Every Mode Robust Stabilization of Jump Linear Systems via LMI

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Abstract

This paper considers the mode-dependent state-feedback control problem of linear systems subject to random Markovian jumps in parameter values. For this kind of systems, the mean-square stability does not ensure the stability of every mode in the deterministic sense. We provide stabilizing solution in both senses. The proposed approach differs from the modified jump regulator approach and has several advantages. The proposed conditions are only sufficient, but less conservative. We can treat uncertainties that can affect modes or the transition probability matrix. The problem is formulated as a convex constraint (LMIs) one.

Key Words. Jump linear system, Markov process, Mean-square stability, Linear Matrix Inequality.

Notation

The matrix M^T denotes the transpose of the real matrix M. For a real matrix P, $P > 0$ (resp. $P \geq 0$) means P is symmetric and positive-definite (resp. positive semidefinite). I denotes the identity matrix, with size determined from context. The symbols **Co** and **E** denote the convex hull, and the expectation operator, respectively.

1 Introduction

The linear system with Markovian jumps is defined by:

$$\frac{dx}{dt} = A(r(t))x + B(r(t))u \qquad (1)$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the control input, and

- the process $r : \mathbf{R}^+ \to \{1, ..., N\}$ is Markovian, with transition probabilities defined by:

$$\mathbf{Prob}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}(t)\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \pi_{ii}(t)\Delta + o(\Delta) & \text{else,} \end{cases}$$

  where $\Pi(t) = (\pi_{ij}(t))$ is a "transition matrix", that is, $\pi_{ij} \geq 0$ for $i \neq j$, and $-\pi_{ii} = \sum_{j \neq i}^{N} \pi_{ij} \leq 0$. The π_{ij} are the transition probability rates from i to j.

- $A(r(t)) = A_i$, $B(r(t)) = B_i$ when $r(t) = i$.

Here $A_i \in \mathbf{R}^{n \times n}$, $B_i \in \mathbf{R}^{n \times m}$, $i = 1, \dots N$, are known matrices. System (1) can be viewed as a Markov process with randomly jumping parameters. This means that the system operates under several known "modes" (A_i, B_i). Due to random changes of the parameter $r(t)$, the system can "jump" from one mode to the other, according to the transition probabilities given above. This kind of systems can be used to model systems subject to failures or structural random changes. They are also used to describe linearized models of a nonlinear system whose operating point changes .

In this paper, it is assume that the transition matrix $\Pi(t)$ is "uncertain". By this it is meant that $\Pi(t)$ is only known to belong to a bounded convex set. We assume that this set is a polytope: $\mathbf{Co}\{\Pi^1, \dots, \Pi^L\}$, where $\Pi^k = [\pi_{ij}^k]_{1 \leq i,j \leq N}$, $k = 1, \dots, L$ are known transition matrices. Note that a convex, linear combination of transition matrices is also a transition matrix.

The system (1) is said to be *mean-square stable* if, when $u = 0$, and for every initial condition $x(0)$, the corresponding trajectory satisfies:

$$\mathbf{E}\, x(t)x(t)^T \to 0 \text{ for } t \to 0.$$

We consider the problem of finding a state-feedback laws which involves jointly mean-square stability, and every mode stability (each pair (A_i, B_i) is stabilizable in the deterministic sense.), using a control law of the form

$$\begin{array}{c} u(t) = K(r(t))x(t), \\ \text{where } K(r(t)) = K_i, \text{when } r(t) = i, \\ K_i \text{ constant matrix, } i = 1, \dots, N. \end{array} \qquad (2)$$

These laws are required to be *robust* to uncertainties in the transition matrix Π, *i.e.* to stabilize the system for every value of Π taken in a bounded convex set. Our conditions are formulated in terms of an LMI problem . For more details on LMI problems and algorithms to solve them, see [3, 4, 11] and references therein.

In the control point of view, both notions of stability are needed. One have to ensure that the plant state of the linearized models of a jump nonlinear system, will not increase and go outside the domain of validity. In the case of linear jump systems, one have to take into account a 'relatively long' evolution in a certain given mode.

In [9, 8], we can find an approach based on a modified jump regulator problem with a prescribed degree of stability. This approach is more conservative than the proposed one here. Futhermore, it assumes a precise knowledge of the transition probability rates which are in general difficult to estimate. In contrast, the proposed approach of this paper is adequate to deal with such problem.

For more details on the jump linear systems, we mention some references which are concentrated on an optimal control approch (with respect to a quadratic criterion), see *e.g.* [10, 12, 5, 7, 2], and references therein.

2 Preliminary

Consider the open-loop system:

$$\frac{dx}{dt} = A(r(t))x. \tag{3}$$

The following result can be found in [1, 6]

Theorem 2.1 *The following properties are equivalent.*

1. *System (3) is mean-square stable.*
2. *There exist matrices $P_1, \ldots, P_N > 0$ such that:*

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{j=N} \pi_{ij} P_j < 0 \quad i = 1, \ldots, N. \tag{4}$$

3. *There exist matrices $Q_1, \ldots, Q_N > 0$ such that:*

$$A_i Q_i + Q_i A_i^T + \sum_{j=1}^{j=N} \pi_{ji} Q_j < 0 \quad i = 1, \ldots, N. \tag{5}$$

Due to the fact that the term $\pi_{ii} P_i$ is negative, the mode A_i may be unstable. To stabilize every mode, the idea in [9, 8] is to cancel the destabilizing influence of $\pi_{ii} P_i$. This idea is based on a modified jump regulator problem with prescribed degree of stability. This leads to a set of coupled Riccati equations, in which the term $\pi_{ii} P_i$ disappears:

$$A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T + \sum_{j \neq i}^{j=N} \pi_{ij} P_j + Q_i = 0, \tag{6}$$

where $Q_i \geq 0, R_i > 0, i = 1, \ldots, N$, are specified weighting matrices.

Under certain assumptions as in [12], the resulting positive solution of (6), involves the existence of matrices $K_i = -R^{-1} B_i^T P_i$ for $i = 1, \ldots, N$, such that:

$$\begin{gathered} (A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + \sum_{j \neq i}^{j=N} \pi_{ji} P_j < 0 \\ P_1 > 0, \ldots, P_N > 0, \quad i = 1, \ldots, N. \end{gathered} \tag{7}$$

Consequently, the resulting closed-loop system is jointly every mode and mean-square stable.

In the next section, we show that the condition (7) is conservative than the proposed (LMIs) condition.

3 Main results

We begin by the following result:

Theorem 3.1 *The two statement are equivalent*

1. *there exist $P_1, \ldots, P_N > 0$ such that*

$$A_i^T P_i + P_i A_i + \sum_{j \neq i}^{j=N} \pi_{ij} P_j < 0, \quad i = 1, \ldots, N. \tag{8}$$

2. *there exist $Q_1, \ldots, Q_N > 0$ such that*

$$A_i Q_i + Q_i A_i^T + \sum_{j \neq i}^{j=N} \pi_{ji} Q_j < 0, \quad i = 1, \ldots, N. \tag{9}$$

Proof:

Let $\tilde{A}_i = A_i - \frac{1}{2}\pi_{ii} I$. The condition (8) can be written as:

$$\begin{gathered} \tilde{A}_i^T P_i + P_i \tilde{A}_i + \sum_{j=1}^{j=N} \pi_{ji} P_j < 0, \\ P_1 > 0, \ldots, P_N > 0, \quad i = 1, \ldots, N, \end{gathered}$$

the result of theorem 3.1 shows that is equivalent to:

$$\begin{gathered} \tilde{A}_i Q_i + Q_i \tilde{A}_i^T + \sum_{j=1}^{j=N} \pi_{ji} Q_j < 0, \\ Q_1 > 0, \ldots, Q_N > 0, \quad i = 1, \ldots, N, \end{gathered}$$

The proof is then straightforward. □

The system (3) is jointly every mode and mean-square stable if and only if there exist $Q_1, \ldots, Q_N > 0$ and $R_1, \ldots, R_N > 0$ such that:

$$\begin{gathered} A_i Q_i + Q_i A_i^T + \sum_{j=1}^{j=N} \pi_{ji} Q_j < 0, \\ A_i R_i + R_i A_i^T < 0, \quad i = 1, \ldots, N. \end{gathered} \tag{10}$$

This is a simple LMI feasibility problem that is easy to check. Contrarily, the problem of finding tractable necessary and sufficient conditions for the state-feedback synthesis (In both stability senses) is still an open problem. That is why we propose only a sufficient condition by setting in (10) $R_i = Q_i$. We show that we can provide a stabilizing control law by the following result:

Theorem 3.2 *There exists a control law of the form* (2) *that stabilizes every mode and the system in the mean-square sense if there exist* $Q_1, \dots, Q_N > 0$ *and* $Y_1, \dots, Y_N$ *such that:*

$$\begin{gathered} A_iQ_i + Q_iA_i^T + B_iY_i + Y_i^TB_i^T + \sum_{j=1}^{j=N} \pi_{ji}Q_j < 0, \\ A_iQ_i + Q_iA_i^T + B_iY_i + Y_i^TB_i^T < 0, \end{gathered} \tag{11}$$

Then, the corresponding stabilizing control law (2) *is given by:*

$$K_i = Y_iQ_i^{-1}, for\ i = 1, ..., N. \tag{12}$$

Proof:
The system (1) is jointly every mode and mean-square stabilizable if there exist matrices $Q_1 > 0, \dots, Q_N > 0$ and $K_1, \dots, K_N$ such that

$$\begin{gathered} (A_i + B_iK_i)Q_i + Q_i(A_i + B_iK_i)^T + \sum_{j=1}^{j=N} \pi_{ji}Q_j < 0, \\ (A_i + B_iK_i)Q_i + Q_i(A_i + B_iK_i)^T < 0, \quad i = 1, \dots, N. \end{gathered}$$

Making the change of variables $Y_i = K_iQ_i$, we obtain (11). □

Now, we establish that if there exist a stabilizing solution of (6), then it yields a solution of (11). In other words our approach is less conservative. Effectively, the resulting closed-loop system corresponding to (6) satisfies:

$$\begin{gathered} (A_i + B_iK_i)^TP_i + P_i(A_i + B_iK_i) + \sum_{j\neq i}^{j=N} \pi_{ij}P_j < 0, \\ i = 1, \dots, N. \end{gathered}$$

The result of theorem 4.1 shows that above inegalities are equivalent to:

$$\begin{gathered} (A_i + B_iK_i)Q_i + Q_i(A_i + B_iK_i)^T + \sum_{j\neq i}^{j=N} \pi_{ji}Q_j < 0, \\ i = 1, \dots, N. \end{gathered}$$

which implies (11).

Furthermore, in the numerical section we give a counterexample (16) for which the approach of [9, 8] does not provide a stabilizing solution. In contrast, the approach adopted here provides a stabilizing solution. Besides that, this solution takes into account uncertainties which can affect the transition probability rates.

In the sequel, we assume that Π is unknown and belongs to a bounded convex set:

$$\mathcal{P} = \left\{ \Pi = \sum_{i=1}^{L} \lambda_k \Pi^k \right\}, \tag{13}$$

where $\lambda_k \geq 0,\ k = 1, \dots, N,\ \sum_{k=1}^{L} \lambda_k = 1$, and $\Pi^k = [\pi_{ij}^k]_{1\leq i,j\leq N},\ k = 1, \dots, L$ are known transition matrices. We have the following result:

Theorem 3.3 *There exists a control law of the form* (2) *that stabilizes every mode and the system in the mean-square sense for every* $\Pi \in \mathcal{P}$ *if there exist* $Q_1, \dots, Q_N > 0$ *and* $Y_1, \dots, Y_N$ *such that:*

$$\begin{gathered} A_iQ_i + Q_iA_i^T + B_iY_i + Y_i^TB_i^T + \sum_{j=1}^{j=N} \pi_{ji}^kQ_j < 0, \\ A_iQ_i + Q_iA_i^T + B_iY_i + Y_i^TB_i^T < 0, \\ i = 1, \dots, N,\ k = 1, \dots, L. \end{gathered} \tag{14}$$

Then, the stabilizing control law (2) *is given by:*

$$K_i = Y_iQ_i^{-1}, fori = 1, \dots, N. \tag{15}$$

4 Numerical examples

Consider the three-mode jump linear system defined by:

$$\begin{gathered} A_1 = \begin{bmatrix} 0.9304 & 0.5269 \\ 0.8462 & 0.0920 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.6539 & 0.7012 \\ 0.4160 & 0.9103 \end{bmatrix}, \\ A_3 = \begin{bmatrix} 0.7622 & 0.0475 \\ 0.2625 & 0.7361 \end{bmatrix},\ B_1 = \begin{bmatrix} 1.1650 \\ 0.6268 \end{bmatrix}, \\ B_2 = \begin{bmatrix} 0.0751 \\ 0.3516 \end{bmatrix},\ B_3 = \begin{bmatrix} -0.6965 \\ 1.6961 \end{bmatrix}, \\ \text{and } \Pi^o = \begin{bmatrix} -1.6667 & 0.6333 & 1.0333 \\ 0.9000 & -1.6000 & 0.7000 \\ 0.4000 & 0.6667 & -1.0667 \end{bmatrix}. \end{gathered} \tag{16}$$

As we have seen before, if each mode stabilizing control law corresponding to the coupled Riccati equation (6) exists, then the closed-loop system satisfies

$$\begin{gathered} (A_i + B_iK_i)Q_i + Q_i(A_i + B_iK_i)^T + \sum_{j\neq i}^{j=N} \pi_{ji}Q_j < 0, \\ i = 1, \dots, N. \end{gathered}$$

Or equivalently

$$\begin{gathered} A_iQ_i + Q_iA_i^T + B_iY_i + Y_i^TB_i^T + \sum_{j\neq i}^{j=N} \pi_{ji}Q_j < 0, \\ i = 1, \dots, N. \end{gathered}$$

For the given example , we have tested the feasibility problem of the above LMIs in $Q_i > 0, Y_i$ using

LMITOOL implemented with the software of [4, 11]. This above LMI contraints have no feasible solution. In this case, we conclude that the approach of [9, 8] cannot provide any solution.

Alternatively, we have solved (11), which provides

$$Q_1 = \begin{bmatrix} 0.2182 & -0.5348 \\ -0.5348 & 1.3108 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.2518 & -0.6171 \\ -0.6171 & 1.5128 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.4111 & -1.0112 \\ -1.0112 & 2.4878 \end{bmatrix},$$
$$Y_1 = [-0.2563 \quad -0.1210] Y_2 = [0.1801 \quad -4.2307], \text{ and } Y_3 = [0.4552 \quad -1.1007].$$

The corresponding every mode stabilizing control law is:

$$K_1 = 10^3 \times [-9.6004 \quad -3.9170], \quad K_2 = 10^3 \times [-41.208 \quad -16.814], \quad \text{and } K_3 = [416.9556 \quad 169.0383].$$

With the same data matrices (A_i, B_i), assume now that Π is unknown and belongs to $\mathbf{Co}\{\Pi^1, \Pi^2, \Pi^3\}$, where the vertices are given by:

$$\Pi_1 = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.7 & -2 & 1.3 \\ 0.7 & 0.5 & -1.2 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} -2 & 0.7 & 1.3 \\ 1 & -1.4 & 0.4 \\ 0 & 1 & -1 \end{bmatrix}, \quad \text{and } \Pi_3 = \begin{bmatrix} -2 & 0.7 & 1.3 \\ 1 & -1.4 & 0.4 \\ 0.5 & 0.5 & -1 \end{bmatrix}.$$

Note that Π^o given above in (15) belongs to the same set.

In this case, we have sought a mode-dependent control law, for which the closed-loop system is jointly every mode and mean-square stable for any value Π whithin $\mathbf{Co}\{\Pi^1, \Pi^2, \Pi^3\}$. We have obtained the following robust mode-dependent control law

$$K_1^{rob} = 10^4 \times [-1.3581 \quad -0.5551], \quad K_2^{rob} = 10^4 \times [-4.6760 \quad -1.9095], \quad \text{and } K_3^{rob} = [359.1818 \quad 144.8945].$$

5 Conclusion

In this paper, we have treated the robust stabilization of jump linear systems. We have presented a new approach which differs from the approach of modified jump regulator problem. The proposed conditions are only sufficient but less conservative. They can take into account different specifications. Precisely, we have provided a mode-dependent control law which stabilizes the system jointly in the mean-square and each mode of operation. In application, this property is important. Regarding to a jump nonlinear system, one have to ensure that the plant state of the linearized model will not grow and go outside the domain of validity. Regarding also to a jump linear system which may have a 'relatively long' evolution in a given mode. Another important aspect, is that the approach presented here takes into account uncertainties that can affect the transition probability rates, which are difficult to estimate. This approach can be extended to the case when the modes are uncertain.

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