Discrete Levels Control of Nonlinear Systems

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Abstract
We study an interesting issue arising in the digital implementation and hybrid control systems frameworks, namely what are the basic characteristics of the behavior of a nonlinear dynamic system when it is driven by input that can assume only discrete levels. In particular, we focus on the controllability of nilpotent systems with and without drift, whose inputs take values in a finite discrete levels set. In both cases, we prove that the constrained system and the corresponding unconstrained system have the same accessibility and controllability properties. An application of these results in motion planning for nonholonomic systems appears in a companion paper.

1. Introduction

A major obstacle in the digital computer realization of control algorithms comes from the finite nature of the computing devices that allows only a finite number of levels in the various signals. Thus, the control of nonlinear systems, a complex affair in its own, gets more complicated when one considers the digital implementation of the desired control policy. It is well known that the behavior and basic properties of a linear system may change drastically when sampling is used in one or more locations in the loop. This paper is an initial attempt to understand the effects of digital implementation on the behavior and properties of a class of nonlinear systems.

Such matters acquire additional importance when one considers control systems in the more general framework of supervisory control systems. Hybrid control systems (see e.g. [1]), as the current modeling term goes, are a combination of continuous-state systems with discrete-event logical controllers. In such an environment, the repercussions from "constrained" realizations may be more far-reaching and, therefore, a better and deeper understanding of behavior is a prerequisite for safe and successful implementations.

In this paper, we consider only piecewise constant inputs, further constrained to correspond to a realistic digital implementation. In a sense, we extend the work of H. J. Sussmann and G. Lafferriere [3], (see also [4, Chapter 8, Section 3.4]). This class of problems contains other members, such as those dealing with the prediction of the plant's behavior after the application of the chosen control law, e.g. [6] and [2]. In [2], the authors examine the issue of continuous systems driven by discrete inputs with low bounded switching intervals. Although some general notions were introduced for the nonlinear case, specific developments were presented for the single input (multiple state) linear case and an optimal control approach was outlined for the single input, single output case.

Our purpose in this paper is to find out the basic characteristics of the behavior of a class of nonlinear dynamic plants when they are driven by inputs that are piecewise constant, quantized, and taking values from a finite set. We focus here on the controllability of a nonlinear system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{x}) v_i$$

where the m components of the input $v_1, v_2, \ldots, v_m$ take values in a discrete levels set $Q = \{q_1, q_2, \ldots, q_r\}$. When the term $\mathbf{f}(\mathbf{x})$ is missing, the system is called drift-free. Further, we assume that the system is nilpotent (or nilpotentizable), that is its flow-equivalent system can be represented, and therefore analyzed, in a simple way. The order of nilpotency is a direct indication of the necessary complexity in the representation. A system is nilpotentizable if it can be made nilpotent by a proper feedback transformation.

In the rest of the paper, we study the controllability of nilpotent systems with drift, whose inputs take values in a discrete level set and we give a fundamental result concerning the Controllability Lie Algebra of the constrained system (part 2.1). Next, we study the case of nilpotent systems without drift, whose inputs take values in a discrete levels set and we prove that these results are independent of the number of levels (part 2.2).

Notation

The following notation will be used throughout:

$\mathcal{L}_S$	the Controllability Lie Algebra (CLA) of the system (S)
$LC(\mathbf{g}_1,\ldots\mathbf{g}_m)$	the Lie Algebra generated by the vector fields $\{\mathbf{g}_1,\ldots,\mathbf{g}_m\}$
$\Re$	the set of real numbers
$\Re^n$	the n-dimensional Euclidean space
I_S	the maximum number of linearly independent vector fields in $\mathcal{L}_S$
$det(\mathbf{A})$	the determinant of the matrix **A**

2. Main Results

We consider a nonlinear system of the form

$$(\Sigma,Q): \qquad \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})v_i, \qquad (1)$$

where the state $x(t)$ belongs to an open subset N of $\Re^n$, while the m components of the input $v_1,v_2,\ldots,v_m$ take values in a discrete input levels set $Q=\{q_1,q_2,\ldots,q_r\}$. We suppose that the vector fields $\mathbf{f}(\mathbf{x}),\mathbf{g}_1(\mathbf{x}),\ldots,\mathbf{g}_m(\mathbf{x})$, which are defined on N, are smooth, complete and linearly independent.

We are going to study the controllability of the system (Σ,Q) assuming that it is nilpotent with degree of nilpotency k. A Lie Algebra $\mathcal{L}$ is nil-potent if there is an integer k such that all the Lie Brackets $[v_1[v_2,\ldots,[v_k,v_{k+1}]\ldots]]$ vanish. The smallest integer k that has this property is called order of nilpotency of $\mathcal{L}$ and $\mathcal{L}$ is said to be nilpotent of order k. The system (Σ,Q) will be called nilpotent if its Controllability Lie Algebra $\mathcal{L}_\Sigma$ is a nilpotent algebra.

The Controllability Lie Algebra (CLA) of the system (Σ,Q) is generated by the vector fields $\mathbf{F}(\mathbf{x},\mathbf{v})=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})v_i$ for all the admissible values of the input vector $\mathbf{v}=[v_1\ v_2\ \ldots v_m]^T$.

If we have r discrete levels then the input vector $\boldsymbol{v}$ can take values in a set consisting of $\mu=r^m$ admissible input vectors. We denote this set by $Q(q_1,q_2,\cdots,q_r)$. Consequently, the vector field $\mathbf{F}(\mathbf{x},\mathbf{v})$ can also take $\mu=r^m$ possible values with each one of them corresponding to a specific value of the input vector $\boldsymbol{v}$. Thus the CLA, $\mathcal{L}_S$, of (Σ,Q) is of the form

$$\mathcal{L}_\Sigma=LC(z_i=\mathbf{F}(\mathbf{x},v_i),\ v_i\in Q(q_1,q_2,\ldots,q_r)).$$

Next, we consider the nonlinear system

$$(\mathrm{S}): \quad \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})u_i \qquad (2)$$

where the m components of the input $u_1,u_2,\ldots,u_m$ take values in $\Re$, without any constraint. In other words, the system (S) is the system (Σ,Q) without the discrete input levels constraint.

The system (S) has a CLA of the form

$$\mathcal{L}_S=LC(\mathbf{f}(\mathbf{x}),\mathbf{g}_1(\mathbf{x}),\ldots,\mathbf{g}_m(\mathbf{x})).$$

2.1 Nilpotent Systems with Drift

Our purpose in this part is to find the relation of the CLAs $\mathcal{L}_S$ and $\mathcal{L}_\Sigma$. As a "comparison tool" we will use the Philip Hall bases W and B of these algebras respectively. At a first glance, we expect that $\mathcal{L}_\Sigma$ is a subalgebra of $\mathcal{L}_S$, but we will prove that under certain circumstances $\mathcal{L}_\Sigma\equiv\mathcal{L}_S$. In particular, we have the following theorem:

Theorem 1

Let the nilpotent systems

$$(\Sigma,Q): \quad \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})v_i$$

and

$$(\mathrm{S}): \quad \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})u_i$$

with order of nilpotency k, where the input vector $\boldsymbol{v}$ takes values in the set $Q(q_1,\ldots,q_r)$ while the input vector $\boldsymbol{u}$ takes, without any constraint, values in $\Re^m$. If we choose q_i, i=1,...,r such the maximum number of linearly independent vector fields in $\mathcal{L}_\Sigma$, I_S, is equal to the maximum number of linearly independent vector field in $\mathcal{L}_S$, I_S, then $\mathcal{L}_\Sigma \equiv \mathcal{L}_S$.

Proof

For simplicity and without any loss of generality we will assume that the degree of nilpotency of the systems (Σ,Q) and (S) is k=2. Let $\boldsymbol{B}$ and $\boldsymbol{W}$ be the P. Hall basis of CLA $\mathcal{L}_\Sigma$ and $\mathcal{L}_S$, respectively. Then the candidate vector fields for the basis $\boldsymbol{W}$, are the following:

$$W = \left\{ \begin{matrix} \mathbf{f}, \mathbf{g}_1, \ldots, \mathbf{g}_m, [\mathbf{f}, \mathbf{g}_1], \ldots, [\mathbf{f}, \mathbf{g}_m], [\mathbf{g}_1, \mathbf{g}_2], \ldots, \\ [\mathbf{g}_1, \mathbf{g}_m], \ldots, [\mathbf{g}_{m-1}, \mathbf{g}_m] \end{matrix} \right\}$$

Assuming that all of these $\theta = m + \frac{(m+1)!}{2(m-1)!}$ vector fields are linearly independent, we have that $\boldsymbol{W} = \boldsymbol{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_\theta\}$, where

$$\mathbf{w}_1 = \mathbf{f},\ \mathbf{w}_2 = \mathbf{g}_1, \ldots,\ \mathbf{w}_{m+1} = \mathbf{g}_m, \ldots,\ \mathbf{w}_\theta = [\mathbf{g}_{m-1}, \mathbf{g}_m]$$

The candidates for the basis $\boldsymbol{B}$ are the following $\mu + \frac{\mu!}{2(\mu-2)!}$ vector fields

$$B = \{z_1, \ldots, z_\mu, [z_1, z_2], \ldots, [z_2, z_3], \ldots, [z_2, z_\mu], \ldots, [z_{\mu-1}, z_\mu]\}$$

.

Observing the vector fields belonging to $\boldsymbol{B}$, we see that they can be written as linear combinations of the vector fields $\mathbf{w}_1, \ldots, \mathbf{w}_\theta$. This means that:

1. The basis $\boldsymbol{B}$ has, at most, θ linearly independent vector fields and
2. Every vector field that belongs to $\mathcal{L}_\Sigma$ belongs also to $\mathcal{L}_S$.

Let us suppose that we have chosen $q_1, \ldots, q_r$ in such a way that in $\boldsymbol{B}$ there are θ linearly independent vector fields. Denoting these vector fields by $\mathbf{b}_1, \ldots, \mathbf{b}_\theta$, we have that $\boldsymbol{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_\theta\}$, where

$$\begin{aligned} \mathbf{b}_1 &= c_{11}(q_i)\mathbf{w}_1 + \ldots + c_{1\theta}(q_i)\mathbf{w}_\theta \\ \mathbf{b}_2 &= c_{21}(q_i)\mathbf{w}_1 + \ldots + c_{2\theta}(q_i)\mathbf{w}_\theta \\ &\vdots \\ \mathbf{b}_\theta &= c_{\theta 1}(q_i)\mathbf{w}_1 + \ldots + c_{\theta\theta}(q_i)\mathbf{w}_\theta \end{aligned} \tag{3}$$

and the coefficients $c_{ij}(q_i), i,j = 1,2,\ldots,\theta$ are functions of $q_1, \ldots, q_r$.

In order to prove that $\mathcal{L}_S \equiv \mathcal{L}_\Sigma$ it is enough to show that

(a) if $s \in \mathcal{L}_\Sigma$ then $s \in \mathcal{L}_S$

(b) if $s \in \mathcal{L}_S$ then $s \in \mathcal{L}_\Sigma$

Proof of (a):

It is obvious and there is no need to prove it.

Proof of (b):

From the assumptions of the theorem, we know that the discrete control levels $q_1, \ldots, q_r$ have been chosen in such a way that

$$\boldsymbol{B} = \mathbf{C}(q_i)\boldsymbol{W} \tag{4}$$

where $\boldsymbol{B} = [\mathbf{b}_1\ \mathbf{b}_2 \ldots \mathbf{b}_\theta]$, $\boldsymbol{W} = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_\theta]^T$ and $\mathbf{C}(q_i) = [c_{ij}(q_i)]$ $i,j = 1,\ldots,\theta$ with the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_\theta$ linearly independent.

This means that the q_i's are such that the matrix $\mathbf{C}(q_i)$ is nonsingular and thus invertible. So we can write that

$$\boldsymbol{W} = \mathbf{C}^{-1}(q_i)\boldsymbol{B} \tag{5}$$

Equation (5) means that the vector field of the basis $\boldsymbol{W}$ can be written as a linear combination of vector fields of the basis $\boldsymbol{B}$, i.e.

$$\begin{aligned} \mathbf{w}_1 &= \Gamma_{11}(q_i)\mathbf{b}_1 + \ldots + \Gamma_{1\theta}(q_i)\mathbf{b}_\theta \\ \mathbf{w}_2 &= \Gamma_{21}(q_i)\mathbf{b}_1 + \ldots + \Gamma_{2\theta}(q_i)\mathbf{b}_\theta \\ &\vdots \\ \mathbf{w}_\theta &= \Gamma_{\theta 1}(q_i)\mathbf{b}_1 + \ldots + \Gamma_{\theta\theta}(q_i)\mathbf{b}_\theta \end{aligned} \tag{6}$$

where the coefficients $\Gamma_{ij}(q_i)$ are functions of the q_i's, i=1,...,m.

Let $s \in \mathcal{L}_S$. Then there are real numbers $\lambda_1, \ldots, \lambda_\theta$ such that

$$\mathbf{s} = \lambda_1\mathbf{w}_1 + \ldots + \lambda_\theta\mathbf{w}_\theta \tag{7}$$

Using (6), equation (7) can be written as follows:

$$\mathbf{s} = \lambda_1(\Gamma_{11}\mathbf{b}_1 + \ldots + \Gamma_{1\theta}\mathbf{b}_\theta) + \ldots + \lambda_\theta(\Gamma_{\theta 1}\mathbf{b}_1 + \ldots + \Gamma_{\theta\theta}\mathbf{b}_\theta)$$
$$= \sum_{k=1}^{\theta}\left(\sum_{i=1}^{\theta}\lambda_i\Gamma_{ik}\right)\mathbf{b}_k \tag{8}$$

Thus $s \in \mathcal{L}_S \Rightarrow s \in \mathcal{L}_\Sigma$.

Having proved (a) and (b) we can say that $\mathcal{L}_\Sigma \equiv \mathcal{L}_S$. During the proof we assumed that all vector fields belonging to W are linearly independent. In case where some of the vector fields of W' are linearly dependent, we have the same result, i.e. $\mathcal{L}_\Sigma \equiv \mathcal{L}_S$. In order to prove that, we follow the previous procedure of proof using the vector fields of W that are linearly independent.

We have just proved that if the discrete input levels $q_1,\ldots,q_r$ satisfy the condition of Theorem 1 then the nilpotent systems (Σ,Q) and (S) have the same CLA. This means that if the system (S) satisfies the Lie Algebra Rank Condition (LARC) [4] at a point $p \in N$ then (Σ,Q) satisfies LARC at $p \in N$, and conversely. Thus (S) has the accessibility property at $p \in N$ if and only if (Σ,Q) has the same property at $p \in N$.

2.2 Nilpotent Systems without drift

We consider a drift-free nilpotent system of the form

$$(\mathrm{S}): \ \dot{\mathbf{x}}(t) = \mathbf{g}_1(\mathbf{x})u_1 + \ldots + \mathbf{g}_m(\mathbf{x})u_m$$

where the state vector $x(t)$ takes values in an open subset N of $\Re^n$, i.e. $N \subset \Re^n$, while the input vector takes values in $\Re^m$, without any constraint. We suppose that the vector fields $\mathbf{g}_1(\mathbf{x}),\ldots,\mathbf{g}_m(\mathbf{x})$ are defined on N, are complete, smooth and linearly independent, $\forall \mathbf{x} \in N$. Also, we suppose that the system (S) is nilpotent with order of nilpotency k. Our main purpose in this paragraph is to study the controllability of (S) when it is excited by an input vector, which satisfies the discrete input levels constraint.

Thus we examine the system

$$(\Sigma,\mathrm{Q}): \ \dot{\mathbf{x}}(t) = \mathbf{g}_1(\mathbf{x})v_1 + \ldots + \mathbf{g}_m(\mathbf{x})v_m$$

where the input vector $v = [v_1,\ldots,v_m]$ belongs to $Q(q_1,\ldots,q_r)$.

We denote with $\mathcal{L}_S$ and $\mathcal{L}_\Sigma$ the CLAs of the systems (S) and (Σ), respectively. In order to find out the relation between $\mathcal{L}_S$ and $\mathcal{L}_\Sigma$ we can use Theorem 1 with $\mathbf{f}(\mathbf{x}) \equiv 0$, $\forall \mathbf{x} \in N$. Thus we have the following:

Corollary 1

Let the drift-free nilpotent systems (Σ,Q) and (S) with degree of nilpotency k. If we choose the discrete levels $q_1,\ldots,q_r$ in such a way that $I_\Sigma \equiv I_S$ then $\mathcal{L}_S \equiv \mathcal{L}_\Sigma$.

Next, we are going to give the conditions that $q_1,\ldots,q_m$ must satisfy in order to $\mathcal{L}_S \equiv \mathcal{L}_\Sigma$. We also prove that these conditions are independent of the choice of r.

Proposition 1

Let the nilpotent drift-free systems (Σ,Q) and (S) with order of nilpotency k and $r=m$. For every choice of the discrete levels $q_1,\ldots,q_m$ such that $q_1=0$ and $q_i \neq 0$, $i=2,\ldots,m$, I_Σ is equal to I_S, i.e. $I_\Sigma \equiv I_S$

Proof

Without any loss of generality we will consider the case where $k=2$.

The system (S) has the form

$$\mathbf{G}_i(\mathbf{x},v_i) = \mathbf{g}_1(\mathbf{x})v_1 + \ldots + \mathbf{g}_m(\mathbf{x})v_m$$

and its CLA $\mathcal{L}_S$ has a P. Hall basis of the form

$$W = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m, [\mathbf{g}_1,\mathbf{g}_2], \ldots, [\mathbf{g}_1,\mathbf{g}_m], \ldots, [\mathbf{g}_{m-1},\mathbf{g}_m]\}$$
$$= \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\theta\}$$

where $\theta = m + \dfrac{(m+1)!}{2(m-1)!}$, assuming that the vector fields $\mathbf{w}_i$, $i = 1,\ldots,\theta$ are linearly independent.

We choose from $\mathcal{L}_\Sigma$ the following vector fields

$$\mathbf{z}_1 = q_{11}\mathbf{g}_1 = q_{11}\mathbf{w}_1, \quad \mathbf{z}_2 = q_2\mathbf{g}_2 = q_2\mathbf{w}_2, \; \dots ,$$
$$\mathbf{z}_m = q_m\mathbf{g}_m = q_m\mathbf{w}_m$$

and the Lie brackets

$$\mathbf{z}_{m+1} = [\mathbf{z}_1, \mathbf{z}_2] = q_{11}q_2\mathbf{w}_{m+1},$$
$$\mathbf{z}_{2m-1} = [\mathbf{z}_1, \mathbf{z}_m] = q_{11}q_m\mathbf{w}_{2m-1},$$
$$\dots,$$
$$\mathbf{z}_\theta = [\mathbf{z}_{m-1}, \mathbf{z}_m] = q_{m-1}q_m\mathbf{w}_\theta$$

where $q_{11} \in \{q_2, \dots, q_m\}$.

Let $W = [\mathbf{w}_1 \; \mathbf{w}_2 \; \dots \; \mathbf{w}_\theta]^T$ and $Z = [\mathbf{z}_1 \; \mathbf{z}_2 \; \dots \; \mathbf{z}_\theta]^T$. Then we can write the previous equations in the form $W = \mathbf{C}(q_i)Z$, where the $\theta \times \theta$ matrix $\mathbf{C}(q_i)$ is given by

$$\mathbf{C}(q_i) = \begin{bmatrix} q_{11} & 0 & \cdots & & & & \cdots & 0 & 0 \\ 0 & q_2 & & & & & & & 0 \\ \cdots & & \cdots & & & & & & \cdots \\ & & q_m & & & \mathbf{0} & & & \\ & & & q_{11}q_2 & & & & & \\ & & & & \cdots & & & & \\ & & & & & q_{11}q_m & & & \cdots \\ \cdots & & \mathbf{0} & & & & q_2q_3 & & 0 \\ 0 & & & & & & & \cdots & 0 \\ 0 & 0 & \cdots & & & & \cdots & 0 & q_{m-1}q_m \end{bmatrix}$$

We can note that $\mathbf{C}(q_i)$ is nonsingular since $\det(\mathbf{C}(q_i)) \neq 0$.

Consequently, there are exactly θ linearly independent vector fields in $\mathcal{L}_\Sigma$, i.e. $I_\Sigma \equiv I_S$.

Proposition 2

We consider the nilpotent drift-free systems (Σ, Q) and (S) with order of nilpotency k and $r>m$. For every choice of the discrete levels $q_1, \dots, q_r$ such that $q_1 = 0$ and $q_i \neq 0$, $i = 2, \dots, m$, I_Σ is equal to I_S, i.e. $I_\Sigma \equiv I_S$

Proposition 3

Let the nilpotent drift-free systems (Σ, Q) and (S) with order of nilpotency k and $r<m$. For every choice of the discrete levels $q_1, \dots, q_r$ such that $q_1 = 0$ and $q_i \neq 0$, $i = 2, \dots, m$, I_Σ is equal to I_S, i.e. $I_\Sigma \equiv I_S$

In the previous proofs we assumed that the vector fields $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_\theta$ are linearly independent. If it is not so, we can show that $I_\Sigma \equiv I_S$ choosing only the linearly independent vector fields out of the whole set.

Proposition 4

Let the nilpotent drift-free systems (Σ, Q) and (S) with order of nilpotency k. For every choice of the discrete levels $q_1, \dots, q_r$ such that $q_1 = 0$ and $q_i \neq 0$, $i = 2, \dots, r$, with $r \geq m$ or $r<m$, the associated CLAs $\mathcal{L}_\Sigma$ and $\mathcal{L}_S$ are such that $\mathcal{L}_\Sigma \equiv \mathcal{L}_S$

Proof

We have proved that if we choose the discrete levels $q_1, \dots, q_r$ in such a way that $q_1 = 0$ and $q_i \neq 0$, $i = 2, \dots, r$, for every $r>1$, then $I_\Sigma \equiv I_S$. But according to Corollary 1 when $I_\Sigma \equiv I_S$ then $\mathcal{L}_\Sigma \equiv \mathcal{L}_S$.

Further, we proved that if we choose the discrete levels in such a way that $q_1 = 0$ and $q_i \neq 0$, $i = 2, \dots, r$, for every $r>1$, the systems (Σ, Q) and (S) have the same CLA. This means that if (S) is controllable then (Σ, Q) is also controllable and conversely.

3. Conclusions

In this paper, we studied an issue arising at the foundation of practical implementations of control systems with repercussions in the hybrid control systems framework as well. We found out the basic characteristics of the behavior of a system when it is driven by a finite number of discrete input levels. In particular, we focused on the controllability of nilpotent systems with and without drift, whose inputs take values in a finite discrete levels set. In both cases, we proved that the constrained system and the corresponding unconstrained system have the same accessibility and controllability properties. An application of these results in motion planning for nonholonomic systems appears in a companion paper [5].

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