Quadratic Stabilizability of Uncertain Continuous-time Systems under State and Control Constraints in the Presence of Disturbances

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Abstract

The problem of finding a linear stabilizing state-feedback control for uncertain continuous-time linear systems with state and control constraints is addressed. Both convex-bounded parametric uncertainties and additive bounded disturbances are considered. Based on the theory of ellipsoidal positively invariant sets, a programming approach is proposed which solves at each step a convex optimization problem involving linear matrix inequalities (LMI's).

Key Words: Uncertain systems, state and control constraints, positive invariance, additive disturbance.

1 Introduction

Many important control problems can be reduced to the problem of finding a stabilizing controller capable of achieving acceptable performances under system uncertainty and design constraints. In the last two decades, several approaches have been proposed for designing robust controllers for the systems subject to structured and/or unstructured uncertainties. A survey of these approaches can be found in [1]. Recently, a large amount of effort has been devoted to the constrained control problem solved by means of the theory of positively invariant sets. The polyhedral positively invariant sets have been studied in [2], [15] and [16]. The ellipsoidal positively invariant sets have been analysed in [7] and [8].

Most of realistic control problems involve both some type of time-domain constraints and model uncertainty [14]. However, in the current literature only a few results are available for the robust constrained control problem. Previous researches into this problem include [4], [9], [13] and [14]. Taking as base the concept of robustness and adopting as performance specification an $\mathcal{H}_\infty$-norm bound, the control synthesis problem has been addressed in [13] for discrete-time systems via the solution of a convex optimization problem. They have considered unstructured uncertainties and only state constraints. Using the same robustness measure, a convex programming approach has been proposed in [14] to solve the control synthesis problem, handling structured model uncertainties and both state and control constraints. The model uncertainties were restricted to the dynamic matrix A and the state constraints involved ellipsoidal and polyhedral sets.

The theory of polyhedral positively invariant sets has been used in [4] to determine a state-feedback control for discrete-time systems subject to structured parametric model uncertainties and polyhedral constraints on both state and control vectors. The results of this theory can be applied to the case of interval matrix uncertainties acting only in the dynamic matrix A of the system. An extension of their results to accommodate convex-bounded uncertainties, including parametric uncertainties in the input matrix B, can be found in [9]. In these two references the robust constrained control problem is reduced to a linear programming problem.

In [5] the state-feedback control design problem for linear systems subject to polyhedral state and control constraints and additive disturbances has been considered. In this work, the important concept of positive D-invariance has been introduced and a linear programming approach has been proposed for both discrete and continuous-time systems with disturbances bounded in a polyhedral set. However, this approach does not take into account parametric uncertainties.

The present paper deals with the linear robust constrained control synthesis problem for continuous-time systems with state and control constraints using the theory of ellipsoidal positively invariant sets. Both the convex-bounded parametric uncertainties and the additive bounded disturbances are studied.

The structure of the paper is as follows. Section 2 presents the preliminary assumptions with respect to the description of the system and the formulation of the linear robust constrained control synthesis problem. In section 3, a sufficient condition for the existence of solutions to this problem is given. A programming procedure for determining a solution is then proposed in section 4. An illustrative example is presented in section 5.

2 Problem Statement

Consider an uncertain continuous-time linear system described by the following state-space equation:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \qquad (1)$$

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where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector and $w(t) \in \Re^l$ is the disturbance vector. A, B_1, and B_2 are real matrices of appropriate dimensions. Assume that the matrices A and B_2 belong to the convex domains defined as

$$\mathcal{D}_A = \left\{ A \in \Re^{n\times n}; A = \sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\}, \tag{2}$$

$$\mathcal{D}_B = \left\{ B_2 \in \Re^{n\times m}; B_2 = \sum_{j=1}^{M} \beta_j B_{2j}, \sum_{j=1}^{M} \beta_j = 1, \beta_j \geq 0 \right\} \tag{3}$$

All pairs (A, B_2) are assumed to be stabilizable.

Furthermore, suppose that the state and control vectors are subject to physical constraints. The set of admissible states is given by the bounded polyhedron

$$D(g,\rho) = \left\{ x \in \Re^n; {g_i}^T x \leq \rho_i,\, \rho_i > 0,\, i = 1, 2, \cdots, r \right\}, \tag{4}$$

where $g_i \in \Re^n$, $g_i \neq 0$, $i = 1, 2, \cdots, r$. By definition, the set $D(g, \rho)$ contains the origin in its interior.

The control vector $u(t)$ is constrained to belong to a polyhedral set defined by:

$$D(h,\mu) = \left\{ u \in \Re^m;\, {h_i}^T u \leq \mu_i,\, \mu_i > 0,\, i = 1, 2, \cdots, p \right\}, \tag{5}$$

where $h_i \in \Re^m$, $h_i \neq 0$, $i = 1, 2, \cdots, p$.

Let us also consider a bounded polyhedral set of admissible initial states $x_0 = x(t_0)$:

$$D(g_0, \rho_0) = \left\{ x_0 \in \Re^n; {g_{0i}}^T x_0 \leq \rho_{0i}, i = 1, 2, \cdots, s \right\}, \tag{6}$$

where $g_{0i} \in \Re^n$, $g_{0i} \neq 0$, $\rho_{0i} > 0$, $i = 1, 2, \cdots, s$.

Finally, suppose that the disturbance vector belongs to the following set:

$$D = \left\{ w \in \Re^l;\, \|w\| \leq \gamma \right\}, \tag{7}$$

where $\|.\|$ denotes the Euclidean norm. Thus, the disturbance $w(t)$ is constrained in a hypersphere of radius γ.

The investigated state-feedback control law is given by

$$u(t) = Kx(t), \tag{8}$$

where $K \in \Re^{m\times n}$. From (5), the set $D(K, h, \mu)$ defined by

$$D(K,h,\mu) = \left\{ x \in \Re^n; {h_i}^T K x \leq \mu_i, i = 1, 2, \cdots, p \right\} \tag{9}$$

is the region in which control saturation does not occur. Hence, from (4) and (9), it is worth noticing that the resulting linear closed-loop system described by

$$\dot{x}(t) \;=\; A_f x(t) + B_1 w(t), \tag{10}$$

where $A_f = A + B_2 K, \forall A \in \mathcal{D}_A, \forall B_2 \in \mathcal{D}_B$, is valid only for the states belonging to $D(g,\rho) \cap D(K,h,\mu)$.

Now we can define the linear robust constrained control synthesis problem.

Problem 1 *Determine a linear state-feedback control $K \in \Re^{m\times n}$ so that for all initial conditions $x_0 \in D(g_0, \rho_0)$ the resulting closed-loop system defined in (10) satisfies the following specifications:*

i) *the constraints (4) and (9) are respected for any admissible disturbance $w \in D$;*

ii) *for $w = 0$, the system is asymptotically stable,* i.e., *every admissible initial state x_0 is transferred to the origin asymptotically.*

Note that the state-feedback gain K is a solution to Problem 1 if and only if the closed-loop system defined in (10) is asymptotically stable without disturbances ($w = 0$) and if no trajectory $x(t; x_0)$ emanating from the region $D(g_0, \rho_0)$ leaves the regions $D(g, \rho)$ and $D(K, h, \mu)$ for any $w \in D$ and $t \geq t_0$.

3 Main Results

The following definition [5] will be useful for establishing some of the results in this paper.

Definition 3.1 *Let D be a compact and convex set containing the origin and let Ω be a non-empty set. Ω is said to be a positively D-invariant set with respect to the system (10) if for every initial state $x(t_0) \in \Omega$ and every disturbance sequence $w(t) \in D$, $t \geq t_0$, $x(t) \in \Omega$ for all $t \geq t_0$.*

The next proposition follows from Definition 3.1.

Proposition 3.1 *The state-feedback gain K is a solution to Problem 1 if and only if $(A+B_2K)$ is asymptotically stable, $\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$, and there exists a positively D-invariant set $\Omega \subseteq \Re^n$ with respect to the closed-loop system (10) such that*

$$D(g_0, \rho_0) \subseteq \Omega \subseteq D(g, \rho), \tag{11}$$

$$\Omega \subseteq D(K, h, \mu). \tag{12}$$

Proof: *Necessity.* Let K be a solution to Problem 1. Define Ω as the set of reachable states $x(t; x_0; w)$ for the uncertain closed-loop system (10), $x_0 \in D(g_0, \rho_0)$ and $w \in D$. First, let us prove that Ω is a positively D-invariant set with respect to the system (10). For any $\overline{x} \in \Omega$, there exist a $x_0 \in D(g_0, \rho_0)$ and a sequence $w(t) \in D$, $t \geq t_0$, such that $\overline{x} = x(t; x_0; w)$. Thus, for every initial state $\overline{x} \in \Omega$ and every sequence $w(t) \in D$, $t \geq \overline{t}$, $x = x(t; \overline{x}; w) \in \Omega$ for all $t \geq \overline{t}$. Consequently, Ω is a positively D-invariant set with respect to the system (10). By definition, $D(g_0, \rho_0)$ is contained in Ω. Furthermore, if there exist a pair (x_0, w), $x_0 \in D(g_0, \rho_0)$, $w \in D$, and a $t \geq t_0$ such that $x(t; x_0; w)$ does not satisfy the constraint (4) or (9), then K is not a solution to Problem 1, because the condition *i)* is violated. Hence, the hypothesis

$\Omega \subseteq D(g,\rho)$ and $\Omega \subseteq D(K,h,\mu)$ is a necessary condition as well. Finally, to satisfy the condition *ii)* of Theorem 1, it is also necessary that $(A + B_2K)$ is asymptotically stable, $\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$.

Sufficiency. The proof of sufficiency is obvious and may be omitted here.

It is well-known that the Lyapunov functions generate positively invariant sets for asymptotically stable systems. In this paper, we are interested in ellipsoidal positively invariant sets generated by quadratic Lyapunov functions of the type $v(x) = x^T Px$, where $P = P^T > 0$ (symmetric positive definite). Thus, consider the ellipsoidal set Ω defined as follows:

$$\Omega = \left\{x \in \Re^n; \quad x^T Px \leq 1, P = P^T > 0\right\}, \tag{13}$$

where $P \in \Re^{n\times n}$.

In order to find a state-feedback gain K, solution to Problem 1, we shall establish the conditions that guarantee the asymptotic stability of the system (10), with $w = 0$, and positive D-invariance of the set Ω defined in (13), with respect to the system (10).

Lemma 3.1 *If there exist $W_1 = W_1{}^T > 0$ and $W_2 \in \Re^{m\times n}$ satisfying the following inequalities:*

$$A_iW_1 + W_1A_i{}^T + B_{2j}W_2 + W_2^T B_{2j}{}^T < 0 \quad ,\forall(A_i, B_{2j}), \tag{14}$$

where $W_1 = P^{-1}$, then the ellipsoid Ω defined in (13) is an asymptotic stability region for the system (10) with $w = 0$. The state-feedback gain K is recovered as

$$K = W_2W_1{}^{-1}. \tag{15}$$

Proof: See [3] and also [6].

Before studying the positive D-invariance of the ellipsoidal set Ω defined in (13), we will present some preliminary results concerning the conditions (11) and (12). First rewrite the set Ω as:

$$\Omega = \left\{x \in \Re^n; \, x^T W_1{}^{-1}x \leq 1, \quad W_1 = W_1{}^T > 0\right\}, \tag{16}$$

where $W_1 = P^{-1}$.

Lemma 3.2 *Consider the ellipsoid Ω and the convex polytope $D(g,\rho)$ defined in (16) and (4) respectively. The ellipsoid Ω is contained in the polytope $D(g,\rho)$ if and only if*

$$g_i^T W_1 g_i \leq \rho_i{}^2, \qquad , i = 1, 2, \cdots, r. \tag{17}$$

Proof: The proof of this result is based on geometric considerations [10] and will be omitted here.

Lemma 3.3 *Consider the ellipsoid Ω defined in (16). Let $v_i \in \Re^n$, $i = 1, 2, \cdots, s$, denote the vertices of the convex polytope $D(g_0, \rho_0)$ defined in (6). The ellipsoid Ω contains the polytope $D(g_0, \rho_0)$ if and only if*

$$v_i{}^T W_1{}^{-1}v_i \leq 1, \qquad , i = 1, 2, \cdots, s. \tag{18}$$

Proof: The proof of this result is also based on geometric considerations and will be omitted as well.

Observe that the inequalities (18) can be expressed as LMI's in W_1 [6]

$$\begin{bmatrix} 1 & v_i{}^T \\ v_i & W_1 \end{bmatrix} \geq 0, \qquad , i = 1, 2, \cdots, s. \tag{19}$$

Hence, if the conditions (17) and (19) hold, the state-feedback gain $K = W_2W_1{}^{-1}$ satisfies the state constraints, *i.e.*, $D(g_0, \rho_0) \subseteq \Omega \subseteq D(g,\rho)$.

The following lemma provides a necessary and sufficient condition that guarantees the linear behavior of the control law $u = Kx$.

Lemma 3.4 *Consider the ellipsoidal set Ω and the region $D(K,h,\mu)$, defined in (16) and (9) respectively. The state-feedback gain $K = W_2W_1{}^{-1}$ satisfies the control constraints,* i.e., *$\Omega \subseteq D(K,h,\mu)$ if and only if*

$$\begin{bmatrix} \mu_i{}^2 & h_i{}^T W_2 \\ W_2{}^T h_i & W_1 \end{bmatrix} \geq 0 \quad , i = 1, 2, \cdots, p, \tag{20}$$

which are LMI's in W_1 and W_2.

Proof: From Lemma 3.2, the ellipsoid Ω is contained in the region $D(K,h,\mu)$ if and only if

$$(K^T h_i)^T W_1 K^T h_i \leq \mu_i{}^2 \quad , i = 1, 2, \cdots, p. \tag{21}$$

Substituting $K = W_2W_1{}^{-1}$ for (21) yields

$$\mu_i{}^2 - h_i^T W_2 W_1{}^{-1} W_2{}^T h_i \geq 0 \quad , i = 1, 2, \cdots, p. \tag{22}$$

Using Schur complements, we conclude that the inequalities (22) are equivalent to the following LMI's:

$$\begin{bmatrix} \mu_i{}^2 & h_i{}^T W_2 \\ W_2{}^T h_i & W_1 \end{bmatrix} \geq 0 \quad , i = 1, 2, \cdots, p. \tag{23}$$

The next result presents a sufficient condition concerning the positive D-invariance.

Lemma 3.5 *Consider the sets Ω and D defined respectively in (16) and (7). Let $\gamma > 0$ be given. If there exist $\alpha \geq 0$, $W_1 = W_1{}^T$ and W_2 satisfying*

$$\begin{bmatrix} A_iW_1 + W_1A_i{}^T + B_{2j}W_2 + W_2{}^T B_{2j}{}^T + \alpha W_1 & \gamma B_1 \\ \gamma B_1{}^T & -\alpha I \end{bmatrix} \leq 0, \tag{24}$$

$i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$, then for $K = W_2W_1{}^{-1}$, Ω is a positively D-invariant set with respect to the closed-loop system (10).

Proof: Let $v(x) = x^T W_1^{-1}x$ be the Lyapunov function associated with the closed-loop system (10). The time derivative of $v(x)$ along any trajectory of the system (10) is given by

$$\dot{v}(x) = x^T \left(A_f^T W_1^{-1} + W_1^{-1}A_f\right) x + x^T W_1^{-1} B_1 w + w^T B_1^T W_1^{-1} x. \tag{25}$$

To prove that Ω is a positively D-invariant set with respect to system (10), it suffices to prove that $\dot{v}(x) \leq$

0 for all x belonging to the boundary of Ω, that is, satisfying $x^T W_1^{-1} x = 1$, and for all admissible w. Note that we suppose all admissible initial states are contained in the ellipsoid Ω. Thus, it can be shown that if $\dot{v}(x) \leq 0$ for all x and w such that $v(x) \geq 1$ and $w \in D$, then Ω is a positively D-invariant set with respect to the system (10).

Hence, if for any pair (x, w) satisfying $x^T W_1^{-1} x \geq 1$ and $w^T w \leq \gamma^2$ the inequality

$$x^T \left(A_f^T W_1^{-1} + W_1^{-1} A_f\right) x + x^T W_1^{-1} B_1 w + w^T B_1^T W_1^{-1} x \leq 0 \tag{26}$$

holds, then Ω is a positively D-invariant set.

Using the S-procedure [17], this condition can be replaced by another one without constraints: if there exist scalars $\alpha \geq 0$ and $\beta \geq 0$, for all pair (x, w), such that

$$x^T \left(A_f^T W_1^{-1} + W_1^{-1} A_f\right) x + x^T W_1^{-1} B_1 w + w^T B_1^T W_1^{-1} x + \\ +\alpha(x^T W_1^{-1} x - 1) + \beta(\gamma^2 - w^T w) \leq 0, \tag{27}$$

then Ω is a positively D-invariant set. Considering the transformation $z = W_1^{-1} x$, the inequality (27) is equivalent to

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A_f W_1 + W_1 A_f^T + \alpha W_1 & B_1 \\ B_1^T & -\beta I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \\ +\beta\gamma^2 - \alpha \leq 0. \tag{28}$$

If we consider $\beta\gamma^2 = \alpha$, the following inequality can be obtained from (28):

$$\begin{bmatrix} A_f W_1 + W_1 A_f^T + \alpha W_1 & \gamma B_1 \\ \gamma B_1^T & -\alpha I \end{bmatrix} \leq 0. \tag{29}$$

Using the change of variables $W_2 = K W_1$, and taking into account the convexity properties of the uncertainties of the system (10), the inequality (29) is equivalent to

$$\begin{bmatrix} A_i W_1 + W_1 A_i{}^T + B_{2j} W_2 + W_2{}^T B_{2j}{}^T + \alpha W_1 & \gamma B_1 \\ \gamma B_1{}^T & -\alpha I \end{bmatrix} \leq 0, \tag{30}$$

$i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$.

Thus, if there exists $\alpha \geq 0$ satisfying the inequalities (30), then the state-feedback gain $K = W_2 W_1{}^{-1}$ guarantees the positive D-invariance of the set Ω with respect to the closed-loop system (10).

Remark 3.1 *Consider the inequalities in (24). Note that for $\alpha > 0$ these inequalities contain the stability condition defined in (14).*

The next theorem establishes a sufficient condition for the existence of solutions to Problem 1.

Theorem 3.1 *Consider the set D and the uncertain closed-loop system defined respectively in (7) and (10). Let $\gamma > 0$ be given. If there exist $\alpha > 0$, $W_1 = W_1^T > 0$ and W_2 such that*

$$g_i{}^T W_1 g_i \leq \rho_i^2 \qquad , i = 1, 2, \cdots, r, \tag{31}$$

$$\begin{bmatrix} 1 & v_i{}^T \\ v_i & W_1 \end{bmatrix} \geq 0 \qquad , i = 1, 2, \cdots, s, \tag{32}$$

$$\begin{bmatrix} \mu_i{}^2 & h_i{}^T W_2 \\ W_2{}^T h_i & W_1 \end{bmatrix} \geq 0 \quad , i = 1, 2, \cdots, p, \tag{33}$$

$$\begin{bmatrix} A_i W_1 + W_1 A_i{}^T + B_{2j} W_2 + W_2{}^T B_{2j}{}^T + \alpha W_1 & \gamma B_1 \\ \gamma B_1{}^T & -\alpha I \end{bmatrix} \leq 0, \tag{34}$$

$i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$, then the state-feedback gain $K = W_2 W_1{}^{-1}$ is a solution to Problem 1 and the suitable set Ω is obtained by the matrix W_1 as defined in (16).

Proof: This result follows directly from Lemmas 3.2, 3.3, 3.4 and 3.5.

4 Synthesis Algorithm

Now, a programming approach is proposed for solving Problem 1. More specifically, given the uncertain closed-loop system defined in (10) and an initial disturbance region defined as

$$D_0 = \left\{w \in \Re^l; \|w\| \leq \gamma_0\right\}, \tag{35}$$

a stabilizing state-feedback gain K and a scalar $\alpha > 0$ are determined in such a way that the state and control constraints are respected for all admissible initial states x_0.

A state-feedback gain K solution to Problem 1 can be found quite simply by testing the feasibility of the set of LMI's given in Theorem 3.1, for different α's. In case several solutions are possible, a synthesis algorithm is proposed to find a gain K, based on the D-K iteration procedure [12]. Before presenting this algorithm, two programming problems shall be announced.

From Theorem 3.1, for a feasible initial α, we can determine a gain K solving the following convex programming problem:

$$\underset{W_1 > 0, W_2, \gamma > 0}{\mathit{minimize}} \quad -\gamma \tag{36}$$

subject to constraints (31)-(34). The gain K that maximizes the disturbance region is recovered as $K = W_2 W_1{}^{-1}$ and the ellipsoidal positively D-invariant set Ω is obtained by the matrix W_1.

Observe that for fixed matrices W_1 and W_2, the inequalities (34) are LMI's in α and γ. Hence, for a stabilizing pair (K, W_1) which satisfies the state and control constraints, γ can be determined by solving the convex programming problem:

$$\underset{\alpha > 0, \gamma > 0}{\mathit{minimize}} \quad -\gamma \tag{37}$$

subject to constraints (34).

Finally, the algorithm proposed for solving Problem 1, with a maximization of the disturbance region, can be stated as follows:

Algorithm

S 1: Choose a feasible initial α for the problem (36);

S 2: For previous α, find $W1$, $W2$ and γ solving the minimization problem (36);

S 3: For previous $W1$ and $W2$, find $\hat{\alpha}$ and $\hat{\gamma}$ solving the minimization problem (37);

S 4: If γ and $\hat{\gamma}$ are close, stop; Otherwise, replace α with $\hat{\alpha}$ and go to step 2.

Several remarks should be made here. First of all, this sequence of minimizations is not guaranteed to converge to the minimum. Nevertheless, this procedure offers a reliable approach for solving Problem 1, since both of the individual problems are handled with efficiency. Moreover, note that to solve Problem 1 it is sufficient to find a scalar γ such that $\gamma \geq \gamma_0$.

Secondly, from the inequalities (34) we can conclude that the closed-loop poles of the uncertain system (10) are placed on the left of the vertical line $-\frac{\alpha}{2}$.

5 Numerical Example

Consider the following uncertain continuous-time system [11]

$$\dot{x}(t) \;=\; Ax(t) + B_1 w(t) + B_2 u(t),$$

where the matrices A and B_2 belong to the convex polytopes D_A and D_B whose vertices are given by

$$A_1 = \begin{bmatrix} -0.9896 & 17.4100 & 96.1500 \\ 0.2648 & -0.8512 & -11.3900 \\ 0 & 0 & -250.0000 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -0.9896 & 17.4100 & 96.1500 \\ 0.0820 & -0.6586 & -10.8100 \\ 0 & 0 & -250.0000 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} -0.6606 & 18.1100 & 84.3400 \\ 0.2648 & -0.8512 & -11.3900 \\ 0 & 0 & -250.0000 \end{bmatrix};$$

$$A_4 = \begin{bmatrix} -0.6606 & 18.1100 & 84.3400 \\ 0.0820 & -0.6586 & -10.8100 \\ 0 & 0 & -250.0000 \end{bmatrix};$$

$$B_{21} = \begin{bmatrix} -97.7798 \\ 0 \\ 250.0000 \end{bmatrix}; \quad B_{22} = \begin{bmatrix} -85.1002 \\ 0 \\ 250.0000 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The set of admissible states is described by the bounded polyhedron $D(G,\rho) = \{x \in \Re^3; Gx \leq \rho\}$ with

$$G = \begin{bmatrix} 0.1 & 0 & 0 \\ -0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & -0.1 \end{bmatrix}; \quad \rho = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The set of initial states x_0 is given by the convex hull of the following vertices:

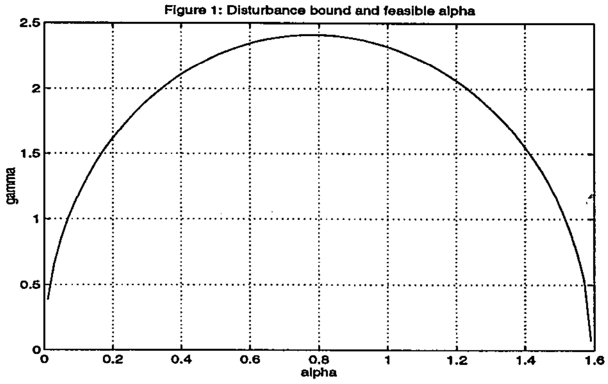
$$v_1 = \begin{bmatrix} 2.5 \\ 0 \\ 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} -2.5 \\ 0 \\ 0 \end{bmatrix}; \quad v_3 = \begin{bmatrix} 0 \\ 2.5 \\ 0 \end{bmatrix};$$

$$v_4 = \begin{bmatrix} 0 \\ -2.5 \\ 0 \end{bmatrix}; \quad v_5 = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix}; \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ -2.5 \end{bmatrix}.$$

The control $u(t)$ is subject to the constraints $-1 \leq u(t) \leq 1$, and the disturbance $w(t)$ is contained in a sphere of radius $\gamma_0 = 1$.

The optimal radius $\gamma_{opt} = 2.411$, corresponding to $\alpha = 0.760$, was achieved after just eight iterations, for the initial $\alpha_0 = 0.1$. Figure 1 shows the feasibility interval for the parameter α and the maximum associated γ.

Figure 1: Disturbance bound and feasible alpha



The state-feedback gain K associated with γ_{opt} is given by:

$$K = \begin{bmatrix} 0.0948 & 0.3083 & 0.0224 \end{bmatrix}.$$

The positively D-invariant set Ω is determined by:

$$W_1^{-1} = \begin{bmatrix} 0.0120 & 0.0176 & -0.0017 \\ 0.0176 & 0.1600 & -0.0034 \\ -0.0017 & -0.0034 & 0.0373 \end{bmatrix}.$$

6 Conclusion

The linear robust constrained control synthesis problem for convex-bounded uncertain systems with additive bounded disturbances and both state and control constraints has been studied. The approach proposed is based on the concept of ellipsoidal positively

D-invariant sets. In other words, the positive invariance of an ellipsoidal set is guaranteed in spite of the additive disturbances acting on the system. The disturbances are supposed to be contained in a hypersphere.

Firstly, a sufficient condition for the existence of solutions to this synthesis problem has been established. This sufficient condition has been obtained in terms of the feasibility of a non-linear problem. Then, based on this result, an algorithm has been proposed to find a linear stabilizing state-feedback gain via the determination of an ellipsoidal positively D-invariant region satisfying the state and control constraints.

At each step of this algorithm a convex programming problem is solved. An optimal solution is not assured by this two-step procedure, however to obtain a solution to our problem it is sufficient to find a disturbance region that contains the initial one. The convex problems can be solved by various effective methods available in the literature, for instance, the interior-point methods.

Finally, due to the convexity properties of this approach, additional performance constraints could be incorporated to the problem, for example, the performance requirements which can be reached by the pole placement. The results of this paper can also be readily extended to the discrete-time case.

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