Adaptive Identification of Distributed Parameter Systems

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Abstract

This work presents synthesis of adaptive identifiers for distributed parameter systems (DPS) described by partial differential equations (PDE's) of parabolic, elliptic, and hyperbolic type. The fundamental concept of identifiability is studied. Adjustable parameters in the adaptive identifiers proposed are shown to admit simultaneous convergence to their nominal space-varying values when an appropriate input signal is used. The class of sufficiently rich input signals referred to as generators of persistent excitation is defined. This class guarantees the existence of a unique zero steady state for the parameter errors, thereby yielding unknown plant parameters.

Keywords: distributed parameter system, identifiability, persistent excitation, adaptive identifier, Lyapunov functional.

I. Introduction

Adaptive identification [1] assumes construction of a model, parameters of which evolve in time and asymptotically converge to the unknown parameters of the plant. Due to the relative simplicity of implementation and some degree of robustness with respect to small perturbations of the plant dynamics adaptive identification of lumped parameter systems found practical applications both by itself and as a part of an adaptive control system.

Some of the general ideas used in the paper, such as the use of Liapunov function for the adaptive identification in distributed parameter systems have appeared in the literature [2]. The first richness-like condition for parameter identifiability in DPS, namely elliptic PDE, was introduced by Baumeister and Scondo in 1987 in [3]. Subsequently, the use of persistency of excitation for parameter identification of parabolic PDE's as a part of an adaptive control law, and an infinite dimensional analog of Barbalat's lemma for convergence proof, was introduced into distributed parameter systems by Bentsman and Hong in 1991 [4]. This topic was further developed by them in a series of papers [5-12], including the case of the spatially and time-varying parameters in the works with Solo [7, 10].

Following this work, Demetriou and Rosen and co-workers in a series of papers (cf. [13, 14] and ref. in [14]) generalized the above results to abstract setting, including hyperbolic and implicit parabolic PDE's.

The previous work on adaptive identification of parameters in parabolic and hyperbolic PDE's, however, utilized the second spatial state derivatives, thereby making the algorithms very sensitive to noise and/or nonsmoothness of solutions and greatly reducing their applicability. The only exceptions where second spatial derivatives are not used are the works [29, 30] and related works by the same authors. These works treat only single parameter in elliptic and parabolic equations and do not mention persistency of excitaion, which is not required for the problems considered therein.

The contribution of the present work is threefold: it introduces the adaptive identification laws that do not require second spatial derivatives of the state, presents explicit persistency of excitation-type conditions for the identifiability of space-varying DPS described by PDE's of parabolic, elliptic and hyperbolic types, and extends the adaptive identification within this framework to a class of implicit hyperbolic pde's, i.e. the ones with an unknown parameter in front of the highest time derivative. Distributed sensing is assumed to be available. The L_2 convergence of parameters is shown, which also leads to the pointwise convergence under additional smoothness assumptions. The synthesis is carried out in the infinite-dimensional setting, yielding algorithms, the numerical approximation of which can be carried out at the implementation stage.

The following mathematical models of DPS are used subsequently:

(i) a heat conduction parabolic PDE with spatially varying coefficients which has the form

$$\rho(x)\dot{Q} = \left[k(x)Q'\right]' - q(x)Q + f(x,t), 0 < x < 1, t > 0, \quad (1.1)$$

$$Q(x,0) = Q_o(x), 0 \le x \le 1$$

with the homogeneous Neumann boundary conditions

$$Q'(0,t) = Q'(1,t) = 0, t > 0 \quad (1.1a)$$

or nonhomogeneous Dirichlet boundary conditions

$$Q(0,t) = \beta_o(t), Q(1,t) = \beta_1(t), t > 0; \quad (1.1b)$$

(ii) a steady state regime of the heat equation which has the form of an elliptic PDE

$$\left[k(x)Q'\right]' - q(x)Q + \varphi(x) = 0, 0 < x < 1, \quad (1.2)$$

$$Q'(0) - Q'(1) = 0, \quad (1.2a)$$

$$Q(0) = \zeta_o, Q(1) = \zeta_1, \quad (1.2b)$$

with

$$\zeta_o = \lim \beta_o(t), \ \zeta_1 = \lim \beta_1(t),$$

$$\lim \int_0^1 \left[\varphi(x) - f(x,t)\right]^2 dx = 0 \ as \ t \to \infty; \quad (1.3)$$

(iii) a vibrating string hyperbolic PDE with spatially varying coefficients which has the form

$$p(x)\ddot{\theta} = \left[æ(x)\theta'\right]' - g(x)\theta + \psi(x,t), \ 0 < x < 1, \ t > 0,$$

$$\theta(x,0) = \theta_o(x),\ \dot{\theta}(x,0) = \theta_1(x),\ 0 < x < 1 \qquad (1.4)$$

with the homogeneous Neumann boundary conditions

$$\theta'(0,t) = \theta'(1,t) = 0,\ t > 0 \qquad (1.4a)$$

or nonhomogeneous Dirichlet boundary conditions

$$\theta(0,t) = \omega_o(t), \theta(1,t) = \omega_1(t),\ t > 0. \qquad (1.4b)$$

Equation (1.1) describes the propagation of heat in a one-dimensional rod, insulated at both ends in the case of boundary conditions (1.1a) or with a given temperature at the ends in the case of boundary conditions (1.1b), where $k(x) > 0$ is a smooth heat conduction coefficient, $q(x) \geq 0$ is a continuous coefficient of the heat exchange with the surroundings, $\rho(x) > 0$ is a continuous heat capacity, $f(x,t), \beta_o(t), \beta_1(t)$ and $Q_o(x,t)$ are sufficiently smooth external and boundary inputs and an initial condition.

Equation (1.2) describes a steady state regime of the temperature field.

Equation (1.4) describes the oscillations of a string with fixed ends in the case of boundary conditions (1.4a) or with the specified ends motion in the case of boundary conditions (1.4b), where æ(x) > 0 is the smooth elasticity coefficient, p(x) > 0 is a continuous density coefficient, g(x) ≥ 0 is a continuous restoring stiffness coefficient, and y(x,t), $w_o(t)$, $w_1(t)$, $q_o(x)$, $q_1(x)$ are sufficiently smooth external and boundary inputs and initial conditions.

The above assumptions guarantee existence, uniqueness, and smoothness of the solutions of the boundary value problems [19]. In the case of the nonhomogeneous boundary conditions the solutions of the above parabolic and hyperbolic equations are understood in a mild sense [17] as a result of the convolutions of the input functions and Green functions of the corresponding boundary value problems.

This work demonstrates that it is in principle possible to simultaneously identify all spatially distributed plant parameters both in the case of heat processes (1.1) and that of mechanical oscillators (1.4). In the steady state (1.2) only one of the two parameters k(x) and q(x) is identifiable under the assumption that the other parameter is known in advance. For the classes of DPS considered, the sufficient identifiability conditions for the unknown parameters are given, which place additional constraints on the external and boundary inputs.

The main result of the present work is the construction of the adaptive identifiers of the spatially varying coefficients for heat process (1.1), steady state regime (1.2), and vibrating string (1.4) under the assumptions that unknown parameters are identifiable and the system state and input functions can be measured at all points of $x \in [0,1]$ and $t \geq 0$.

The adaptive identifiers are represented as error systems describing the evolution of the state error and the parameter error. The state and the parameter error systems take the form of PDE and ODE (ordinary differential equation), respectively. The justification of the convergence of the tunable parameters of the adaptive identifier to the parameters of the plant is based on the extension to the infinite-dimensional case [20] of the method of Lyapunov functions with nonpositive derivative along the system trajectories. For the boundary conditions considered, Lyapunov functional with nonpositive time derivative along the solution of the coupled identifier-plant system is constructed. It takes zero values on a certain manifold in the state space. A sufficiently rich input signal, called a generator of persistent excitation guarantees the absence of the nontrivial trajectories on this manifold, and thereby ensures the existence of a unique zero steady state for the parameter errors. Since the identification problems considered are ill-posed [2,16], the regularization principle [21,22] is used to justify the well-posedness of the algorithms proposed.

The identification algorithms proposed here for Neumann and Dirichlet boundary conditions do not admit easy generalization to nonhomogeneous Neumann boundary conditions and to the mixed homogeneous and nonhomogeneous boundary conditions, since in these cases the expressions for the derivatives of the corresponding Lyapunov functionals contain the terms linearly dependent on the parameter errors which can take positive as well as negative values. The numerical simulation of the adaptive identifier of the heat process with mixed boundary conditions confirmed the lack of the parameter convergence in the case of mixed Neumann and Dirichlet problem.

For simplicity, the presentation here is limited to one space variable; however, the extension to the case of several spatial variables is straightforward. This, in particular, permits the use of the results presented in this paper for the identification of the parameters of geophysical and electromagnetic fields, described by the two- and three-dimensional elliptic PDE's. We also note that for a vibrating beam described by fourth-order hyperbolic PDE [19] a similar plant identifier can be designed as well.

The paper has the following structure. Section II introduces formal definitions of the parameter identifiability for systems (1.1), (1.2), and (1.4). The class of sufficiently rich signals which ensure parameter identifiability is presented in Section III. Design of adaptive identifiers is given in Section IV. Finally, Section V presents the conclusions.

II. Parameter Identifiability

In the present work identifiability is understood as a uniqueness of the solution of the parameter identification problem for a given set of measurements. We restrict our consideration to the full state information case when distributed sensing is available. Below, the identifiability concept is specialized for each class of the DPS considered.

A. Heat Processes

Along with the system (1.1)-(1.1a) ((1.1)-(1.1b)) consider its model in which k(x), r(x), q(x) are replaced by $\tilde{k}(x), \tilde{\rho}(x), \tilde{q}(x)$, respectively. The output error ΔQ between the system state Q(x,t) and the model state $\tilde{Q}(x,t)$ and parameter mismatch, Δk, Δr, and Δq are given by

$$\Delta Q(x,t) = Q(x,t) - \tilde{Q}(x,t),\ \Delta k(x) = k(x) - \tilde{k}(x),$$
$$\Delta\rho(x) = \rho(x) - \tilde{\rho}(x),\ \Delta q(x) = q(x) - \tilde{q}(x),\ 0 \le x \le 1,\ t \ge 0. \quad (2.1)$$

Definition 2.1 A set of parameters {k(x), r(x), q(x)} of the heat process (1.1)-(1.1a) ((1.1)-(1.1b)) is identifiable under external input f(x,t) (and boundary inputs $b_0(t)$, $b_1(t)$) if the relation

$$\Delta Q(x,t) = 0 \ \textit{for all}\ x \in [0,1],\ t \ge 0, \quad (2.2)$$

or, equivalently

$$\Delta\rho(x)\dot{Q} = \left[\Delta k(x)Q'\right]' - \Delta q(x)Q,\ 0 \le x \le 1,\ t \ge 0, \quad (2.2a)$$

implies that

$$\Delta k(x) = 0,\ \Delta\rho(x) = 0,\ \Delta q(x) = 0 \ \textit{for all}\ x \in [0,1]. \quad (2.3)$$

The identifiability of a subset of parameters {k(x), r(x), q(x)} can be defined similarly. The next section presents the sufficient conditions for the simultaneous identification of parameters k(x), r(x), and q(x).

B. Steady State Regimes

Unlike in the case of heat processes, simultaneous identification of parameters k(x), q(x) in the steady state (1.2) - (1.2a) or (1.2) - (1.2b) is impossible in principle irrespectively of which external inputs $\varphi(x)$ (and boundary conditions x_0, x_1) are used. Indeed, let us choose an interval (x_0, x_1), on which a solution Q(x) of a boundary values problem (1.2) - (1.2a) or (1.2) - (1.2b) differs from zero. Note, that the identically equal to zero solution $Q(x) \equiv 0$ under zero external excitation a priori carries no information about coefficients k(x), q(x) and admits arbitrary choice of these coefficients. Then, according to Eq. (1.2) the value of parameter q(x) on this interval is defined uniquely:

$$q(x) = \left\{\left[k(x)Q'(x)\right]' + \varphi(x)\right\} / Q(x),\ x_o \le x \le x_1$$

via the value of parameter k(x) which in turn can be smoothly varied on this interval.

Referring to the above discussion, consider the steady state (1.2) in which one of the parameters k(x) or q(x) is replaced by $\tilde{k}(x)$ *or* $\tilde{q}(x)$, respectively. The output error ΔQ(x) and parameter mismatch Δk(x) or Δq(x) are given as before.

Definition 2.2 Parameter k(x) is identifiable in a steady state regime (1.2) - (1.2a) ((1.2) - (1.2b)) under external input $\varphi(x)$ (and boundary conditions x_0, x_1) if the relation

$$\Delta Q(x) = 0 \ \textit{for all}\ x \in [0,1], \quad (2.4)$$

or, equivalently

$$\left[\Delta k(x)Q'\right]' = 0,\ \ 0 \le x \le 1, \quad (2.4a)$$

implies that

$$\Delta k(x) = 0 \ \textit{for all}\ x \in [0,1]. \quad (2.5a)$$

The identifiability of parameter q(x) is defined similarly.

Definition 2.3 Parameter q(x) is identifiable in a steady state regime (1.2) - (1.2a) ((1.2) - (1.2b)) under external input $\varphi(x)$ (and boundary conditions x_0, x_1) if the relation (2.4) or, equivalently

$$\Delta q(x)\,Q(x) = 0,\ \ 0 \le x \le 1 \quad (2.4b)$$

implies that

$$\Delta q(x) = 0 \ \textit{for all}\ x \in [0,1]. \quad (2.5b)$$

Fairly simple necessary and sufficient conditions of the parameter identifiability in the steady state regime will be presented in Section III.

C. Vibrating Strings

Along with the system (1.4) - (1.4a) ((1.4) - (1.4b)) consider its model in which æ(x), p(x), g(x) are replaced by $\tilde{æ}(x)$, $\tilde{p}(x)$, $\tilde{g}(x)$, respectively. Let us introduce the following notations:

$$\Delta\theta(x,t) = \theta(x,t) - \tilde{\theta}(x,t),\ \Delta æ(x) = æ(x) - \tilde{æ}(x),$$
$$\Delta p(x) = p(x) - \tilde{p}(x),\ \Delta g(x) = g(x) - \tilde{g}(x),\ 0 \le x \le 1,\ t \ge 0 \quad (2.6)$$

where q(x,t) is the system output, and $\tilde{\theta}(x,t)$ is the model output.

Definition 2.4 A set of parameters (æ(x), p(x), g(x)) of the vibrating string Eq. (1.4) - (1.4a) ((1.4) - (1.4b))are identifiable under external input y(x,t) (and boundary inputs $w_0(t)$, $w_1(t)$) if the relation

$$\Delta\theta(x,t) = 0 \ \ \textit{for all}\ x \in [0,1],\ \ t \ge 0, \quad (2.7)$$

or, equivalently

$$\Delta p(x)\ddot{\theta} = \left[\Delta æ(x)\theta'\right]' - \Delta g(x)\theta,\ 0 \le x \le 1,\ t \ge 0, \quad (2.7a)$$

implies that

$$\Delta æ(x) = 0, \Delta p(x) = 0,\ \Delta g(x) = 0 \ \textit{for all}\ x \in [0,1]. \quad (2.8)$$

For the vibrating strings as well as for heat processes it is possible to identify the whole set of parameters {æ(x), p(x), g(x)}.

III. Sufficient Conditions of the Parameter Identifiability

Identification of the spatially varying parameters places certain requirements on the input functions. The class of sufficiently rich input signals referred to as generators of persistent excitation is defined in this section. This class guarantees the identifiability of the plant parameters.

A. Persistent Excitation of the Heat Process

The concept of persistent excitation of the heat process is introduced as follows.

Definition 3.1. External input $f(x,t)$ (and boundary inputs $\beta_o(t), \beta_1(t)$) generates (generate) persistent excitation of the heat process (1.1), (1.1a) ((1.1), (1.1b), respectively if Fourier coefficients $\mathbf{l}_n(t), \dot{\mathbf{l}}_n(t), n = 1,2,\ldots$ of the solution

$$Q(x,t) = \sum_{n=1}^{\infty} \mathbf{l}_n(t) r_n(x) \quad (3.1)$$

and its time derivative

$$\dot{Q}(x,t) = \sum_{n=1}^{\infty} \dot{\mathbf{l}}_n(t) r_n(x) \quad (3.1a)$$

of the Neumann (Dirichlet) boundary value problem (1.1), (1.1a) ((1.1), (1.1b)) are linearly independent functions.

The definition given above does not specify the particular orthonormal basis of functions

$$r_n(x) \in \tilde{H}_2^o(0,1) = \{r(x): r'(0) = r'(1) = 0, r''(x) \in L_2(0,1)\}, n = 1,\ldots$$

$$\left(r_n(x) \in H_2^0(0,1) = \{r(x): r(0) = r(1) = 0, r''(x) \in L_2(0,1)\}\right)$$

used in the Fourier series, since, as it is shown below, the choice of basis can be arbitrary.

Proposition 3.1. If $f(x,t)$ (and $\beta_o(t)$, $\beta_1(t)$) generates (generate) persistent excitation of the heat process with respect to orthonormal basis of functions $r_n(x) \in \tilde{H}_2^0(0,1)\left(H_2^0(0,1)\right)$,, n=1,2, ... then it generates (they generate) persistent excitation with respect to arbitrary orthonormal basis of functions $\tilde{r}_m(x) \in \tilde{H}_2^0(0,1)\left(H_2^0(0,1)\right), m = 1,2,\ldots$

Proof: Along with (2.9) let there be a representation

$$Q(x,t) = \sum_{m=1}^{\infty} \tilde{\mathbf{I}}_m(t)\tilde{r}_m(x). \qquad \text{Then}$$

$$\tilde{\mathbf{I}}_m(t) =$$

$$\int_0^1 Q(x,t)\tilde{r}_m(x)dx = \int_0^1 \sum_{n=1}^{\infty} \mathbf{I}_n(t)r_n(x)\tilde{r}_m(x)dx = \sum_{n=1}^{\infty} \alpha_{mn}\mathbf{I}_n(t)$$

and

$$\tilde{\mathbf{I}}(t) = \mathrm{A}\mathbf{I}(t), \tag{3.2}$$

where $\alpha_{mn} =$

$$\int_0^1 \tilde{r}_m(x)r_n(x)dx, \mathrm{A} = \{\alpha_{mn}\}_{m,n=1}^{\infty}, \tilde{\mathbf{I}} = \left(\tilde{\mathbf{I}}_1, \tilde{\mathbf{I}}_2, \ldots\right)^{\mathrm{T}}, \mathbf{I} = \left(\mathbf{I}_1, \mathbf{I}_{2,\ldots}\right)^{\mathrm{T}}.$$

Analogously to (3.2) obtain

$$\mathbf{I}(t) = \mathrm{A}^T\tilde{\mathbf{I}}(t),$$

which obviously implies that $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{T}}$.

Assume now that functions $\tilde{\mathbf{I}}_m(t), \dot{\tilde{\mathbf{I}}}_m(t), m = 1,2,\ldots,$ are linearly dependent, i.e.

$$\sum_{m=1}^{\infty}\left[\tilde{\mu}'_m\tilde{\mathbf{I}}_m(t) + \tilde{\mu}''_m\dot{\tilde{\mathbf{I}}}_m(t)\right] = 0,$$

where constants $\tilde{\mu}'_m, \tilde{\mu}''_m, m = 1,2\ldots$ are not equal to zero simultaneously. Then, taking into account (3.2) yields

$$\sum_{m=1}^{\infty}\left[\tilde{\mu}'_m\tilde{\mathbf{I}}_m(t) + \tilde{\mu}''_m\dot{\tilde{\mathbf{I}}}_m(t)\right] = \sum_{m,n}^{\infty} \alpha_{mn}\left[\tilde{\mu}'_m\mathbf{I}_n(t) + \tilde{\mu}''_m\dot{\mathbf{I}}_n(t)\right] =$$

$$\sum_{n=1}^{\infty}\left[\mu'_n\mathbf{I}_n(t) + \mu''_n\dot{\mathbf{I}}_n(t)\right] = 0, n = 1,2,\ldots,$$

where constants

$$\sum_{m=1}^{\infty}\left[\tilde{\mu}'_m\tilde{\mathbf{I}}_m(t) + \tilde{\mu}''_m\dot{\tilde{\mathbf{I}}}_m(t)\right] = \sum_{m,n}^{\infty} \alpha_{mn}\left[\tilde{\mu}'_m\mathbf{I}_n(t) + \tilde{\mu}''_m\dot{\mathbf{I}}_n(t)\right] =$$

$$\sum_{n=1}^{\infty}\left[\mu'_n\mathbf{I}_n(t) + \mu''_n\dot{\mathbf{I}}_n(t)\right] = 0, n = 1,2,\ldots,$$

are not equal to zero simultaneously by assumption and nonsingularity of the infinite-dimensional matrix A. But this contradicts the linear independence of functions $\mathbf{I}_n(t), \dot{\mathbf{I}}_n(t)$, n = 1,2, ... Consequently, functions $\tilde{\mathbf{I}}_m(t), \dot{\tilde{\mathbf{I}}}_m(t)$, m = 1, 2, ... are also linearly independent. Thus, the proposition is proven.

The property of the input functions to generate persistent excitation of the heat process is a sufficient condition of the parameter identifiability.

Theorem 3.2. If external input f(x,t) (and boundary inputs $b_0(t)$, $b_1(t)$) generates (generate) persistent excitation of the system (1.1) - (1.1a) ((1.1) - (1.1b)) then parameters k(x), r(x), q(x) are identifiable under f(x,t) (and $b_0(t)$ and $b_1(t)$).

Proof. Representing solution of the plant equation (1.1) - (1.1a) ((1.1) - (1.1b)) as a Fourier series

$$Q(x,t) = \sum_{n=0}^{\infty} \mathbf{I}_n(t) \cos \pi nx$$

and substituting it into (2.2a) yields

$$\sum_{n=0}^{\infty} \Delta\, \rho(x) \cos \pi nx\, \dot{\mathbf{I}}_n(t) =$$

$$-\sum_{n=0}^{\infty} \left\{\left[\Delta q(x) + (\pi n)^2 \Delta k(x)\right]\cos \pi nx + \pi n\, \Delta k'(x) \sin \pi nx\right\}\mathbf{I}_n(t),$$

where Fourier coefficients $\mathbf{I}_n(t)$ and their derivatives $\dot{\mathbf{I}}_n(t)$ are linearly independent by virtue of Proposition 3.1. Hence,

$$\left[\Delta q(x) + (\pi n)^2 \Delta k(x)\right]\cos \pi nx = \pi n\, \Delta k'(x) \sin \pi nx,$$

$$\Delta\rho(x) \cos \pi nx = 0,\ n = 0,1,2,\ldots$$

and validity of (2.3) follows due to the fact that function sets $\{\cos \pi nx\}$, $\{\sin \pi nx\}$ have nonintersecting everywhere dense zero sets. This proves Theorem 3.2.

Since solution (3.1) of the heat conduction Eq. (1.1) admits the explicit mode representation [15] given by

$$\begin{aligned} \mathbf{I}_n(t) &= \int_0^1 Q_o(x)r_n^o(x)dx e^{-\lambda_{on}^2 t} + \\ &\int_0^t e^{-\lambda_{on}^2(t-\tau)}\int_0^1 f(x,\tau)r_n^o(x)dxd\tau, n = 1,2,\ldots \end{aligned} \tag{3.3a}$$

for Neumann boundary value problem (1.1a) and

$$\mathbf{I}_n(t) = \int_0^1 Q_o(x)r_n^1(x)dx e^{-\lambda_{1n}^2 t} + \int_0^t e^{-\lambda_{1n}^2(t-\tau)}\int_0^1 f_o(x,\tau)r_n^1(x)dxd\tau$$

$$+ r_n^{1'}(0)\int_0^t e^{-\lambda_{1n}^2(t-\tau)}\beta_o(\tau)d\tau - r_n^{1'}(1)\int_0^t e^{-\lambda_{1n}^2(t-\tau)}\beta_1(\tau)d\tau, n = 1,\ldots$$

(3.3b)

for Dirichlet boundary value problem (1.1b), where $\left\{\lambda_{in}^2\right\}_{n=1}^{\infty}$, and $\left\{r_n^i(x)\right\}_{n=1}^{\infty}, i = 0,1$ form a set of eigenvalues and orthonormal basis of eigenfunctions of the corresponding Sturm-Liouville problems

$$\begin{aligned} &\left[k(x)r_n^{o'}(x)\right]' - q(x)r_n^o(x) = -\lambda_{on}^2\rho(x)r_n^o(x), \\ &r_n^{o'}(0) = r_n^{o'}(1) = 0, \end{aligned} \tag{3.4a}$$

and

$$\begin{aligned} &\left[k(x)r_n^{1'}(x)\right]' - q(x)r_n^1(x) = -\lambda_{1n}^2\rho(x)r_n^1(x), \\ &r_n^1(0) = r_n^1(1) = 0, \end{aligned} \tag{3.4b}$$

the construction of the generator of the persistent excitation of the heat process is not difficult. In particular, for the zero initial condition $Q_o(x) = 0$ one can choose the following inputs as the generators of the persistent excitation:

i) time-invariant external input

$$f(x,t) = f_o(x) = \sum_{n=1}^{\infty} f_{n-1}^{o} \cos \pi(n-1)\, x, \qquad (3.5)$$

for nonzero Fourier coefficients

$$f_o^o = \int_o^1 f_o(x)dx \neq 0,\ f_n^o = \sqrt{2}\int_o^1 f_o(x)\cos \pi nx \neq 0,\ n = 1,2\ldots \qquad (3.5a)$$

for Neumann boundary value problem (1.1a); in this case function

$$\mathbf{l}_m(t) = \bar{f}_m^o\left(1 - e^{-\lambda_{om}^2 t}\right) / \lambda_{om}^2, \dot{\mathbf{l}}_m(t) = \bar{f}_m^o e^{-\lambda_{om}^2 t}, m = 1,2,\ldots$$

are linearly independent, since input Fourier coefficients

$$\bar{f}_m^o = \int_o^1 f_o(x) r_m^0(x)dx = \sum_{n=1}^{\infty} f_{n-1}^0 \int_o^1 \cos\left[\pi(n-1)x\right] r_m^o(x)\, dx,$$

m = 1, 2, ... for eigenfunctions $r_m^o(x)$ are also nonzero due to the nonsingularity $A^{-1} = A^T$ of matrix

$$A = \left(\alpha_{mn}^o\right)_{m,n=1}^{\infty},\quad \alpha_{mn}^o = \int_o^1 \cos\left[\pi(n-1)x\right] r_m^o(x)dx;$$

ii) time-invariant external input

$$f(x,t) = f_1(x) = \sum_{n=1}^{\infty} f_n^1 \sin \pi nx \qquad (3.6)$$

with nonzero Fourier coefficients

$$f_n^1 = \sqrt{2}\int_o^1 f_1(x) \sin \pi nx\, dx,\ n = 1,2,\ldots \qquad (3.6a)$$

and zero boundary inputs

$$\beta_o(t) = \beta_1(t) = 0 \qquad (3.6b)$$

for Dirichlet boundary value problem (1.1b); as in the previous case, functions

$$\mathbf{l}_m(t) = \int_o^1 f_1(x) r_m^1(x)dx\left(1 - e^{-\lambda_{1m}^2 t}\right) \Big/ \lambda_{1m}^2,$$

$$\dot{\mathbf{l}}_m(t) = \int_o^1 f_1(x) r_m^1(x)dx\, e^{-\lambda_{1m}^2 t},\ m = 1,2,\ldots$$

are linearly independent;

iii) time-invariant boundary inputs

$$\beta_o(t) = v^o, \beta_1(t) = v^1$$

and zero external input

$$f(x,t) = 0 \qquad (3.7a)$$

for Dirichlet boundary value problem (1.1b), if

$$v_m = v^o r_m^{1'}(0) - v^1 r_m^{1'}(1) \neq 0,\ m = 1,2,\ldots; \qquad (3.7b)$$

in this case

$$\mathbf{l}_m(t) = v_m\left(1 - e^{-\lambda_{1m}^2 t}\right) \Big/ \lambda_{1m}^2, \dot{\mathbf{l}}_m(t) = v_m e^{-\lambda_{1m}^2 t},\ m = 1,$$

2, ..., are also linearly independent.

Remark 3.1 It should be noted that $r_m^{1'}(0) \neq 0$ and $r_m^{1'}(1) \neq 0$, since otherwise $r_m^1(x) \equiv 0$ according to (3.4b). Hence, it is always possible to select constants n^o and n^1 such that relations (3.7b) are satisfied.

Remark 3.2 The persistent excitation property of the input can be extended to the case of the multi-dimensional spatial variable, however the construction of the generator of persistent excitation in this case becomes more complicated if the corresponding Sturm-Liouville problem has roots with multiplicity. In this case, persistent excitation cannot be generated by boundary input or time-invariant external input, however time-varying external input with linearly independent Fourier coefficients can be the generator of the persistent excitation for a heat process with zero initial conditions.

B. Identifiability Conditions for Steady State Regimes

Identifiability of parameter q(x) of a steady state (1.2) is equivalent to the uniqueness of solution of the linear algebraic Eq. (1.2) with respect to q, which takes place if and only if

$$Q(x) \neq 0 \text{ for almost all } x \in [0,1]. \qquad (3.8)$$

Since according to (1.2) condition (3.8) holds only for functions j(x) which differ from zero almost everywhere, the identifiability of q(x) is equivalent to condition

$$j(x) \neq 0 \text{ for almost all } x \in [0,1]. \qquad (3.9)$$

Identifiability of parameter k(x) of Eq. (1.2) is equivalent to the uniqueness of solution of linear differential Eq. (1.2) with respect to k(x). Integrating this equation yields

$$v_m = v^o r_m^{1'}(0) - v^1 r_m^{1'}(1) \neq 0,\ m = 1,2,\ldots \qquad (3.10a)$$

under Neumann boundary conditions (1.2a) and

$$k(x)Q'(x) = \int_o^x \left[q(\xi)Q(\xi) + \varphi(\xi)\right]d\xi + const \qquad (3.10b)$$

under Dirichlet boundary conditions (1.2b). Equation (3.10a) is uniquely solvable with respect to k(x) only in the case when

$$Q'(x) \neq 0 \text{ almost for all } x \in [0,1], \qquad (3.11)$$

which due to (1.2) is equivalent to the linear independence

$$\varphi(x) \neq c_o q(x),\ c_o = const,\ x \in (x_1, x_2) \subset [0,1] \qquad (3.12)$$

of functions $\varphi(x)$ and q(x) on any subinterval (x_1, x_2).

For the unique solvability of Eq. (3.10b) under condition (3.12) it is necessary to assume additionally that

$$\exists\, x_o \in [0,1]: Q'(x_o) = 0. \qquad (3.13)$$

Obviously, condition (3.13) can be always satisfied via a special choice of one of the variables x_o or x_1. Thus, the identifiability of parameter k(x) of Eq. (1.2) is equivalent to condition (3.12) in the case of Neumann boundary conditions (1.2a) and equivalent to conditions (3.12), (3.13) in the case of Dirichlet boundary conditions (1.2b).

The conditions obtained are formulated below in the form of a theorem.

Theorem 3.3 Parameter q(x) is identifiable in the steady state (1.2) - (1.2a) or (1.2) - (1.2b) under external input j(x) if and only if condition (3.9) is satisfied. Parameter k(x) is identifiable in the steady state (1.2) - (1.2a) under external input j(x) if and only if condition (3.12) is satisfied. Parameter k(x) is identifiable in the steady state (1.2) - (1.2b) under external input j(x) and boundary conditions x_o, x_1 if and only if conditions (3.12) and (3.13) are satisfied.

C. Persistent excitation of the vibrating string.

The persistent excitation condition for the vibrating string unlike that for the heat process requires the linear independence of the time derivatives of the solution Fourier coefficients up to the second order.

Definition 3.2 External input y(x,t) (and boundary inputs $w_o(t)$, $w_1(t)$) generates (generate) persistent excitation of the vibrating string (1.4) - (1.4a) ((1.4) - (1.4b), respectively) if Fourier coefficients $\xi_n(t), \dot{\xi}_n(t), \ddot{\xi}_n(t)$, n = 1, 2, ..., of the solution

$$\theta(x,t) = \sum_{n=1}^{\infty} \xi_n(t)\, r_n(x) \tag{3.14}$$

and its time derivatives

$$\dot{\theta}(x,t) = \sum_{n=1}^{\infty} \dot{\xi}_n(t)\, r_n(x),\ \ddot{\theta}(x,t) = \sum_{n=1}^{\infty} \ddot{\xi}_n(t)\, r_n(x) \tag{3.14a}$$

of the Neumann (Dirichlet) boundary value problem (1.4) - (1.4a) ((1.4) - (1.4b)) are linearly independent functions.

Similar to Prosition 3.1 and Theorem 3.2, the following results take place.

Proposition 3.4 If y(x,t) (and $w_o(t)$, and $w_1(t)$) generates (generate) persistent excitation of the vibrating string with respect to orthonormal basis of functions $r_n(x) \in \tilde{H}_2^o(0,1)\left(H_2^o(0,1)\right)$, n = 1, 2,..., then it generates (they generate) persistent excitation with respect to arbitrary orthonormal basis of functions $\tilde{r}_m(x) \in \tilde{H}_2^o(0,1)\left(H_2^o(0,1)\right)$, m = 1, 2, ...

Theorem 3.5 If external input y(x,t) (and boundary inputs $w_o(t)$, $w_1(t)$) generates (generate) persistent excitation of the system (1.4) - (1.4a) ((1.4) - (1.4b)), then parameters æ(x), p(x), g(x) are identifiable under y(x,t) (and $w_o(t)$, and $w_1(t)$).

The proofs of Proposition 3.4 and Theorem 3.5 are similar to those of Proposition 3.1 and Theorem 3.2, respectively, and therefore are omitted.

Since solution (3.14) of the vibrating string Eq. (1.4) as well as solution (3.1), (3.3), and (3.4) of the heat conduction Eq. (1.1) admits the explicit mode representation [17]

$$\xi_n(t) = \int_o^1 \theta_o(x) z_n^o(x)\,dx \cos\gamma_{on}t + \frac{1}{\gamma_{on}} \int_o^1 \theta_1(x)\, z_n^o(x)\,dx \sin\gamma_{on}t$$

$$+ \frac{1}{\gamma_{on}} \int_o^t \sin\gamma_{on}(t-\tau) \int_o^1 \psi(x,\tau) z_n^o(x)dx\,d\tau,$$

$$n=1, 2, \ldots \tag{3.15a}$$

for Neumann boundary value problem (1.4a) and

$$\xi_n(t) = \int_o^1 \theta_o(x) z_n^1(x)\,dx \cos\gamma_{1n}t + \frac{1}{\gamma_n} \int_o^1 \theta_1(x)\, z_n^1(x)\,dx \sin\gamma_{1n}t$$

$$+ \int_o^t \sin\gamma_{1n}(t-\tau) \int_o^1 \psi(x,\tau) z_n^1(x)dx\,d\tau + z_n^{1'}(0) \int_o^t \sin\gamma_{1n}(t-\tau)\times$$

$$\omega_o(\tau)d\tau - z_n^{1'}(1) \int_o^t \sin\gamma_{1n}(t-\tau)\omega_1(\tau)d\tau,$$

$$n=1, 2, \ldots \tag{3.15b}$$

for Dirichlet boundary value problem (1.4b), where $\left\{\gamma_{in}^2\right\}_{n=1}^{\infty}$ and $\left\{z_n^i(x)\right\}$, i = 0,1, form set of eigenvalues and orthonormal basis of eigenfunctions of the corresponding Sturm-Liouville problems

$$\left[æ(x) z_n^{o'}(x)\right]' - g(x) z_n^o(x) = -\gamma_{on}^2 p(x)\, z_n^o(x),$$
$$z_n^{o'}(0) = z_n^{o'}(1) = 0, \tag{3.16a}$$

and

$$\left[æ(x) z_n^{1'}(x)\right]' - g(x) z_n^1(x) = -\gamma_{1n}^2 p(x)\, z_n^1(x), z_n^1(0) = z_n^1(1) = 0, \tag{3.16b}$$

the construction of the generator of the persistent excitation of the string also is not difficult. For instance, the following inputs generate persistent excitations of the string (1.4) under the zero initial conditions, $q_o(x) = 0$, $q_1(x) = 0$:

i) time-invariant external input

$$\psi(x,t) = \psi_o(x) = \sum_{n=1}^{\infty} \psi_{n-1}^o \cos\pi(n-1)x, \tag{3.17}$$

with nonzero Fourier coefficients

$$\psi_o^o = \int_o^1 \psi_o(x)dx \neq 0,\ \psi_n^o = \sqrt{2}\int_o^1 \psi_o(x)\cos\pi nx dx \neq 0,$$
$$n=1,2,\ldots \tag{3.17a}$$

for Neumann boundary value problem (1.4a); in this case functions

$$\xi_m(t) = \tilde{\psi}_m^o\left(1 - \cos\gamma_{om}t\right)/\gamma_{om}^2,\ \dot{\xi}_m(t) = \tilde{\psi}_m^o \sin\gamma_{om}t/\gamma_{om},$$
$$\ddot{\xi}_m(t) = \tilde{\psi}_m^o \cos\gamma_{om}t,\ m = 1,2,\ldots$$

are linearly independent, since input Fourier coefficients

$$\tilde{\psi}_m^o = \int_o^1 \psi_o(x) z_m^o(x)dx = \sum_{n=1}^{\infty} \psi_{n-1}^o \int_o^1 \cos\left[\pi(n-1)x\right] z_m^o(x)dx,$$

$$m = 1,2,\ldots$$

for eigenfunctions $z_m^o(x)$ are also nonzero due to the nonsingularity $A^{-1} = A^T$ of matrix

$$A = \left(v_{mn}^o\right)_{m,n=1}^{\infty},\ v_{mn}^o = \int_o^1 \cos\left[\pi(n-1)x\right] r_m^o(x)\,dx;$$

ii) time-invariant external input

$$\psi(x,t) = \psi_1(x) = \sum_{n=1}^{\infty} \psi_n^1 \sin\pi nx \tag{3.18}$$

with nonzero Fourier coefficients

$$\psi_n^1 = \sqrt{2}\int_o^1 \psi_1(x)\sin\pi nx\,dx \neq 0,\ n = 1,2,\ldots \tag{3.18a}$$

and zero boundary inputs

$$\omega_o(t) = \omega_1(t) = 0 \tag{3.18b}$$

for Dirichlet boundary value problem (1.4b); as in the previous case, functions

$$\xi_m(t) = \int_o^1 \psi_1(x) z_m^1(x)\,dx\left(1 - \cos\gamma_{1m}t\right)/\gamma_{1m}^2,$$
$$\dot{\xi}_m(t) = \int_o^1 \psi_1(x) z_m^1(x)\,dx \sin\gamma_{1m}t/\gamma_{1m},$$
$$\ddot{\xi}_m(t) = \int_o^1 \psi_1(x) z_m^1(x)\,dx \cos\gamma_{1m}t,\ m = 1,2,\ldots$$

are linearly independent ;

iii) time-invariant boundary inputs

$$\omega_o(t) = \mu^o,\ \omega_1(t) = \mu^1 \tag{3.19}$$

and zero external input

$$y(x,t) = 0 \tag{3.19a}$$

for Dirichlet boundary value problem (1.4b), if

$$\mu_m = \mu^o z_m^{1'}(0) - \mu^1 z_m^{1'}(1) \neq 0, \quad m = 1,2,\ldots; \tag{3.19b}$$

in this case,

$$\xi_m(t) = \mu_m \left(1 - \cos\gamma_{1m} t\right) / \gamma_{1m}^2,$$
$$\dot{\xi}_m(t) = \mu_m \sin\gamma_{1m} t / \gamma_{1m}, \quad \ddot{\xi}_m(t) = \mu_m \cos\gamma_{1m} t, \; m = 1,2,\ldots$$

are also linearly independent.

Remark 3.3 Similarly to Remark 3.1 one can check that there exist such constants μ^o and μ^1 that relations (3.19b) are satisfied.

Remark 3.4 In order to generate the persistent excitation of the hyperbolic system (1.4) with multi-dimensional spatial variable one should use time-varying external input with linearly independent Fourier coefficients (cf. Remark 3.2 for details).

IV. Adaptive Identifier Design

As discussed earlier, it is theoretically possible to determine unknown distributed plant parameters based on the noise-free measurements under the assumption that the parameters are identifiable. This task can be carried out by the adaptive identifiers proposed below.

A. Adaptive Identification of a Heat Process

The following identification law is proposed for Neumann boundary value problem (1.1), (1.1a) in order to identify the spatially varying plant parameters $\rho(x), k(x),$ *and* $q(x)$:

$$\tilde{\rho}(x,t)\dot{\tilde{Q}} = \left[\tilde{k}(x,t)\tilde{Q}'\right]' - \tilde{q}(x,t)\tilde{Q} + f(x,t) + v_o(Q - \tilde{Q}),$$
$$\tilde{Q}(x,0) = \tilde{Q}_o(x), \quad 0 < x < 1, \tag{4.1}$$
$$\tilde{Q}'(0,t) = \tilde{Q}'(1,t) = 0, t > 0, \tag{4.1a}$$
$$\dot{\tilde{k}}(x,t) = -v_1\left(Q - \tilde{Q}\right)' \tilde{Q}', \tilde{k}(x,0) = k_o(x), \tag{4.2a}$$
$$\dot{\tilde{q}}(x,t) = -v_2\left(Q - \tilde{Q}\right)\tilde{Q}, \tilde{q}(x,0) = q_o(x), \tag{4.2b}$$
$$\dot{\tilde{\rho}}(x,t) = -v_3\left(Q - \tilde{Q}\right)\dot{\tilde{Q}}, \tilde{\rho}(x,0) = \rho_o(x) \tag{4.2c}$$

where $v_i > 0, i = 0,1,2,3$ are adaptation gains, $k_o(x) > 0$ and $\tilde{Q}_o(x)$ are smooth functions, $q_o(x) > 0$ and $\rho_o(x) > 0$ are continuous functions. The law, as shown below, ensures the necessary asymptotic convergence

$$\lim_{t\to\infty} \int_0^1 \left\{ \begin{matrix} \left[\Delta Q(x,t)\right]^2 + \left[\Delta k(x,t)\right]^2 + \\ \left[\Delta q(x,t)\right]^2 + \left[\Delta\rho(x,t)\right]^2 \end{matrix} \right\} dx = 0, \tag{4.3}$$

$$\Delta Q = Q - \tilde{Q}, \Delta k = k - \tilde{k}, \Delta q = q - \tilde{q}, \Delta\rho = \rho - \tilde{\rho}, \tag{4.3a}$$

for arbitrary adaptation gains, initial distributions and generator of persistent excitation of heat process (1.1), (1.1a). Since state and parameter errors (4.3a) are continuous functions the quadratic convergence (4.3) implies their pointwise convergence almost everywhere in $x \in [0,1]$ and consequently permits identification of unknown coefficients $k(x), \rho(x), q(x)$ everywhere in $x \in [0,1]$.

Theorem 4.1. Let the external input $f(x,t)$ generate persistent excitation of the heat process (1.1), (1.1a). Then the limiting relation (3.3) holds with the adaptive identification law (4.1), (4.1a) and parameters $\tilde{k}(x,t), \tilde{\rho}(x,t), \tilde{q}(x,t)$ tuned as (4.2a)-(4.2c).

Proof. First, let us prove the local existence of the unique solution of the overall system (1.1), (4.1), (4.2). For this purpose, we integrate equations (4.2a)-(4.2c)

$$\tilde{k}(x,t) = k_o(x) - v_1 \int_0^t \left(Q - \tilde{Q}\right)' \tilde{Q}' d\tau,$$
$$\tilde{q}(x,t) = q_o(x) - v_2 \int_0^t \left(Q - \tilde{Q}\right)\tilde{Q} d\tau,$$
$$\tilde{\rho}(x,t) = \rho_o(x) - v_3 \int_0^t \left(Q - \tilde{Q}\right)\dot{\tilde{Q}} d\tau = \rho_o(x) + \tag{4.4}$$
$$+ v_3 \int_0^t \dot{Q}\tilde{Q} d\tau + \frac{1}{2} v_3 \left\{\left[2Q(x,0) - \tilde{Q}(x,0)\right]\tilde{Q}(x,0) - \right.$$
$$\left. -\left[2Q(x,t) - \tilde{Q}(x,t)\right]\tilde{Q}(x,t)\right\},$$

and substitute the outputs of the integrators into (4.1). Since functions (4.4) are positive at initial time moment t = 0 and they are continuous in $\left(x, t, Q, \tilde{Q}\right)$ the resulting equation is locally parabolic [17]. Hence, the results of [18] can be applied to show the existence of a unique local solution of (4.1), (4.1a).

Now, using Lyapunov functional

$$V(t) = \frac{1}{2} \int_0^1 \left\{ \rho(x)\left[\Delta Q(x,t)\right]^2 + \frac{1}{v_1}\left[\Delta k(x,t)\right]^2 + \right.$$
$$\left. + \frac{1}{v_2}\left[\Delta q(x,t)\right]^2 + \frac{1}{v_3}\left[\Delta\rho(x,t)\right]^2 \right\} dx, \tag{4.5}$$

where, according to (4.2) and (4.3a), variables $\Delta Q, \Delta k, \Delta q, \Delta\rho$ satisfy equations

$$\rho(x)\Delta\dot{Q} + \Delta\rho(x,t)\dot{\tilde{Q}} = \left[k(x)\Delta Q'\right]' + \left[\Delta k(x,t)\tilde{Q}'\right]' -$$
$$-\left[q(x) + v_o\right]\Delta Q - \Delta q(x,t)\tilde{Q}, 0 < x < 1, t > 0, \tag{4.6a}$$
$$\Delta Q'(0,t) = \Delta Q'(1,t) = 0,$$
$$\Delta\dot{k} = v_1 \Delta Q' \tilde{Q}', \Delta\dot{q} = v_2 \Delta Q \tilde{Q}, \Delta\dot{\rho} = v_3 \Delta Q \dot{\tilde{Q}}, \tag{4.6b}$$

we can show that solution of (4.1), (4.1a) is well-posed for all t ≥ 0. Indeed, the computation of the derivative of Lyapunov functional along the trajectories of (4.6) yields

$$\dot{V}(t) = -\int_0^1 k\left[\Delta Q'\right]^2 dx + k\Delta Q \Delta Q' \Big|_o^1 - \int_o^1 \Delta k \Delta Q' \tilde{Q}' dx$$
$$+ \Delta k \Delta Q \tilde{Q}' \Big|_o^1 - \int_o^1 \left(q + v_o\right)\left[\Delta Q\right]^2 dx - \int_o^1 \Delta q \Delta Q \tilde{Q} dx -$$
$$- \int_o^1 \Delta\rho \Delta Q \dot{\tilde{Q}} dx + \int_o^1 \Delta k \Delta Q' \tilde{Q}' dx + \int_o^1 \Delta q \Delta Q \tilde{Q} dx +$$
$$+ \int_o^1 \Delta\rho \Delta Q \dot{\tilde{Q}} dx = -\int_o^1 k\left[\Delta Q'\right]^2 dx - \int_o^1 \left(q + v_o\right)\left[\Delta Q\right]^2 dx \le 0,$$

which implies the boundedness of Lyapunov functional $V(t) \le V(0) < \infty$ for all t ≥ 0, L_2-boundedness of the solutions of (4.6) and their stability. Since the principal term $\partial/\partial x\left[k(x)\partial/\partial x\right]$ in Eq. (4.6) has a compact resolvent in $L_2(0,1)$ and the outputs of heat process (1.1) and dynamic model (4.1) are smooth functions, every trajectory of system (4.6) is precompact due to its boundedness [16]. Therefore, due to the invariance principle [20, Theorem 4.3.4], there must be a convergence of the trajectories of system (4.6) to the maximal invariant subset of a set of solutions of (4.6) for which

$$\dot{V}(t) = -\int_0^1 k(x)[\Delta Q']^2 dx - \int_0^1 [q(x) + v_o][\Delta Q]^2 dx = 0.$$

Taking into account (4.6) this leads to the expressions

$$\Delta k(x,t) = \Delta k(x), \Delta q(x,t) = \Delta q(x), \Delta \rho(x,t) = \Delta \rho(x), \quad (4.7a)$$

$$\Delta \rho(x)\dot{Q}(x,t) = [\Delta k(x) Q'(x,t)]' - \Delta q(x) Q(x,t), \quad (4.7b)$$
$$Q'(0,t) = Q'(1,t) = 0, 0 < x < 1, t > 0.$$

Therefore to complete the proof it remains to show that (4.7b) holds if and only if

$$\Delta k(x) = \Delta q(x) = \Delta \rho(x) = 0 \text{ for all } x \in [0,1]. \quad (4.8)$$

This is indeed true by virtue of Theorem 3.2 and an assumption that f(x,t) generates the persistent excitation of the heat process. This proves Theorem 4.1.

In order to identify the plant parameters for the case of Dirichlet boundary value problem (1.1), (1.1b) we need to modify boundary conditions (4.1a) for the identification law proposed above and to consider Dirichlet boundary conditions

$$\tilde{Q}(0,t) = \beta_o(t), \tilde{Q}(1,t) = \beta_1(t), t > 0 \quad (4.1b)$$

as well.

Theorem 4.2. Let the external and boundary inputs $f(x,t), \beta_o(t), \beta_1(t)$ generate persistent excitation of the heat process (1.1), (1.1b). Then the limiting relation (4.3) holds with the adaptive identification law (4.1), (4.1b) and parameters $\tilde{k}(x,t), \tilde{\rho}(x,t), \tilde{q}(x,t)$ tuned as (4.2a)-(4.2c).

The proof of this theorem is similar to that of Theorem 4.1 and therefore it is omitted here.

Remark 4.1 If some of the plant parameters are known a priori or are space-invariant, then the corresponding equations (4.2) can be omitted or, respectively, replaced by the equations with respect to the corresponding lumped variables

$$k^*(t) = \int_0^1 \tilde{k}(x,t)\, dx, q^*(t) = \int_0^1 \tilde{q}(x,t)\, dx, \rho^*(t) = \int_0^1 \tilde{\rho}(x,t) dx:$$

$$\dot{k}^*(t) = -v_1 \int_0^1 (Q - \tilde{Q})' \tilde{Q}' dx, \quad k^*(0) = k_o^* > 0, \quad (4.2a')$$

$$\dot{q}^*(t) = -v_2 \int_0^1 (Q - \tilde{Q}) Q dx, \quad q^*(0) = q_o^* > 0, \quad (4.2b')$$

$$\dot{\rho}^*(t) = -v_3 \int_0^1 (Q - \tilde{Q}) \dot{\tilde{Q}} dx, \quad \rho^*(0) = \rho_o^* > 0. \quad (4.2c')$$

The identifier convergence $k^*(t) \to k$, $q^*(t) \to q$, $r^*(t) \to r$ as $t \to \infty$ for spatially-invariant coefficients k(x)=k, q(x)=q, r(x)=r is a consequence of Theorems 4.1 and 4.2.

Remark 4.2 Identifier (4.1) - (4.2) does not require higher order temporal and spatial derivatives of the state, which lead to the loss of robustness in the presence of measurement noise and dynamic nonidealities. Certainly, employing the first order spatial state derivative in identifier Eq. (4.2a) implies the use of the regularization method [21] for its calculation because of ill-posedness of heat conduction coefficient identification problem [22]. However, due to smoothness of the heat equation solution that takes place under our assumptions on the plant parameters, the external and boundary inputs, and the initial distribution, an implementation of the derivative in the identification algorithm admits simple realization by an appropriate differential filter or via its approximation by first order difference with a sufficiently small step.

B. Adaptive Identification of the Steady State Regime.

The following identification law

$$\dot{\tilde{Q}} = [k(x,t)\tilde{Q}']' - \tilde{q}(x,t)\tilde{Q} + \varphi(x) + v_o(Q - \tilde{Q}), \quad (4.9)$$
$$\tilde{Q}(x,0) = \tilde{Q}_o(x), \quad 0 < x < 1,$$

$$\tilde{Q}'(0,t) = \tilde{Q}'(1,t) = 0, \quad t > 0, \quad (4.9a)$$

$$\dot{\tilde{q}}(x,t) = -v_2(Q - \tilde{Q})\tilde{Q}, \quad \tilde{q}(x,o) = q_o(x) \quad (4.10)$$

is proposed in order to identify the spatially varying parameter q(x) in the steady state mode (1.2) under Neumann boundary condition (1.2a). The law ensures the necessary asymptotic convergence

$$\lim_{t \to \infty} \int_0^1 \left\{ [\Delta Q(x,t)]^2 + [\Delta q(x,t)]^2 \right\} dx = 0, \quad (4.11)$$

$$\Delta Q = Q - \tilde{Q}, \quad \Delta q = q - \tilde{q} \quad (4.11a)$$

for arbitrary $n_o > 0$, $n_2 > 0$, and continuous initial distributions $\tilde{Q}_o(x)$ and $q_o(x)$, and for arbitrary function $\varphi(x)$, different from zero almost everywhere. Since state and parameter errors (4.11a) are continuous functions the quadratic convergence (4.11) implies their pointwise convergence almost everywhere in $x \in [0,1]$ and consequently permits identification of an unknown coefficient q(x) for all $x \in [0,1]$.

Theorem 4.3 Let the parameter k(x) be known a priori and let the external input $\varphi(x)$ satisfy the condition (3.9). Then, the limiting relation (4.11) holds with the adaptive identification law (4.9), (4.9a), and parameter $\tilde{q}(x,t)$ tuned as (4.10).

Proof. First, note that the proof of the unique solution existence of the overall system (4.9), (4.9a), (4.10) is similar to that of Theorem 4.1. Now, let us introduce Lyapunov functional

$$V(t) = \frac{1}{2} \int_0^1 \left\{ [\Delta Q(x,t)]^2 + \frac{1}{v_2} [\Delta q(x,t)]^2 \right\} dx \quad (4.12)$$

and calculate its time derivative

$$\dot{V}(t) = -\int_0^1 k[\Delta Q']^2 dx + k\Delta Q \Delta Q' \Big|_o^1 - \int_0^1 (q + v_o)[\Delta Q]^2 dx$$
$$-\int_0^1 \Delta q \Delta Q\, \tilde{Q} dx + \int_0^1 \Delta q \Delta Q\, \tilde{Q} dx = -\int_0^1 k[\Delta Q']^2 dx - \quad (4.13)$$
$$-\int_0^1 (q + v_o)[\Delta Q]^2 dx \le 0$$

along the trajectories of the equations

$$\Delta \dot{Q} = [k(x)\Delta Q']' - [q(x) + v_o]\Delta Q - \Delta q(x)\tilde{Q}, \quad (4.14a)$$
$$\Delta Q'(0,t) = \Delta Q'(1,t) = 0,$$

$$\Delta \dot{q}(x,t) = v_2 \Delta Q \tilde{Q}, \quad 0 < x < 1, \quad t \ge 0 \quad (4.14b)$$

with respect to variables ΔQ, Δq. Since the principal term $\partial[k(x)\, \partial/\partial x]/\partial x$ in Eq. (4.14) has a compact resolvent in

$L_2(0,1)$ and the outputs of system (1.2), (1.2a) and model (4.9) (4.9a) are smooth functions, every trajectory of system (4.14) is precompact due to its boundedness [16]. Therefore, by virtue of the invariance principle [20, Theorem 4.3.4], there must be a convergence of the trajectories of system (4.14) to the maximal invariant subset of a set of solutions of (4.14) for which

$$\dot{V}(t) = -\int_0^1 k(x)\left[\Delta Q'\right]^2 dx - \int_0^1 \left(q(x) + v_o\right)\left[\Delta Q\right]^2 dx = 0 \quad (4.13a)$$

Due to the continuity of the integrands it leads to the expression $\Delta Q \equiv 0$ that holds by Theorem 3.3 if and only if $\Delta q \equiv 0$. Thus, Theorem 4.3 is proven.

In order to identify parameter q(x) for the case of Dirichlet boundary value problem (1.2), (1.2b) it is necessary to modify boundary conditions (4.9) for the identification laws proposed above and to consider Dirichlet boundary conditions as well. The formulation and the proof of the corresponding result can be carried out with no difficulty and therefore is omitted.

Next, consider the problem of the identification of parameter k(x) for Dirichlet boundary value problem (1.2) - (1.2b). The identification law

$$\dot{\tilde{Q}} = \left[\tilde{k}(x,t)\tilde{Q}'\right]' - q(x)\tilde{Q} + \varphi(x) + v_o\left(Q - \tilde{Q}\right), \quad \tilde{Q}(x,0) = \tilde{Q}_o(x),\ 0 < x < 1 \quad (4.15)$$

$$\tilde{Q}(0,t) = \zeta_o,\ \tilde{Q}(1,t) = \zeta_1,\ t > 0, \quad (4.15a)$$

$$\dot{\tilde{k}}(x,t) = -v_1\left(Q - \tilde{Q}\right)'\tilde{Q}',\ t > 0,\ \tilde{k}(x,0) = k_o(x), \quad (4.16)$$

ensures the necessary asymptotic convergence

$$\lim_{t\to\infty} \int_0^1 \left\{\left[\Delta Q(x,t)\right]^2 + \left[\Delta k(x,t)\right]^2\right\} dx = 0, \quad (4.17)$$

$$\Delta Q = Q - \tilde{Q},\ \Delta k = k - \tilde{k} \quad (4.17a)$$

for arbitrary $n_o > 0$, $n_1 > 0$ and smooth initial distributions $\tilde{Q}_o(x)$ and $k_o(x) > 0$, for arbitrary boundary values z_o, z_1, and for arbitrary function j(x) linearly independent on q(x) inside any subinterval $(x_1, x_2) \subset [0,1]$ that provides equality $Q'(x_o) = 0$ at some point $x_o \in [0,1]$.

Theorem 4.4 Let parameter q(x) be known a priori and let conditions (3.12), (3.13) be satisfied. Then, the limiting relation (4.17) holds with the adaptive identification law (4.15) and parameter $\tilde{k}(x,t)$ tuned as (4.16).

Proof. The existence of the unique solution of the overall system (4.15), (4.16) is proven similarly to that of Theorem 4.1. Setting $\Delta k = k - \tilde{k}$ let us define Lyapunov functional

$$V(t) = \frac{1}{2}\int_0^1 \left\{\left[\Delta Q(x,t)\right]^2 + \frac{1}{v_1}\left[\Delta k(x,t)\right]^2\right\} dx, \quad (4.18)$$

along the trajectories of the equations

$$\Delta\dot{Q} = \left[k(x)\Delta Q'\right]' + \left[\Delta k(x,t)\tilde{Q}'\right]' - \left[q(x) + v_o\right]\Delta Q, \quad \Delta Q(0,t) = \Delta Q(1,t) = 0, \quad (4.19a)$$

$$\Delta\dot{k} = v_1 \Delta Q'\tilde{Q}',\ t > 0,\ \ 0 < x < 1 \quad (4.19b)$$

with respect to variables ΔQ, Δk, and compute its time derivative

$$\dot{V}(t) = -\int_0^1 k\left[\Delta Q'\right]^2 dx + k\,\Delta Q\,\Delta Q'\Big|_o^1 - \int_0^1 \Delta k\,\Delta Q'\,\tilde{Q}'dx$$

$$+\Delta k\,\Delta Q\,\tilde{Q}'\Big|_o^1 - \int_0^1 \left(q - v_o\right)\left[\Delta Q\right]^2 dx + \int_0^1 \Delta k\,\Delta Q'\,\tilde{Q}'dx$$

$$= -\int_0^1 k\left[\Delta Q'\right]^2 dx - \int_0^1 \left(q + v_o\right)\left[\Delta Q\right]^2 \le 0.$$

Similarly to the proof of Theorem 4.3, the invariance principle [20, Theorem 4.3.4] applied to system (4.19) guarantees a convergence of the system trajectories to the maximal invariant subset of a set of solutions of (4.19) for which relation (4.13a) is satisfied. Taking into account (4.19) equality (4.13) leads to the expressions

$$\Delta k(x,t) = \Delta k(x),\ \left[\Delta k(x)Q'\right]' = 0,\ 0 \le x \le 1. \quad (4.20)$$

Due to (3.13) it follows that $\Delta k(x)Q' = 0$ which by virtue of (3.12) implies that $\Delta k(x) = 0$ almost for all $x \in [0,1]$. This proves Theorem 4.4.

To identify parameter k(x) for the case of Neumann boundary value problem (1.2) - (1.2a) it is necessary to replace boundary conditions (4.15a) in identification law (4.15) - (4.16) by the appropriate Neumann boundary conditions and omit condition (3.13).

Remark 4.3 If parameters q(x) or k(x) are space-invariant then identification Eq. (4.10) or (4.16) can be replaced by Eq. (4.2b) or (4.2a) with respect to variable $q^*(t) = \int_0^1 \tilde{q}(x,t)dx$ or $k^*(t) = \int_0^1 \tilde{k}(x,t)dx$, respectively.

Remark 4.4 The identification of parameter k(x) is ill-posed problem both for the heat process and the steady state regime. Therefore, Remark 4.2 is still valid for the steady state regime identification.

C. Adaptive Identification of the Vibrating String

The following identification law is proposed for Dirichlet boundary value problem (1.4) - (1.4b) in order to identify the spatially varying plant parameters æ(x), p(x), and g(x):

$$\tilde{p}(x,t)\ddot{\tilde{\theta}} = \left[\tilde{æ}(x,t)\tilde{\theta}'\right]' - \tilde{g}(x,t)\theta + \psi(x,t) + \mu\left(\dot{\theta} - \dot{\tilde{\theta}}\right) + \mu_o\left(\theta - \tilde{\theta}\right),$$

$$0 < x < 1,\ t > 0,$$

$$\tilde{\theta}(x,0) = \tilde{\theta}_o(x),\ \dot{\tilde{\theta}}(x,0) = \tilde{\theta}_1(x),\quad 0 \le x \le 1, \quad (4.21a)$$

$$\tilde{\theta}(0,t) = \omega_o(t),\ \tilde{\theta}(1,t) = \omega_1(t),\quad t \ge 0, \quad (4.21b)$$

$$\dot{\tilde{æ}}(x,t) = -\mu_1\left(s\,\Delta\theta' + \Delta\dot{\theta}'\right)\tilde{\theta}',\ \tilde{æ}(x,0) = æ_o(x), \quad (4.22a)$$

$$\dot{\tilde{g}}(x,t) = -\mu_2\left(s\,\Delta\theta + \Delta\dot{\theta}\right)\tilde{\theta},\ \tilde{g}(x,0) = g_o(x), \quad (4.22b)$$

$$\dot{\tilde{p}}(x,t) = -\mu_3\left(s\,\Delta\theta + \Delta\dot{\theta}\right)\ddot{\tilde{\theta}},\ \tilde{p}(x,0) = p_o(x), \quad (4.22c)$$

where $s > 0$, $\mu > 0$, $\mu_i > 0$, $i = 0, 1, 2, 3$ are adaptation gains such that

$$a(x) = \mu s + g(x) + \mu_o - p(x)s^2 > 0,\ \ \mu - p(x)s > 0, \quad \text{for all } x \in [0,1], \quad (4.22d)$$

$æ_o(x) > 0$, and $\tilde{\theta}_o(x)$, and $\tilde{\theta}_1(x)$ are smooth functions, $g_o(x) > 0$ and $p_o(x) > 0$ are continuous functions, $\Delta\theta = \theta - \tilde{\theta}$. The law ensures the desired asymptotic convergence

$$\lim_{t\to\infty}\int_o^1 \left\{ \begin{matrix} [\Delta\theta(x,t)]^2 + [\Delta\dot\theta(x,t)]^2 + [\Delta\theta']^2 \\ +[\Delta æ(x,t)]^2 + [\Delta g(x,t)]^2 + [\Delta p(x,t)]^2 \end{matrix} \right\} dx = 0 \tag{4.23}$$

$$\Delta\theta = \theta - \tilde\theta,\ \Delta æ = æ - \tilde{æ},\ \Delta g = g - \tilde g,\ \Delta p = p - \tilde p, \tag{4.23a}$$

for arbitrary initial distributions, adaptation gains and generators of persistent excitation of vibrating string (1.4) - (1.4b), which due to continuity of state and parameter errors (4.23a) implies their pointwise convergence almost everywhere in $x\in[0,1]$ and consequently identification of unknown parameters everywhere in $x\in[0,1]$.

Theorem 4.5 Let the external and boundary inputs $y(x,t)$, $w_o(t)$, and $w_1(t)$ generate persistent excitation of the vibrating string (1.4) - (1.4b). Then, the limiting relation (4.23) holds for the adaptive identification law (4.21) and parameters $\tilde{æ}(x,t)$, $\tilde g(x,t)$, $\tilde p(x,t)$ tuned as (4.22).

Proof. First, we prove the local existence of the unique solution of the overall system (1.4), (4.21), and (4.22). For this purpose, we integrate Eqs. (4.22)

$$\tilde{æ}(x,t) = æ_o(x) - \mu_1\int_o^t \left(s\Delta\theta' + \Delta\dot\theta'\right)\tilde\theta'\, d\tau,$$

$$\tilde g(x,t) = g_o(x) - \mu_2\int_o^t \left(s\Delta\theta + \Delta\dot\theta\right)\theta\, d\tau,$$

$$\tilde p(x,t) = p_o(x) - \mu_3\int_o^t \left(s\Delta\theta + \Delta\dot\theta\right)\ddot{\tilde\theta}\, d\tau = p_o(x) +$$

$$+\mu_3\int_o^t\left[s\Delta\dot\theta + \ddot\theta\right]\dot{\tilde\theta}\, d\tau + \frac{1}{2}\mu_3\left\{\left[2s\Delta\theta(x,0) + 2\dot\theta(x,0) - \dot{\tilde\theta}(x,0)\right]\right\}\times$$

$$\times\dot{\tilde\theta}(x,0) - \left[2s\Delta\theta(x,t) + 2\dot\theta(x,t) - \dot{\tilde\theta}(x,t)\right]\dot{\tilde\theta}(x,t)\Big\}, \tag{4.24}$$

and substitute the outputs of the integrators into (4.21). Since functions (4.24) are positive at initial time moment t = 0 and they are continuous in $(x,t,\theta,\tilde\theta)$ the resulting equation is locally hyperbolic [19]. Hence, the results of [19] can be applied to show the existence of a unique local solution of (4.21).

Now, let us define Lyapunov functional

$$V(x) = \frac{1}{2}\int_o^1 \left\{ \begin{bmatrix} p(x)[s\Delta\theta + \Delta\dot\theta]^2 + a(x)[\Delta\theta]^2 + æ(x)[\Delta\theta']^2 \\ +\dfrac{1}{\mu_1}[\Delta æ]^2 + \dfrac{1}{\mu_2}[\Delta g]^2 + \dfrac{1}{\mu_3}[\Delta p]^2 \end{bmatrix} \right\} dx \tag{4.25}$$

on solutions of equations

$$p(x)\Delta\ddot\theta + \Delta p(x,t)\ddot{\tilde\theta} = \left[æ(x)\Delta\theta'\right]' + \left[\Delta æ(x,t)\tilde\theta'\right]'$$

$$-\mu\Delta\dot\theta - \left(g + \mu_o\right)\Delta\theta - \Delta g\tilde\theta,$$

$$\Delta\theta(0,t) = 0,\ \Delta\theta(1,t) = 0, \tag{4.26a}$$

$$\Delta\dot{æ} = \mu_1\left(s\Delta\theta' + \Delta\dot\theta'\right)\tilde\theta',$$
$$\Delta\dot g = \mu_2\left(s\Delta\theta + \Delta\dot\theta\right)\tilde\theta, \tag{4.26b}$$
$$\Delta\dot p = \mu_3\left(s\Delta\theta + \Delta\dot\theta\right)\ddot{\tilde\theta}$$

with respect to variables Δq, $\Delta æ$, Δg, Δp to demonstrate that solution of (4.21) and (4.22) is well-posed for all $t \geq 0$. The computation of the time derivative of the Lyapunov functional along the trajectories of (4.26) yields

$$\dot V(t) = \int_o^1\Big[p\left(s\Delta\dot\theta + \Delta\ddot\theta\right)\left(s\Delta\theta + \Delta\dot\theta\right) + a\Delta\theta\Delta\dot\theta + æ\Delta\theta'\Delta\dot\theta' +$$

$$+\frac{1}{v_1}\Delta æ\Delta\dot{æ} + \frac{1}{v_2}\Delta g\Delta\dot g + \frac{1}{v_3}\Delta p\Delta\dot p\Big]dx = \int_o^1\Big\{ps\Delta\dot\theta\left(s\Delta\theta + \Delta\dot\theta\right) +$$

$$+\left(s\Delta\theta + \Delta\dot\theta\right)\left[-\Delta p\ddot{\tilde\theta} + \left(æ\Delta\theta'\right)' + \left(\Delta æ\tilde\theta'\right)' - \mu\Delta\dot\theta - \left(g + \mu_o\right)\Delta\theta\right.$$

$$\left.-\Delta g\tilde\theta + a\Delta\theta\Delta\dot\theta + +æ\Delta\theta'\Delta\dot\theta' + \Delta p\ddot{\tilde\theta} + \Delta g\tilde\theta\right] + \Delta k\left(s\Delta\theta' + \Delta\dot\theta'\right)\tilde\theta'\Big\}dx$$

$$= \int_o^1\Big\{ps\left(\Delta\dot\theta\right)^2 + ps^2 \times$$

$$\times\Delta\theta\Delta\dot\theta - \left(s\Delta\theta' + \Delta\dot\theta'\right)\left(æ\Delta\theta' + \Delta æ\tilde\theta'\right) - \left(s\Delta\theta + \Delta\dot\theta\right)$$

$$\left[\mu\Delta\dot\theta + \left(g + \mu_o\right)\Delta\theta\right] +$$

$$a\ \Delta\theta\Delta\dot\theta + æ\Delta\theta'\ \Delta\dot\theta' + \Delta æ\left(s\Delta\theta' + \Delta\dot\theta'\right)\tilde\theta'\Big\}dx$$

$$+\left(s\Delta\theta + \Delta\dot\theta\right)\left(æ\Delta\theta' + \Delta æ\tilde\theta'\right)\Big|_o^1 = \int_o^1\Big\{ps\Delta\dot\theta^2$$

$$+p\mu^2\Delta\dot\theta\Delta\theta - sæ\left(\Delta\theta'\right)^2 - æ\Delta\theta'\Delta\dot\theta' - \mu s\Delta\theta\Delta\dot\theta - \mu\Delta\dot\theta^2$$

$$-\left(g + \mu_o\right)\Delta\theta\Delta\dot\theta - gs\Delta\theta^2 + a\Delta\theta\Delta\dot\theta + æ\Delta\theta'\Delta\dot\theta'\Big\}dx =$$

$$= -\int_o^1\left(\mu - ps\right)\left[\Delta\dot\theta\right]^2 dx - \int_o^1 sæ\left(\Delta\theta'\right)^2 dx$$

$$-\int_o^1\left(g + \mu_o\right)s\left[\Delta\theta\right]^2 dx \leq 0,$$

that guarantees the boundedness of Lyapunov functional $V(t) \leq V(0) < \infty$, for all $t \geq 0$, L_2-boundedness

$$\int_o^1\Big\{\left[\Delta\theta(x,t)\right]^2 + \left[\Delta\theta'(x,t)\right]^2 + \left[\Delta\dot\theta(x,t)\right]^2\Big\}dx \tag{4.27}$$
$$\leq M < \infty, M = const,\ t \geq 0$$

of the solutions of (4.26) and their time and spatial derivatives, and their stability.

Furthermore, (4.27) ensures L_2-uniform boundedness of every trajectory of (4.26) and equicontinuity property in Banach space $L_2(0,1)$ for its values $\Delta Q(x,t)$ under $t \geq 0$. Hence, by virtue of Ascoli-Arzela theorem [28] every trajectory of (4.26) is precompact in $L_2(0,1)$. Therefore, according to the invariance principle [20, Theorem 4.3.4] the trajectories of system (4.26) tend to the maximal invariant subset of a set of solutions of (4.26) for which

$$\dot V(t) = -\int_o^1\Big\{\left(\mu - ps\right)\left[\Delta\dot\theta\right]^2 + sæ\left[\Delta\theta'\right]^2 + \left(g + \mu_o\right)s\left[\Delta\theta\right]^2\Big\}dx = 0$$

By virtue of (4.26) it implies that

$$\Delta æ(x,t) = \Delta æ(x),\ \Delta g(x,t) = \Delta g(x),\ \Delta p(x,t) = \Delta p(x) \tag{4.28a}$$

$$\Delta p(x)\ddot\theta = \left[\Delta æ(x)\theta'\right]' - \Delta g(x)\theta,\ \theta(0,t) = \omega_o(t),$$
$$\theta(1,t) = \omega_1(t),\ 0 < x < 1,\ t > 0. \tag{4.28b}$$

But due to the assumption of the theorem relations (4.28) hold if and only if

$$\Delta æ(x) = \Delta g(x) = \Delta p(x) = 0 \text{ for all } x\in[0,1]. \tag{4.29}$$

This justifies the convergence (4.23) and completes the proof of the theorem.

The similar identification problem solution for the case of Neumann boundary conditions (1.4a) is obtained by replacing boundary conditions in the identification law proposed above by the appropriate Neumann boundary conditions.

Remarks 4.1 and 4.2 are valid for the identification of the vibrating string as well.

V. Conclusions

This work presents construction of the adaptive identifier for heat processes, their steady state regimes, and vibrating strings, described by partial differential equations of parabolic, elliptic, and hyperbolic type, respectively, with either homogeneous Neumann or nonhomogeneous Dirichlet boundary conditions. Distributed sensing of the system state and knowledge of the input is assumed to be available. The whole adaptive identifier is represented as two error systems describing the evolution of the state error and the parameter error. In the adaptive identifiers of the distributed parameter systems, the state and the parameter error systems take the form of a partial differential equation and an ordinary differential equation respectively. The identifier developed does not require higher order temporal and spatial derivatives of the state, which lead to the loss of robustness in the presence of measurement noise. Employing the first-order spatial state derivative in the identification law implies the use of the regularization method for its calculation because of ill-posedness of the identification problem. Due to assumptions on plant parameters and input functions, the implementation of the derivative in the identification algorithms admits simple realization via an appropriate differential filter or an approximation by first-order difference with a sufficiently small step. Adjustable parameters in the adaptive identifier are shown to simultaneously converge to their nominal space-varying values when an appropriate input signal is used. The sufficiently rich input signals, referred to as the generators of the persistent excitation are defined for the problem at hand. They ensure the existence of a unique zero steady state for the parameter errors thereby yielding unknown spatially varying plant parameters. The identification algorithms proposed admit a generalization to the case of the multi-dimensional spatial variable. The validity of the algorithms proposed is limited by the boundary conditions. The numerical simulation shows the loss of convergence under boundary conditions of mixed type.

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