ck^2 z_2 & 1 + bkz_2 \\ 1 + bk\bar{z}_2 & ak + ck^2 \bar{z}_2 \end{bmatrix} \quad (13)$$

Then

$$det\Delta_{2N_1}(k, z_2) =$$

$$= k^2\left(a^2 + c^2k^2 + 2ackx\right) - \left(1 + b^2k^2 + 2bkx\right) \quad (14)$$

where $x = cos\,\phi_1$. One also obtains that $det\Delta_{2N_1}(k, z_2)$ is also linear in x. So, for a certain k the minimum value of $det\Delta_{N_2}(k, z_1)$ is obtained for $x = \pm 1$. Thus, for the minimum k with $det\Delta_{2N_1}(k, z_2) = 0$ the determinant $det\Delta_{2N_1}(k, z_2)$ will be zero for $x = \pm 1$. Therefore, we obtain for the minimum k:

$$k^2\left(a^2 + c^2k^2 + 2ack\right) - \left(1 + b^2k^2 + 2bk\right) = 0 \quad (15.1)$$

or

$$k^2\left(a^2 + c^2k^2 - 2ack\right) - \left(1 + b^2k^2 - 2bk\right) = 0 \quad (15.2)$$

Solving (15.1), (15.2) we find k = *minimum* of the real positive values of the set

$$\left\{ \frac{a+b \pm \sqrt{(a+b)^2 - 4c}}{2c}, \frac{-a+b \pm \sqrt{(-a+b)^2 + 4c}}{2c}, \right.$$

$$\left. \frac{a-b \pm \sqrt{(a-b)^2 + 4c}}{2c}, \frac{-a-b \pm \sqrt{(-a-b)^2 - 4c}}{2c} \right\}$$

From which $\sigma = k - 1$. All the results agree with those derived in [3]÷[8], but here they derived in an easier manner.

Example 2 [6]: Consider $B(z_1, z_2) = 3 - z_1 - z_2$. Following the above procedure we obtain $\sigma_1 = 1, \sigma_2 = 1$ as well as $\sigma = 0.5$. The latter can be obtained from (15.1) and (15.2) if we put $a = b = -1/3$ **and** $c = 0$.
Remark: An interesting genaralization of the Definitions of $\sigma_1, \sigma_2, \sigma$ could be the following:
Definition of the stability margin σ *with weights* λ_1, λ_2 ($\lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0$ & $\lambda_2 \geq 0$) Given a 2-D discrete system described by the transfer function (1), we call stability margin σ the supremum of the positive real numbers for which $B\left((1+\lambda_1\sigma)\cdot z_1, (1+\lambda_2\sigma)\cdot z_2\right)$ is a *(Hurwitz) Stable Polynomial.*

Taking into account this definition, we can consider Definitions 1÷3 as special cases of the previous definition (Definition 3 needs a slight modification). Moreover, modifying the above method, one can easily derive a general algorithm for evaluating the stability margin σ with weights λ_1, λ_2.

3 Conclusion

The stability margin for 2-D discrete systems has been considered. A new method for computing the stability margins have been proposed. The method is based on a constrained optimization problem of a real positive parameter. Since the formulation of the inners determinant [9] is more "direct" than the formulation of the Schour-Cohn matrix [1], [12], the method, offering a more direct computation of the stability margin, is better than the method of [3].

The significance of the proposed computational method and the improvement with respect to previous work in [3]÷[8] is, first, that we use the inners determinant instead of the method of *Schur-Cohn* which is used in [3]. The method of the *inners* determinant has the same multiplexity as the method of the Schur-Cohn ([9]) but it is actually an essential simplification of the Schur-Cohn method as far as the formulation of the various matrices is concerned [9], [12]. For this reason the proposed method is better than that of [3].

Work is in progress by the author in the area of 2-D stability margin formulating analogous methods for 2-D continuous systems. Other recent results can also be found in [2].

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Appendix

Consider the mapping δ: $k_1 \rightarrow \delta(k_1)$ where $\delta(k_1) = \Delta_{2N_1}(k_1, z_2)$. This is a *continuous* mapping since the matrix $\Delta_{2N_1}(k_1, z_2)$ consists of polynomials in k_1, z_2. Also, consider the mapping det : $\delta(k_1) \rightarrow det\,\delta(k_1)$. This is also a continuous mapping.

Therefore, their synthesis $det\,\delta$: $k_1 \rightarrow det\,\delta(k_1)$ is also a continuous mapping. We denote S, the set S =

$\{\delta(k_1) \text{ with } \delta(k_1) > 0\}$, where $>$ denotes positive innerwise for all z_2 with $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$, [9]. We also denote *det*{S} the subset of the real numbers which consists of all the determinants of $\delta(k_1)$ that belong to S. Evidently, *det*{S} is the set of all the (strictly) positive real numbers. Thus, the only limit point of *det*{S} is the 0.

S is an *open* set and because of the continuity of the mapping δ, the corresponding set of k_1 will also be *open* (see any standard textbook of *Real Analysis* or *Topology*, [11]). Thus, the supremum of k_1 is a limit point of this set and because of the continuity of the mapping δ, for this k_1, $\delta(k_1)$ is also a limit point of S. Furthermore by the continuity of the mapping $det: \delta(k_1) \to det\,\delta(k_1)$, $det\delta(k_1)$ is limit point in the set *det*{S}, for this k_1. Since the only limit point of *det*{S} is the 0, we conclude that for this k_1, we have $det\delta(k_1)=0$. As a result, we obtain that for the supremum of k_1 for which $B(k_1 z_1, z_2)$ is *(Hurwitz) Stable*, the inners matrix $\Delta_{2N_1}(k_1, z_2)$ will be singular (for some z_2, $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$).

A brief biography of the author is given in the paper: "New Stability Test For 2-D Systems" which is also presented in these Proceedings.