MOTION PLANNING FOR DRIFT-FREE NONHOLONOMIC SYSTEMS UNDER A DISCRETE LEVELS CONTROL CONSTRAINT

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Abstract
The motion planning problem for non-holonomic systems without drift is solved under the constraint that the control inputs can take values in a set that contains a finite number of allowable levels. If the system to be steered is also nilpotent or nilpotentizable, the steering can be exact. The results are applied in the steering of a simple system.

1. Introduction

In this paper we propose an algorithm that solves the Motion Planning Problem (MPP) of a drift-free nonholonomic system, whose inputs take values in a discrete levels set. Motion planning problems are concerned with obtaining open-loop controls which steer a system from an initial point $\boldsymbol{x}_0$ to a final point $\boldsymbol{x}_f$ over a given finite time interval. In other words, the MPP is the problem of finding reasonable algorithms producing for every pair $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$ of points, an open-loop control $t \to \boldsymbol{u}(t) = (u_1(t) \dots u_m(t))$ that steers $\boldsymbol{x}_0$ to $\boldsymbol{x}_f$.

The MPP has been studied by several researchers and the literature is quite extensive. Most of the proposed strategies deal with nonholonomic systems whose inputs take values in $\Re$ without any constraint and work well in a number of special cases. A variety of motion planning techniques are described in the book [6], which is a collection of research articles on nonholonomic motion planning. Besides [6], an excellent introduction to motion planning for nonholonomic systems can be found in the book by Murray, Li and Sastry [7]. The book by Latombe [5] also contains a nice chapter on nonholonomic motion planning. The motion planning methodologies can be loosely classified into three approaches: differential-geometric and differential-algebraic techniques, geometric phase methods and control parametrization approaches.

Our method uses tools from the differential-geometric control theory and is based on the strategy proposed by Lafferriere and Sussmann in [1-4], where they define a general framework for solving the MPP for drift-free non-holonomic systems, which has the following characteristics: (a) it does not in principle require special assumptions on the spans of the Lie brackets; (b) it does not use optimal control; (c) it works exactly for nilpotent and nilpotentizable systems; (d) it produces an iterative algorithm for completely general systems which converges quickly to a solution; (e) admits extensions to some systems with drift.

This paper is organized as follows: In section 2 we present our main results concerning the trajectories of a drift-free nilpotent system, whose inputs take values in a discrete levels set and in addition we state the algorithm that solves the MPP for such a system. Finally, in section 3 we explore the details of the proposed algorithm using a specific example.

Notation

$\mathcal{L}_S$ the Controllability Lie

	Algebra of the system (S)
$LC(g_1,\dots,g_m)$	the Lie Algebra generated by the vector fields $\{g_1,\dots,g_m\}$
$\Re$	the set of real numbers
$\Re^n$	the n-dimensional Euclidean space
e^g	the formal exponential of the vector field g

2. Main Results

We consider a drift-free nonholonomic system of the form

$$(\Sigma,\mathrm{Q}): \dot{\mathbf{x}}(t)=\mathbf{g}_1(\mathbf{x})v_1+\dots+\mathbf{g}_m(\mathbf{x})v_m \quad (1)$$

where the state $x(t)$ belongs to an open subset N of $\Re^n$, while the m inputs $v_1, v_2, \dots, v_m$ take values in a discrete levels set $\mathrm{Q}=\{q_1,\dots,q_r\}$. We assume that the vector fields $g_1(x), g_2(x), \dots, g_m(x)$, which are defined on N, are smooth, complete and linearly independent. We also suppose that the system (Σ,Q) is nilpotent with order of nilpotency k. The Controllability Lie Algebra (CLA) of the system (Σ,Q), $\mathcal{L}_\Sigma$, is generated by the vector fields

$$\mathbf{G}(\mathbf{x},\mathbf{v})=\mathbf{g}_1(\mathbf{x})v_1+\mathbf{g}_2(\mathbf{x})v_2+\dots+\mathbf{g}_m(\mathbf{x})v_m$$

for all the admissible values of the input vector $v=[v_1 \ \ v_2 \ \ \dots \ \ v_m]^T$. Since we have r discrete levels, the control vector v can take values in a set containing $\mu=r^m$ admissible input vectors. We denote with $\mathrm{Q}(q_1,\dots,q_r)$ this set. Then, the CLA of (Σ,Q) is given by $\mathcal{L}_\Sigma=LC(z_i=G(x,v_i),\ \ v_i\in \mathrm{Q}(q_1,q_2,\dots,q_r))$.

We also consider the corresponding unconstrained system

$$(\mathrm{S}): \dot{\mathbf{x}}(t)=\mathbf{g}_1(\mathbf{x})u_1+\dots+\mathbf{g}_m(\mathbf{x})u_m \quad (2)$$

where the m components of the input $u_1,\dots,u_m$ take values in $\Re$ without any constraint. The CLA of (S), denoted with $\mathcal{L}_\mathrm{S}$, is given by

$$\mathcal{L}_S=LC(g_1(x),g_2(x),\dots,g_m(x)).$$

The controllability of (Σ,Q) has been studied in [8] and is described by the following proposition:

Proposition 1 ([8], Proposition 4)
Let us consider the nilpotent drift-free systems (Σ,Q) and (S) with order of nilpotency k. For every choice of the discrete levels $q_1,\dots,q_r$ such that $q_1=0$ and $q_i\neq 0$, $i=2,\dots,r$, with $r\geq m$ or $r<m$, the associated CLAs $\mathcal{L}_\Sigma$ and $\mathcal{L}_\mathrm{S}$ are such that $\mathcal{L}_\Sigma\equiv\mathcal{L}_\mathrm{S}$

In this section we are going to propose an algorithm that solves the Motion Planning Problem of the system (Σ,Q). In particular, we will propose a control algorithm that produces, for every pair x_0, x_f of points, an open loop control

$$t\to v(t)=[v_1(t) \ \ v_2(t) \ \ \dots \ \ v_m(t)]^T$$

that steers the system (Σ,Q) from the initial point x_0 to the final point x_f. The proposed algorithm is based on the algorithm of G. Lafferriere and H. J. Sussmann presented in references [1-4].

If the system (S) can be steered from x_0 to x_f then the controllability condition $\mathcal{L}_\Sigma\equiv\mathcal{L}_\mathrm{S}$ guarantees that there is a control $v(t)$ that steers the system (Σ,Q) from x_0 to x_f. Thus if the system (S) can be steered from x_0 and x_f and the discrete levels $q_1,\dots,q_r$ have been chosen in such a way that $q_1=0$ and $q_i\neq 0$, $i=2,\dots,r$ then the system (Σ,Q) can also be steered from x_0 to x_f.

According to G. Lafferriere and H.J. Sussmann in [2], the control algorithm for the system (S) involves two basic steps:

Algorithm 1: ([2])
STEP I: Find a control $w(t)$ that steers x_0 to x_f for the extended system S_e.
STEP II: Use $w(t)$ to compute a control $u(t)$ that steers x_0 to x_f for the system S .

The method that is proposed in [1-4], can mainly be applied to nilpotent or nilpotentizable drift-free systems providing exact steering. The main point of this strategy is the use of the extended system (S_e), given by

(S_e): $\dot{x}(t) = g_1(x)w_1 + \ldots + g_m(x)w_m + \ldots + g_p(x)w_p$

where $g_1(x), \ldots, g_m(x), \ldots, g_p(x)$ are higher order Lie brackets of the $g_1(x), \ldots, g_m(x)$ chosen so that they span $\Re^n$ for all x, or at least for all x in some prescribed bounded region R.

Definition

We say that the system (S) can be steered from x_0 to x_f in:

- M polynomial moves if the corresponding trajectory has the form $e^{\alpha_1 G_1} e^{\alpha_2 G_2} \ldots e^{\alpha_M G_M}$ where α_i, i=1,...,M are real and the vector fields G_i are $G_i = u_{i1} g_1 + \ldots + u_{im} g_m$ for some real u_{ij}, j=1,...,m;
- M_{bb} bang-bang moves if the corresponding trajectory has the form $e^{\alpha_1 G_1} e^{\alpha_2 G_2} \ldots e^{\alpha_{M_{bb}} G_{M_{bb}}}$ where $\alpha_i \in \Re$ and $G_i \in \{g_1, \ldots, g_m\}$ for i=1,...,m.

Proposition 2

Let us consider the nilpotent systems (Σ, Q) and (S) where the discrete levels $Q = \{q_1, \ldots, q_r\}$ are such that:

a) q_1=0

b) $q_i \neq 0$, i=2,...,r

Then the system (Σ, Q) can be steered from x_0 to x_f in at least M moves, where M is the minimum number of moves that (S) needs for the same movement.

Proof

We consider the systems (Σ, Q) and (S), while the discrete levels Q satisfy conditions (a) and (b). Let us assume that the system (S) can be steered from x_0 to x_f in, at least, M moves. Then the corresponding trajectory has the form

$$e^{\alpha_1 G_1} e^{\alpha_2 G_2} \ldots e^{\alpha_M G_M} \quad (3)$$

where α_i, i=1,...,M are real and G_i are vector fields of the form $G_i = u_{i1} g_1 + \ldots + u_{im} g_m$ for some real u_{ij}, j=1,...,m.

Next, let us assume that the system (Σ, Q) can be steered from x_0 to x_f in M moves where $M < M$. Then the corresponding trajectory has the form

$$e^{b_1 F_1} e^{b_2 F_2} \ldots e^{b_M F_M} \quad (4)$$

where b_i are real and $F_i = Q_{i1} g_1 + \ldots + Q_{im} g_m$ for i=1,...,M and $Q_{ij} \in Q$ j=1,...,m

However, trajectory (4) can also be followed by (S). This means that (S) can be steered from x_0 to x_f in $M < M$ moves, which is in contradiction with the assumptions of the proposition. Thus, (Σ, Q) needs at least M moves in order to be steered from x_0 to x_f.

Proposition 3

We consider the nilpotent systems (Σ, Q) and (S) where the discrete levels $\{q_1, \ldots, q_r\}$ are such that:

a) q_1=0

b) $q_i \neq 0$, $i = 2, \ldots, r$ and

c) at least one of q_i is negative.

Then the system (Σ, Q) can be steered from x_0 to x_f in exactly M_{bb} bang-bang moves, where M_{bb} is the minimum number of bang-bang moves that (S) needs for the same movement.

Proof

Assuming that the system (S) can be steered from x_0 to x_f in at least M_{bb} bang-bang moves, then the corresponding trajectory has the form

$$e^{\alpha_1 G_1} e^{\alpha_2 G_2} \dots e^{\alpha_{M_{bb}} G_{M_{bb}}} \qquad (5)$$

where $\alpha_i \in \Re$ and $G_i \in \{g_1, \dots, g_m\}$ for i = 1,...,m.

But we can write that $\alpha_i G_i = \frac{\alpha_i}{Q_j}(Q_j G_i)$ i,j = 1,...,M_{bb}

where $Q_j \in \{q_2, \dots, q_r\}$ and is such that $\alpha_i Q_j > 0$

Thus we can write (11) in the form

$$e^{\frac{\alpha_1}{Q_1}(Q_1 G_1)} e^{\frac{\alpha_2}{Q_2}(Q_2 G_2)} \dots e^{\frac{\alpha_{M_{bb}}}{Q_{M_{bb}}}(Q_{M_{bb}} G_{M_{bb}})} \qquad (6)$$

Relation (6) means that the system (Σ, Q) can be steered from x_0 to x_f in M_{bb} moves.

Let us assume that (Σ, Q) can be steered from x_0 to x_f in less than M_{bb} moves, i.e. in $\mu < M_{bb}$ moves. Then the corresponding trajectory has the form

$$e^{T_1(Q_1 G_1)} e^{T_2(Q_2 G_2)} \dots e^{T_\mu(Q_\mu G_\mu)} \qquad (7)$$

where T_i are positive real numbers, $Q_i \in Q$ and $G_i \in \{g_1, \dots, g_m\}$ for i=1,...,μ

But relation (7) can be written in the form

$$e^{(T_1 Q_1) G_1} e^{(T_2 Q_2) G_2} \dots e^{(T_\mu Q_\mu) G_\mu} \qquad (8)$$

which means that (S) can be steered from x_0 to x_f in $\mu < M_{bb}$ moves which is impossible. Thus (Σ, Q) can be steered from x_0 to x_f in exactly (or more) M_{bb} bang-bang moves. ■

Now, we are ready to present the algorithm that can be used to solve the MPP for the case of the system (Σ, Q). If the unconstrained system (S) can be steered for the initial point x_0 to the final point x_f then according to Proposition 1 the constrained system (Σ, Q) can also be steered from x_0 to x_f. The proposed algorithm provides exact steering and this is guaranteed by Corollary 1.

Algorithm 2

Step 1: Compute the control $u(t)$ that steers (S) from x_0 to x_f, or compute the control $w(t)$ that steers (S_e) from x_0 to x_f.

Step 2: Using $u(t)$ or $w(t)$ from the above step find the control $v(t)$ that steers (Σ, Q) from x_0 to x_f

Corollary 1

If the discrete levels satisfy the conditions of Proposition 3 then there is at least one control $v(t)$ computed by the above algorithm that steers (Σ, Q) from x_0 to x_f.

Proof

If the discrete levels satisfy the conditions (a), (b) and (c) of Proposition 3 then according to it the system (Σ, Q) can be steered from x_0 to x_f in exactly M_{bb} bang-bang moves. Thus, there is at least one control $v(t)$ that steers x_0 to x_f. In order to compute $v(t)$ we can follow the method used in the proof of Proposition 3, which is Algorithm 2 ■

3. Example

We will explore the details of our strategy through a specific example. Consider a disk rolling on a plane without slipping (pl. see Fig. 1). The configuration space is given by (x,y,ψ,θ), where (x,y) are the coordinates of the contact point, θ is the steering angle with respect to the x-axis and ψ is the contact point in wheel coordinates. The controls are the driving speed and the steering speed. The kinematics of the disk are given by the equations ([1])

$$\begin{aligned}\dot{x} &= cos(\theta)u_1\\ \dot{y} &= sin(\theta)u_1\\ \dot{\theta} &= u_2\end{aligned} \tag{9}$$

The system is locally nilpotentizable near the origin and using the feedback

$$u_1 = \frac{1}{\cos(x_3)}w_1,\ u_2 = \cos^2(x_3)w_2$$

can be transformed to

$$\begin{aligned}\dot{x}_1 &= w_1\\ \dot{x}_2 &= tan(x_3)w_1\\ \dot{x}_3 &= cos(x_3)w_2\end{aligned} \tag{10}$$

where $(x_1,x_2,x_3)=(x,y,\theta)$. System (10) is a nilpotent system with degree of nilpotency k=2, since

$$[g_1,g_2]=[0\ \ \text{-1}\ \ 0]^T$$

and $[g_1,[g_1,g_2]]=[g_2,[g_1,g_2]]=[0\ \ 0\ \ 0]^T$, where $g_1(x_1,x_2,x_3)=[1\ \ \tan(x_3)\ \ 0]^T$ and $g_2(x_1,x_2,x_3)=[0\ \ 0\ \ \cos^2(x_3)]^T$.

Our purpose is to steer (10) from the initial point $\boldsymbol{x_0}=[0\ \ 0\ \ 0]^T$ to the final $\boldsymbol{x_f}=[2\ \ 1\ \ 0]^T$. Given the backwards P. Hall coordinates $(h_1,h_2,h_3)=(2,0,-1)$ of $\boldsymbol{x_f}$ we can find numbers $\alpha_1,\alpha_2,\alpha_3$ such that

$$e^{\alpha_1 g_2}e^{\alpha_2 g_1}e^{\alpha_3 g_2} = e^{h_3[g_1,g_2]}e^{h_2 g_2}e^{h_1 g_1}.$$

Using the Sussmann formula we get

$$\alpha_1= h_2 - \frac{h_3}{h_1} = 0.5, \alpha_2= h_1=2 \text{ and } \alpha_3=\frac{h_3}{h_1}=-0.5$$

Thus, using the controls

$$\begin{aligned}(w_1,w_2)&=(0\ \ \alpha_1)=(0\ \ 0.5) && \text{for unit time}\\ (w_1,w_2)&=(\alpha_2\ \ 0)=(2\ \ 0) && \text{for unit time}\\ (w_1,w_2)&=(0\ \ \alpha_3)=(0\ \ -0.5) && \text{for unit time}\end{aligned}$$

we can steer (10) from $\boldsymbol{x_0}$ to $\boldsymbol{x_f}$.

Next, we consider the same system where the inputs w_1 and w_2 take values in the discrete levels set $\mathrm{Q}=\{0,+5,-5\}$. The q_i's are such that $q_1=0, q_i \neq 0,\quad i=2,3$ and, at least, one of them is negative. Note that the set Q satisfies the conditions of Proposition 6. Thus, the const-rained system can be steered from $\boldsymbol{x_0}$ to $\boldsymbol{x_f}$ in three moves. In fact, using the controls

$$\begin{aligned}(v_1,v_2)&=(0\ \ 5) && \text{for time } 0.1\\ (v_1,v_2)&=(5\ \ 0) && \text{for time } 0.4,\\ (v_1,v_2)&=(0\ \ -5) && \text{for time } 0.1\end{aligned}$$

the system follows the trajectory

$$e^{0.1(5g_2)}e^{0.4(5g_1)}e^{0.1(-5g_2)}$$

that steers it from $\boldsymbol{x_0}$ to $\boldsymbol{x_f}$.

4. Conclusions

The motion planning problem for nonholonomic systems without drift is solved under the constraint that the control inputs can take values in a set that contains a finite number of allowable levels. The proposed algorithm can be applied to nilpotent or nilpotentizable drift-free systems providing exact steering. In particular, if the corresponding unconstrained system can be steered from an initial point $\boldsymbol{x_0}$ to a final point $\boldsymbol{x_f}$ then the constrained system can also be steered from $\boldsymbol{x_0}$ to $\boldsymbol{x_f}$. The results are applied in the steering of a simple system.

References

[1] Lafferriere, G., "A general strategy for computing steering controls of systems without drift", Proc. of the 30th Conference on Decision and Control, Brighton, England, Dec. 1991.

[2] Lafferriere, G., and Sussmann H.J., "A differential geometric approach to motion planning", in Z. Li and J. F. Canny, (eds), Nonholonomic Motion Planning, Kluwer, pp. 235-270, 1993.

[3] Lafferriere, G., and Sussmann H.J., "Motion Planning of controllable systems without drift: a preliminary report" SYCON report 90-04, Rutgers Center for Systems and Control, Rutgers University, New Brunswick, New Jersey, July 1990.

[4] Lafferriere, G., and Sussmann H.J., "Motion Planning of controllable systems without drift", Proc. IEEE International Conference on Robotics and Automation, Sacramento, CA, April 1991.

[5] Latombe, J. C., Robot Motion Planning, Kluwer, Boston, 1991.

[6] Li, Z., and Canny, J. F. (eds), Nonholonomic Motion Planning, Kluwer, 1993.

[7] Murray, R. M., Li, Z., and Sastry, S. S., A Mathematical Introduction to Robotic Manipulation, CRC Press, 1994.

[8] Skiadas, Ph., and Koussoulas, N. T., "Discrete Levels Control of Nonlinear Systems", submitted to the 5th IEEE Mediterranean Conference on Control and Systems, Paphos, Cyprus, July 1997.

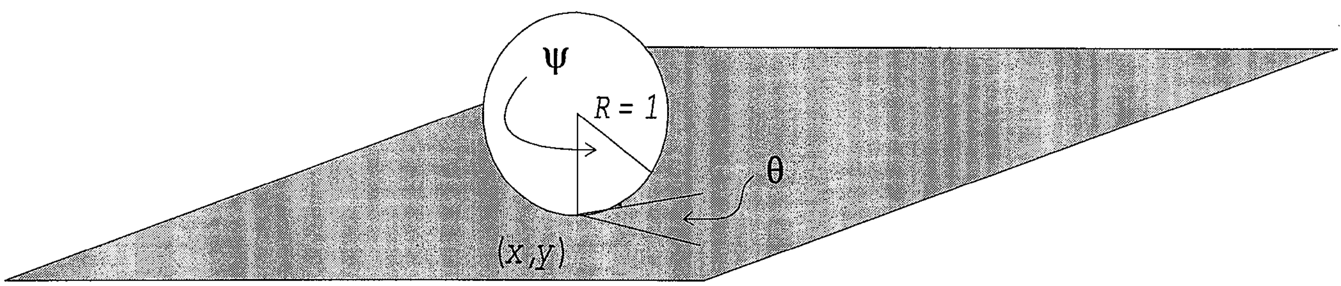

Fig. 1 A disk rolling on the plane