ℓ^1-Optimal Control for Multirate Systems under Full State Feedback [1]

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Abstract

This paper considers the minimization of the ℓ^∞-induced norm of the closed loop in linear multirate systems when full state information is available for feedback. A state-space approach is taken and concepts of viability theory and controlled invariance are utilized. The essential idea is to construct a set such that the state may be confined to that set and that such a confinement guarantees that the output satisfies the desired output norm conditions. Once such a set is computed, it is shown that a memoryless nonlinear controller results, which achieves near-optimal performance. The construction involves the solution of several finite linear programs and generalizes to the multirate case earlier work on linear time-invariant (LTI) systems.

1. Introduction

Multirate sampled data systems arise in many applications in which it is desirable to use multiple sampling rates for controlling a continuous-time system. The impetus to use multiple sampling rates could result from, for instance, differing bandwidths of input signals or differing limitations of the physical sensors and actuators used to implement a control algorithm. In addition, if the exogenous inputs or the regulated outputs are continuous signals, a multirate model can be used to approximate these continuous signals to any degree of accuracy. As a result, it is important to be able to design controllers for multirate sampled data systems that perform optimally in some sense.

In this paper the notion of optimality is with respect to ℓ^∞ performance. In particular, we are interested in minimizing the ℓ^∞-induced norm of the closed loop map. In the linear time invariant (LTI) case this amounts to minimizing the corresponding ℓ^1 norm. This ℓ^1 problem can be solved using input-output techniques and duality theory (e.g., [5]). For linear multirate sampled data (LMRSD) systems the problem is solved in [3] using again an input-output viewpoint and lifting techniques developed in [6,7,8] that convert the problem to an LTI, however nonstandard, problem.

Although the problem of ℓ^∞-gain minimization is solved in the input-output framework for both LTI and LMRSD systems certain characteristics of their solutions may not be desirable. In particular, considering the ℓ^1-optimal control problem with full state feedback it was shown [4] that, unlike the $\mathcal{H}^\infty$-optimal case, optimal as well as near-optimal controllers can be dynamic and of arbitrarily high order. This result motivated a new, state-space, approach to the ℓ^1 problem when the state is available for feedback. Recent work in [9,10] towards this direction has shown that static *nonlinear* state feedback performs as well as linear dynamic feedback. In other words, full state feedback ℓ^1-optimal control need not require dynamics if nonlinear controllers are admissible. Moreover, a constructive, finite-step, algorithm for near-optimal nonlinear state feedback is furnished. The approach in the work of [9,10] is to construct controlled invariant sets in the context of viability theory and differential inclusions. It is precisely this work that we generalize to the multirate case in this paper. We show that a memoryless nonlinear controller can be constructed to achieve near-optimal performance.

The remainder of this paper is organized as follows. Section 2 presents some background material. Section 3 presents the problem formulation. Section 4 discusses the notion of a multirate controlled invariance kernel. Section 5 introduces machinery necessary for the construction of an ℓ^1-optimal multirate controller, and outlines an algorithm to construct such a controller. Finally, Section 6 presents an example.

2. Mathematical Preliminaries

First, we give some basic notation : $\mathcal{R}^+$ denotes the set of nonnegative real numbers and $\mathcal{Z}^+$ denotes the set of nonnegative integers. For $M \in \mathcal{R}^{m\times n}$, let $M_{(i,j)}$ denote the ij^{th} element of M, let $M_{(i,:)}$ denote the i^{th} row of M, and let $M_{(:,j)}$ denote the j^{th} column of M. Define $\left|M_{(i,:)}\right| := \sum_{j=1}^{n} \left|M_{(i,j)}\right|$, and $|M| = \max_i \left|M_{(i,:)}\right|$. Similarly for $x \in \mathcal{R}^n$, let x_i denote the i^{th} component of x and define $|x| = \max_i |x_i|$. The appropriate definition of $|\cdot|$ will be apparent from context. Let $\ell_n^\infty(\mathcal{Z}^+)$ denote the set of bounded one-sided sequences in $\mathcal{R}^n$. For $f = \{f(0), f(1), f(2), \ldots\} \in \ell_n^\infty(\mathcal{Z}^+)$, define $\|f\| := \sup_{t\in\mathcal{Z}^+} |f(t)|$. A causal operator $H : \ell_n^\infty(\mathcal{Z}^+) \to \ell_m^\infty(\mathcal{Z}^+)$ is called stable if $\|H\| := \sup_{f\in\ell_n^\infty, f\neq 0} \frac{\|Hf\|}{\|f\|} < \infty$.

A set-valued map $F : X \rightsquigarrow Y$ is a mapping from individual points $x \in X$ to sets $F(x) \subset Y$. The domain

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of a set-valued map F is defined as $dom(F) = \{x \in X : F(x) \text{ is non-empty}\}$.

In the sequel, some elements of viability theory will be required. For a more complete treatment of viability theory, the interested reader should consult [1,2].

3. Problem Formulation

In this paper, the ℓ^1-optimal control problem for a linear multirate system with state feedback available is considered. The system equations are given by

$$\begin{aligned} x(t+1) &= Ax(t) + Ew(t) + Bu(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \end{aligned} \quad (1)$$

where: $x \in \mathcal{R}^n$ contains the state of the system, $w(t) \in \mathcal{R}^q$ contains exogenous inputs, $u(t) \in \mathcal{R}^m$ contains control inputs, $z(t) \in \mathcal{R}^p$ contains regulated outputs. The measured inputs of the system (i.e. the states) are sampled at rates of l_1T, l_2T,...,l_nT. The control inputs are delivered to the system at rates of k_1T, k_2T,...,k_mT. It is assumed that T is the least common sampling interval among all of the inputs and the outputs. The noise enters the system discretely at a rate of T time units. The assumption that the noise enters at the fastest sampling rate simplifies the solution of the ℓ^1-optimal control problem, but it is removal through straightforward extensions of the algorithms which appear in this paper. It is also assumed that the sampling intervals of the inputs and the outputs are all synchronized, such that the jump discontinuities in the inputs and the outputs occur at the same time instant. State feedback is assumed to be available such that the measured outputs (denoted as $y \in \mathcal{R}^n$) are the states.

The controllers, $\mathcal{K}_{multi}$, which are **admissible** for this system are memoryless multirate controllers which are, in general, a nonlinear function of the state. By memoryless, it is meant that the controllers may be defined without introducing additional state variables to the system. And, multirate refers to the above stipulation that the inputs and the regulated outputs may appear at different rates. Given an admissible controller, $\mathcal{K}_{multi}$, define $T_{zw}(\mathcal{K}_{multi})$ to be the forced dynamics from w to z with zero initial conditions. Similarly define $T_{xw}(\mathcal{K}_{multi})$ and $T_{uw}(\mathcal{K}_{multi})$.

Definition 3.1 *An admissible p-periodic controller, $\mathcal{K}_{multi}$, is said to be internally stabilizing with a performance (resp., strict performance) of γ if 1) the unforced dynamics ($w = 0$) are globally exponentially stable and 2) the forced dynamics with zero initial conditions satisfy $||T_{zw}(\mathcal{K}_{multi})|| \le \gamma$, (resp., $||T_{zw}(\mathcal{K}_{multi})|| < \gamma$), with both $||T_{xw}(\mathcal{K}_{multi})||$, $||T_{uw}(\mathcal{K}_{multi})|| < \infty$.*

The optimum performance problem can now be postulated as

$$\gamma_{opt} = \inf_{\mathcal{K}_{multi}} \{||T_{zw}(k)|| : \mathcal{K}_{multi} \text{ is admissible and internally stabilizing}\}$$

We point out that arbitrary time variation does not offer any advantage over multirate if the controller is linear [3]. Moreover, it can be deduced from the developments of Section 5 that a memoryless nonlinear controller can at least match the performance of any linear one. Finally, it also can be concluded from the results of Section 5 that a dynamic controller does not outperform a memoryless periodic one. Hence, the class of admissible controllers is not restrictive.

4. Multirate Controlled Invariance

In this section, the concept of the multirate controlled difference inclusion is introduced. For a particular multirate controlled difference inclusion, the structures which are of particular interest are multirate controlled invariant sets. If a multirate system begins within such a set, then it will be confined to that set for all time under the action of the associated controlled difference inclusion. This invariance property will be exploited in the construction of a controller which solves the stated ℓ^1-optimal control problem. Due to the requirements of this control law, which will be discussed in the next section, it is necessary to consider simultaneously the behavior of the multirate system at each step of a time interval of R steps (i.e. RT time units), where $R = LCM(l_1, l_2, \ldots, l_n, k_1, k_2, \ldots, k_n)$. As a consequence, the definition of a controlled difference inclusion must be appropriately adapted in order to be used to model multirate systems. Specifically, it must be altered such that the behavior of the multirate system for R steps is described. This requirement is met by the following definition.

Definition 4.1 *Let $F : \mathcal{R}^n x \mathcal{R}^{Rm} \leadsto \mathcal{R}^n$ be a set valued map. Define*

$$\tilde{F}(x) = \left\{ \bigcup_{\substack{u^i \in R^m \\ i \in \{0,\ldots,R-1\}}} F(x, u^0, \ldots, u^{R-1}) \right\}.$$

Then, $x(j+R) \in \tilde{F}(x(j))$ is the **multirate controlled difference inclusion** *defined by F.*

In the above definition, the variables $u^0, \ldots, u^{R-1}$ represent the control inputs at times $Rj, \ldots, Rj+(R-1)$. Also, the time interval described by a multirate controlled difference inclusion will always begin and end at time steps at which the system has access both to all the states and to all the controls. Note that the shortest length of time between such time steps is in fact R time steps. Another important detail of the above definition is that the output of the multirate system can only be considered every R steps when modeled with a multirate controlled difference inclusion. Therefore, when applied to an ℓ^1-optimal control problem, multirate controlled difference inclusions clearly must also incorporate the required bounds on the outputs of intermediate steps. This will be accomplished by appropriately defining the set-valued map $F(x, u^0, \ldots, u^{R-1})$.

As previously indicated, the concept of the controlled invariance of a multirate controlled difference inclusion is integral to the construction of an ℓ^1-optimal control law. The essential idea is to define a set which will insure that the required output ℓ^∞-norm bounds are met and to then search for the largest subset to which the multirate system can be confined under some admissible control law for all time. If such a set exists, then an ℓ^1-optimal controller can be constructed. Formally, a controlled invariant set for a multirate controlled difference inclusion satisfies the following definition.

Definition 4.2 *Consider the multirate controlled difference inclusion defined by F. A set $K \subset \mathcal{R}^n$ is* **multirate controlled invariant** *under F if $\forall x \in K$, there exist $u^i \in \mathcal{R}^m, i \in \{0, \ldots, R-1\}$, such that $F(x, u^0, \ldots, u^{R-1}) \subset K$.*

An important type of multirate controlled invariant set is the multirate controlled invariance kernel, which is the largest multirate controlled invariant set in the sense given by the below definition.

Definition 4.3 *Consider the multirate controlled difference inclusion defined by F. Let the set K be a subset of $\mathcal{R}^n$. The* **multirate controlled invariance kernel** *of K for F, denoted as $\textsc{Cinv}(K)_R$, is the largest closed subset of K such that for all $x \in \textsc{Cinv}(K)_R$, there exist $u^i \in \mathcal{R}^m, i \in \{0, \ldots, R-1\}$, such that $F(x, u^0, \ldots, u^{R-1}) \subset \textsc{Cinv}(K)_R$. Here, the term largest implies that $\textsc{Cinv}(K)_R$ contains all other closed subsets of K with the above invariance property.*

An algorithm for the construction of the multirate controlled invariance kernel is given in the following proposition, which follows almost immediately from the Controlled Invariance Kernel Algorithm contained in [11].

Proposition 4.1 *Let $F : \mathcal{R}^n x \mathcal{R}^{Rm} \rightsquigarrow \mathcal{R}^n$ be a lower semicontinuous set valued map. Also, assume that the set $\bigcup_n F(s_n, u_n)$ is bounded if and only if the sequences $\{u_{i_n}\} \in \mathcal{R}, i \in \{0, \ldots, R-1\}$ and $x_n \in \mathcal{R}^n$ are bounded. Let $K \subset \mathcal{R}^n$ be a compact set. Define $K_0 = K$, and recursively define the subsets K_{Rj} of K, for $j = 1, 2, \ldots$, by*

$$K_{Rj} = \{x \in K_{R(j-1)} : F(x, u^0, u^1, \ldots, u^{R-1}) \subset K_{R(j-1)}, \quad \text{with } u^i \in \mathcal{R}^m, i \in \{0, \ldots, R-1\}\}$$

Then

$$\textsc{Cinv}(K)_R = \bigcap_{j=0}^{\infty} K_{Rj}$$

The construction of a multirate controlled invariance kernel $\textsc{Cinv}(K)_R$ is integral to the construction of the ℓ^1-optimal control law developed in this paper for multirate systems. It is important to note that in the most general sense, the definition of multirate controlled difference inclusions allows the control input to be non-causal and to depend upon unavailable state information. As discussed in the following section, this potential difficulty can be avoided by imparting to the multirate controlled difference inclusion a form which depends upon the particular multirate system of interest.

5. Construction of Multirate Control Laws

In this section, the multirate controlled invariant set $\textsc{Cinv}(\textsc{Object}^0_\gamma)$ is defined and it's role in the construction of an admissible multirate controller which achieves a performance arbitrarily close to the optimum is described. The following assumptions are made for the remaining portion of this paper in order to simplify the construction of the controller and the arguments of the proofs which follow.

Assumption 5.1

1. *$rank(E) = rank(C_1(t)) = n$*
2. *$rank(B(t)) = m$*

These two assumptions simplify greatly the construction of the control law. It should be noted that is possible to remove the rank assumption on E with arbitrarily small perturbations to E. The rank assumption on each C_1 may also be removed, but this must be done by introducing new, non-trivial outputs in order to avoid numerical difficulties and to insure a reasonable bound on the plant states. The final assumption insures that there will be no control redundancies, and it may be removed by arbitrarily small perturbations to B.

For the remainder of this paper, it also will be assumed that *both the states and control input only have rates of T and $2T$*, such that $R = 2$. The states and control variables with the same sampling rates will be grouped together, such that x_1 (u_1) contains all states (control inputs) which appear at rates of T, and x_2 (u_2) contains all states (control inputs) which appear at rates of $2T$. This assumption that the multirate system possesses only sampling rates of T and $2T$ will greatly simplify and clarify the presentation of the multirate control law construction algorithm. But the algorithm may be straightforwardly extended to the general multirate problem, as will be briefly discussed at the close of this section.

The first step in constructing an ℓ^1-optimal controller is to connect the state equations given by (1) to a multirate controlled difference inclusion which will be suitable for use in Proposition 4.1. As previously indicated, this multirate controlled difference inclusion must be peculiarly defined in order to insure that the resulting controller is causal and that only available state information is used to produce control inputs. Writing $x(j)$ as x^j, the form of this multirate controlled difference inclusion is as follows

$$F_\gamma(x^{2j}, u^{2j}, u^{2j+1}) = \Big\{ f^1(x_1^{2j+1}, \tilde{x}_2^{2j+1}, u_1^{2j+1}, x^{2j}, u^{2j}, w^{2j+1}), \forall \left|w^{3j}\right| \le \tfrac{1}{\gamma}, \forall \left|w^{2j+1}\right| \le \tfrac{1}{\gamma} : . \left|C_1 x^{2j} + D_{11} w^{2j} + D_{12} u^{2j}\right| \le 1 \text{ and } \left|C_1 x^{2j+1} + D_{11} w^{2j+1} + D_{12} u^{2j+1}\right| \le 1 \Big\}$$

where

$$x^{2j+1} = f^0(x^{2j}, u^{2j}, w^{2j}) = Ax^{2j} + Bu^{2j} + Ew^{2j}$$

and

$$f^1(\cdot) = \left[\begin{array}{c|c} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \begin{pmatrix} x_1^{2j+1} \\ --- \\ \tilde{x}_2^{2j+1} \end{pmatrix} + \left[\begin{array}{cc|cc} B_{11} & 0 & 0 & B_{12} \\ B_{21} & 0 & 0 & B_{22} \end{array}\right] \begin{pmatrix} u_2^{2j} \\ u_1^{2j} \\ --- \\ u_2^{2j+1} \\ u_1^{2j+1} \end{pmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w^{2j+1}$$

and

$$\tilde{x}_2^{2j+1} \in Est(x_1^{2j+1}, x^{2j}, u^{2j})$$

where $Est(x_1^{2j+1}, x^{2j}, u^{2j})$ is the "estimating set" for x_2^{2j+1}. The estimating set is the smallest set to which x_2^{2j+1} can be guaranteed to belong, given the available information (i.e. x_1^{2j+1} and the control inputs). This set may be constructed by using the techniques used in [11] to contruct ℓ^1-optimal estimators. It is necessary to use this estimating set in place of the actual value of x_2^{2j+1} in the definition of the multirate controlled difference inclusion in order to unsure that the controller does not depend on x_2 at the odd time steps (at which x_2 is not measured). By using $Est(\cdot)$, the controller is forced to achieve simultaneously the control objective for any value of x_2^{2j+1} in $Est(\cdot)$. That is, the lack of information about x_2^{2j+1} induces an extra measure of conservativeness. Also note that u_2^{2j+1} has been replaced by u_2^{2j} in F^γ. This insures that u_2^{2j+1} (which is not available) is not used as a control input. For the above definition of f^1, the A and B system matrices are divided into blocks which correspond to the dimensions of x_1, x_2, u_1 and u_2.

The multirate controlled difference inclusion defined by F_γ is equivalent to two time steps of the system equations (1) for $\|w\|_\infty \le 1/\gamma$ and $\|z\|_\infty \le 1$. Also, note thatWs discussed f_1 does not include x_2^{2j+1}or u_2^{2j+1}, since x_2 and u_2 are only available at times $t = 2j, j \in \{0, 1, \ldots\}$. Now, define the set $\textsc{Object}_\gamma^0$ such that

$$\textsc{Object}_\gamma^0 = \{x \in \mathcal{R}^n : |C_1 x + D_{11} w + D_{12} u| \le 1, \text{ for some } u \in \mathcal{R}^m \text{ and } \forall |w| \le \tfrac{1}{\gamma}\}$$

In can be shown that F^γ satisfies the assumptions of Proposition 4.1, such that the algorithm detailed in Proposition 4.1 may be used to construct the multirate controlled invariance kernel $\textsc{Cinv}(\textsc{Object}_\gamma^0)$, when it exists, of $\textsc{Object}_\gamma^0$ when subject to F_γ.

As indicated in Section 4, the concept of the multirate controlled invariance kernel is integral to the formation of an ℓ^1-optimal controller. Specifically, the controlled invariance kernel of interest is $\textsc{Cinv}(\textsc{Object}_\gamma^0)$. Clearly, if the state is confined at time steps $2j$ to $\textsc{Object}_\gamma^0$, then *the ℓ^∞-norm of the output at time steps $2j$ will be less than or equal to one.* The ℓ^∞-norm of the output at all intermediate times will also be less than one due to the definition of the multirate controlled difference inclusion F_γ. This ability to bound the ℓ^∞-norm of the output at all times, suggests the following algorithm for the construction of an optimal control law.

The first step is to construct the multirate controlled invariance kernel $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ for a particular $\gamma > 0$, using the algorithm described in Proposition 4.1. Practically, $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ will be difficult to form if the infinite intersection $\bigcup_{j=0}^\infty K_{2j}$ does not converge within a finite and suitably small number steps. An alternative is to truncate the invariance kernel algorithm at a point such that adding sets to the aforementioned intersection only produces an incremental change which is small in some sense, as was done for the LTI, single-rate problem in [10]. If it is determined that $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ does not exist (i.e. $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ is empty), then γ has been chosen too small. In fact, it can be shown that if $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ does not exist for a particular γ, then it is not possible to find a controller with a performance level of γ. Therefore, if $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ does not exist, γ should be increased, and the algorithm should be re-run.

If γ is not too small, then the second step of the algorithm may be run. This second step is to determine the set of all controls by which the state can be confined within the multirate controlled invariant set $\textsc{Cinv}(\textsc{Object}_\gamma^0)$. This may be done by utilizing one step of the algorithm in Proposition 4.1, in which K_0 is initialized as $\textsc{Object}_\gamma^0$. Clearly, K_1 will equal $\textsc{Object}_\gamma^0$, and the set of all control values which forces the state from K_1 in to K_0 may be recorded. A memoryless multirate controller may then be chosen from this set of potential controls. By construction, this controller will have a performance level of γ. If this performance level is not small enough or a performance level closer to the optimal value is desired, then γ should be decreased by an appropriate value and the algorithm should be re-run from the first step.

The extension to the multirate problem is straightforward, and primarily involves redefining F^γ to cover R time steps of the system, rather than just two time steps. Care must be taken, as in the two rate case, to insure that neither unavailable states nor unavailable controls appear in the definition of F^γ.

6. Example

As an example of the construction of a memoryless multirate controller, consider the multirate system

$$x(t+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w(t)$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t)$$

Where

$$\begin{array}{ll} l_1 = 1 & k_1 = 1 \\ l_2 = l_3 = 2 & k_2 = 2 \end{array}$$

such that x_2, x_3, and u_2 appear at a rate of 2T; and x_1 and u_1 appear at a rate of T. Choosing $\gamma = 4$, which implies that $\|w\|_\infty \le \frac{1}{4}$, the algorithm converges to $\mathrm{CINV}(\mathrm{OBJECT}^0_\gamma)$, which is given by the following expression

$$\mathrm{CINV}(\mathrm{OBJECT}^0_\gamma) = \left\{ x : |x_1| \le 1, |x_2| \le \frac{3}{4}, |x_3| \le 1 \right\}$$

Having obtained $\mathrm{CINV}(\mathrm{OBJECT}^0_\gamma)$ for $\gamma = 4$, a controller must be found which insures that the system remains within this controlled invariant set. As previously indicated, a memoryless controller may be constructed by initializing $K_0 = \mathrm{CINV}(\mathrm{OBJECT}^0_\gamma)$ and applying one step of the algorithm in Proposition 4.1. One possible controller, which results from this type of construction and which achieves a performance of $\gamma = 4$ has the form $u(x, j) = (u_1(x)u_2(x))'$, where

$$\begin{array}{rcll} u_1(j) & = & g_{21}(x(j)) & \text{:for j } \textbf{even} \\ & = & g_{21}(x(j-1)) & \text{:for j } \textbf{odd} \end{array}$$

$$\begin{array}{rcll} u_2(j) & = & g_{11}(x(j)) & \text{:for j } \textbf{even} \\ & = & g_{12}(x_1(j), x(j-1)) & \text{:for j } \textbf{odd} \end{array}$$

and

$$g_{21}(j) = \tfrac{1}{2}\max\left\{-1, -\tfrac{1}{2} - x_3(j)\right\} + \tfrac{1}{2}\min\left\{1, \tfrac{1}{2} - x_3(j)\right\}$$

$$g_{11}(j) = \tfrac{1}{2}\max\left\{-\tfrac{3}{4} - x_1(j), -\tfrac{1}{4} - x_1(j) - u_1(j)\right\} + \tfrac{1}{2}\min\left\{\tfrac{3}{4} - x_1(j), \tfrac{1}{4} - x_1(j) - u_1(j)\right\}$$

$$g_{12}(j) = \tfrac{3}{4} - x_1(j)$$

Using the above controller, simulations were run in which the disturbance $w(t)$ was chosen with a uniform distribution such that $\|w\|_\infty \le \frac{1}{4}$. In these simulations, it was found that $\|z\|_\infty$ remains less than one, thereby confirming the efficacy of the controller.

7. Conclusions

A state-space approach was taken and the concepts of viability theory and controlled invariance were used to produce a method for the construction of near optimal control laws for multirate systems when full state information is available for feedback. As previously discussed, the algorithm which was described in this paper may be extended straightforwardly to the general multirate system. It should be noted, that the control laws resulting from the algorithm presented here only guarantee a performance of γ if the noise is fixed at a level of $\frac{1}{\gamma}$. The difficulty arises from the formulation of the problem as the controlled invariance problem. However, a controller with a guaranteed induced norm level can be formulated by scaling the state as was done in [10] for LTI, single-rate systems. The resulting optimum control laws are static and contain R different piecewise linear elements, where R is the least common multiple of all the sampling rates, which are sequentially applied to the multirate system. This construction method is attractive due to the desirable static nature of the resulting control laws. Thus, it can potentially serve as an alternative to the well-known input-output synthesis methods.

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