Dynamic Control of a Kanban System in Dioid Algebra

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Abstract

After recalling kanban system model in dioid algebra , we propose dynamic controls of such systems in order to both reduce work in process and keep the same performance as the classical kanban system.

1 Introduction

The main objective of the Just In Time (JIT) manufacturing approach is both to limit the stock and to meet the customer's demand, i.e., to compute as close as possible the latest control dates of the process input as the output part is obtained (at the latest) before the customer's desired dates. The optimal control has been given when the manufacturing process can be represented as a Timed Event Graph (TEG) (a subclass of Petri nets of which each place admits one and only one transition upstream and downstream), and under the following assumptions : 1) the customers demand is initially known (all the reference input is used to compute the optimal control) 2) the process is perfectly modelized and is not perturbated by some exogenous events (fault parts, machine failures). Previously, Toyota had organized its manufacturing system in production stage with a kanban method, which many searchers have studied [3], [8],[9]. The idea behind this method is the following. A production line is divided into several stages and at every stage there is a fixed number of tags (tickets) called kanban. An arriving job receives a kanban at the entrance of the stage and keeps it until it exits the stage. If an arriving job does not find an available kanban at the entrance, it is forced to wait in the previous stage until a kanban is freed.
The success of this just in time manufacturing approach is certainly due to the following characteristics: 1) the part number in each stage is bounded by the kanban number, denoted K. 2) for each stage an output buffer (just downstream the machines) is able to admit a number of finished parts (FP). This allows to minimize the influence of exogenous events (machine failure, fault parts). Moreover, if the output buffer is full (FP=K) a customer's demand of K parts can be instantaneously satisfied. The aim of many works about this method is to establish the number of kanbans in order to guarantee the customer's demand in spite of stochastic perturbations [9]. Another searcher objective is to obtain the optimal kanban number which respects the usual trade-off between the greatest production rate and the least work-in-process (WIP) [8].
In this paper kanban system is studied in a deterministic case by using the dioid $\mathcal{M}_{in}^{ax}[[\gamma,\delta]]$ (see [4]).
Afterward we establish a control strategy in order to both compute the latest input dates of the production line and to keep the same output behavior as a manufacturing system organized in kanban stages. This first approach is a particular case of the control using a reference model and developed in [5]. The second control strategy is developed in order to both limit the WIP and to keep the same stock evolution of finished parts as a classical kanban stage.

2 TEG Representation in Dioid Algebra

2.1 Dioid Algebra

Definition 1 (Dioid) *Dioid $\mathcal{D}$ is a set with two inner operations $(\oplus,\otimes)$, both associative and both having neutral elements denoted ε and e respectively, such that $\oplus$ is commutative and idempotent $(a\oplus a=a)$, $\otimes$ is distributive with respect to $\oplus$ and ε is absorbing for $\otimes$.*

Definition 2 (Natural Order) *Dioid can be endowed with a natural order $a\oplus b=a \Leftrightarrow a\succeq b$. $a\oplus b$ is the least upper bound of a and b.*

Definition 3 (Complete Dioid) *A dioid is complete iff it is closed for infinite sums and if $\otimes$ distributes over infinite sums too. In particular, the sum of all the elements of the dioid is denoted T.*

Example 1 ($\overline{\mathbb{Z}}_{min}$, $\overline{\mathbb{Z}}_{max}$) *The set $\mathbb{Z}\cup\{+\infty\}$ endowed with the* min *operator as $\oplus$ and the classical addition as $\otimes$ is a dioid called $\mathbb{Z}_{min}$. Its neutral elements are $\varepsilon=+\infty$ and $e=0$. Moreover, by considering the set $\mathbb{Z}\cup\{-\infty,+\infty\}$ the dioid becomes $\overline{\mathbb{Z}}_{min}$ (with $T=-\infty$). The dual Dioid $\overline{\mathbb{Z}}_{max}$ is the set $\mathbb{Z}\cup\{-\infty,+\infty\}$, with* max *as $\oplus$, $+$ as $\otimes$, $\varepsilon=-\infty$, $e=0$ and $T=+\infty$.*

Definition 4 (Lower bound) *In a complete dioid, the greatest lower bound always exists for an arbitrary* *set S. In particular, if $S = (a, b)$, this bound is denoted $a \wedge b = \bigoplus_{x \preceq a,\, x \preceq b} x$. One obtains the following equivalences $a \succeq b \Leftrightarrow a \oplus b = a \Leftrightarrow a \wedge b = b$.*

Definition 5 (Residuation) *A mapping $f : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are ordered sets, is residuated if for all $y \in \mathcal{D}$, the least upper bound of the subset $\{x \in \mathcal{C} | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted $f^\sharp(y)$. The mapping $f^\sharp : \mathcal{D} \to \mathcal{C}$ is called the residual of f. By definition, $f[f^\sharp(y)] \preceq y$, $\forall y \in \mathcal{D}$.*

Example 2 *The mapping $f : \begin{array}{c} \mathcal{D} \to \mathcal{D} \\ x \to a \otimes x \end{array}$, where $\mathcal{D}$ is a complete dioid, is residuated. Its residual will be denoted $f^\sharp(y) = a \setminus y$.*

Property 2.1.1 *Let $a, b, x \in \mathcal{D}$, the following relation is always verified*

$$a \setminus (xb) \succeq (a \setminus x)b \qquad (1)$$

Proof: see [1] section 4.4.4.

Theorem 1 *The equation $x = ax \oplus b$ admits a least solution $x = a^*b$ where the Kleene star operator is defined as : $a^* = \bigoplus_{k \geq 0} a^k$.*

Definition 6 (Dioid modulo z) *Let $a, b, x \in \mathcal{D}$ a commutative dioid. a and b are said to be equivalent modulo z (denoted $a \equiv b(mod\ z)$) iff $az^* = bz^*$. Let $[a] = \{x |\ x \equiv a(mod\ z)\}$ denote the equivalence class of a according to equivalence relation mod z. Let us denote $\mathcal{D}_{/modz}$ the quotient dioid $\mathcal{D}$, i.e., the set of equivalence class modulo z of $\mathcal{D}$ elements.*

2.2 TEG Modeling in Dioid Algebra

Baccelli *et al.* [1], Cohen *et al.* [2] have shown that a TEG can be represented by linear equations over dioid algebra. To obtain a linear model, we can associate with the transition labelled x_i either a counter function which is a map $\mathbb{Z} \to \overline{\mathbb{Z}}_{min}$, $t \to x_i(t)$ defined by: $x_i(t) = k \iff$ the firing number of x_i at or after the time t is k, or dually a dater function which is a map $\mathbb{Z} \to \overline{\mathbb{Z}}_{max}$, $k \to x_i(k)$ defined by: $x_i(k) = t \iff$ the firing numbered k of x_i occurs at time t. Then, a relation between counter or dater functions associated to TEG transitions may be established. For example, according to *fig. 1*: $x_2(t) = 2 \otimes x_1(t-1)$. Other representations may also be obtained by introducing shift operators γ and δ where γ is a backward shift operator in event domain (formally, $\gamma x(k) = x(k-1)$) and δ is a backward shift operator in time domain (formally, $\delta x(t) = x(t-1)$). Then, an input-output representation may be given in dioid $\mathbb{Z}_{min}[[\delta]]$ (resp. $\mathbb{Z}_{max}[[\gamma]]$) of formal power

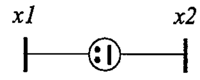

Figure 1: Example of elementary TEG

series in δ (resp. γ) with exponents in $\mathbb{Z}$ and coefficients in $\overline{\mathbb{Z}}_{min}$ (resp. $\overline{\mathbb{Z}}_{max}$). TEG behavior is naturally described by non decreasing trajectories which leads us to establish simplification rules for shift operators manipulation :

$$\gamma^k \oplus \gamma^l = \gamma^{min(k,l)} \quad \delta^t \oplus \delta^s = \delta^{max(t,s)} \qquad (2)$$

Finally, a two dimensional domain representation manipulating power series in both γ and δ is obtained. Let $\mathbb{B}[[\gamma, \delta]]$ be the dioid of formal power series in γ and δ with boolean coefficients and exponents in $\mathbb{Z}$ (an element may be written as $s = \bigoplus_{i,j \in \mathbb{Z}} s(i,j)\gamma^i\delta^j$, $s(i,j) \in \{\varepsilon, e\}$). By adding the simplification rules (2), a new dioid called $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ is defined and corresponds to the quotient dioid $\mathbb{B}[[\gamma, \delta]]_{/mod(\gamma \oplus \delta^{-1})}$, i.e., each element of $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ represents an equivalence class modulo $(\gamma \oplus \delta^{-1})$, see [2] for an exhaustive presentation. Then, we can associate with each transition of a TEG an element x of $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ which codes the set of information available about the sequence of events related to this transition. For example, considering *fig. 1*, one obtains

$$x_2 = \gamma^2 \delta x_1$$

More generally, a TEG yields standard equations in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ of the form

$$\begin{cases} X = AX \oplus BU \\ Y = CX \oplus DU \end{cases}$$

where X, U and Y are vectors of elements in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ representing respectively internal, inputs and outputs transitions of TEG. Considering state representation previously exposed and resolving first state equation according to *Theorem 1*, we obtain $X = A^*BU$ and therefore, $Y = (CA^*B \oplus D)U$. Then $H = CA^*B \oplus D$ characterizes the transfer relation between TEG input and output (Let us note that each entry of H is a series s). In order to interpret the transfer matrix H, we recall that a counter function is associated with a series s and is the unique non decreasing function $\mathcal{C}_s$ such that $s = \bigoplus_{t \in \mathbb{Z}} \gamma^{\mathcal{C}_s(t)}\delta^t$ (see [7]).

2.3 Rational Computation in Dioid Algebra

The power series which arise in the transfer matrix H are rational. Let us recall some properties about rationals.

Definition 7 (Rational closure) *The rational closure, denoted $\mathcal{E}^*$, of a subset $\mathcal{E}$ of a complete dioid $\mathcal{D}$ is the smallest subdioid $\mathcal{F}$ such that $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}$ is rationally stable (i.e., stable for the operators $(\oplus, \otimes, ^*)$).*

Definition 8 (Rational series) *An element of $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ is a rational series if it belongs to the rational closure of $\mathcal{R} := \{e, \varepsilon, \gamma, \delta\}$.*

Theorem 2 *A series is rational iff it is periodic. (partial result of theorem 5.39. of [1])*

Definition 9 (Periodic series) *A series of $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ is periodic iff it can be written as $s = p \oplus qr^*$ where $p = \bigoplus_{i=0}^{\kappa} \gamma^{n_i}\delta^{t_i}$ and $q = \bigoplus_{j=0}^{\theta} \gamma^{N_j}\delta^{T_j}$ are polynomials and $r = \gamma^{\nu}\delta^{\tau}$ is monomial. The ratio ν/τ is called the production rate of s.*

Definition 10 (Causality) *A series s of $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ is causal either if $s = \varepsilon$ or if $val(s) \geq 0$ and $s \succeq \gamma^{val(s)}$, with $val(s)$ the valuation in γ of any representative of s, i.e., the lower bound of $\{i \in \mathbb{Z} \mid \forall j \in \mathbb{Z}, s(i,j) \neq \varepsilon\}$*

Theorem 3 (Operations over periodic series) *$\oplus, \otimes, \wedge$ and Residuation of periodic series are periodic series.*

See [4],[5],[6] where the algebra of periodic series is investigated.

3 Kanban System Modelized in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$

3.1 Kanban Model

A kanban production line is constituted of multiple elementary stages like the one represented by the TEG of *fig. 2.* where K_i,n_i,t_i are respectively the kanbans

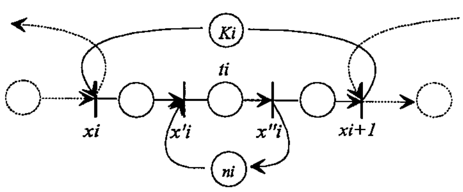

Figure 2: TEG of kanban stage i

number, machines number and the processing times in the stage number i.

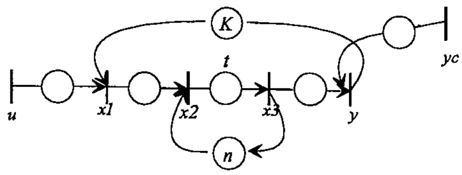

Figure 3: TEG of one kanban stage

To simplify let us first consider the case of one kanban stage of production line, its TEG is represented by *fig. 3*.

Input u designs the stock arrival of unprocessed parts and the input y_c describes the customer's request which can be given from the downstream stage in case of multiple stages as it is suggested in dotted lines *fig. 2*. In $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$, the TEG of a single stage leads to the following state model (see [4] for an exhaustive development).

$$\left\{ \begin{array}{ll} X & = \begin{bmatrix} \varepsilon & \varepsilon & \gamma^K \\ e & \varepsilon & \gamma^n \\ \varepsilon & \delta^t & \varepsilon \end{bmatrix} X \oplus \begin{bmatrix} \gamma^K & e \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \begin{bmatrix} y_c \\ u \end{bmatrix} \\ y & = \begin{bmatrix} \varepsilon & \varepsilon & e \end{bmatrix} X \oplus \begin{bmatrix} e & \varepsilon \end{bmatrix} \begin{bmatrix} y_c \\ u \end{bmatrix} \end{array} \right.$$

By using theorem 1 the transfer relation between inputs u, y_c and output y corresponds to

$$y = \left(e \oplus \gamma^K \delta^t (\gamma^{min(K,n)} \delta^t)^*\right) y_c \oplus \delta^t (\gamma^{min(K,n)} \delta^t)^* u \tag{3}$$

Remark: According to production rate definition (*Def. 9*), transfer relation (3) shows that only n kanbans are necessary to have a production rate equal to n/t. Later on , we assume that $K \geq n$, *i.e*, the stage production rate is not decreased by the kanban policy.

3.2 Internal Behavior of the Stage

Let us note that transition x_1 represents the input of the unprocessed part in the stage and x_3 the input of the finished part in the output buffer of the stage.
The transfer relations between the inputs and internal transitions of a kanban stage, under the assumption proposed in previous remark, are given by

$$\begin{aligned} x_1 &= \gamma^K \left(e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*\right) y_c \qquad (4) \\ &\quad \oplus \left(e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*\right) u \\ x_2 &= \gamma^K (\gamma^n \delta^t)^* y_c \oplus (\gamma^n \delta^t)^* u \\ x_3 &= \gamma^K \delta^t (\gamma^n \delta^t)^* y_c \oplus \delta^t (\gamma^n \delta^t)^* u \qquad (5) \end{aligned}$$

Let us note that the counter function $\mathcal{C}_{x1}(t)$ characterizes the parts number introduced in the stage at or after time t, and that $\mathcal{C}_y(t)$ represents the parts number put out of the stage. By considering stock functions (see [7]), we can introduce the next definitions.

Definition 11 (WIP) *It is the instantaneous number of parts waiting for or in processing in the stage:*

$$\mathcal{S}_{x1,x3}(t) = \mathcal{C}_{x1}(t) - \mathcal{C}_{x3}(t) \ \forall t$$

Definition 12 (FP) *It is the instantaneous parts number in the output buffer, i.e.,*

$$\mathcal{S}_{x3,y}(t) = \mathcal{C}_{x3}(t) - \mathcal{C}_y(t) \ \forall t$$

Definition 13 (Internal stock) *It is the instantaneous number of parts in the stage (WIP+FP), defined as :*

$$\mathcal{S}_{x1,y}(t) = \mathcal{C}_{x1}(t) - \mathcal{C}_y(t) \ \forall t$$

3.3 Kanban Stage Properties

Internal Stability: A kanban stage is internally stable, *i.e.*, for all inputs, the internal stock $\mathcal{S}_{x1,y}(t)$ remains bounded. More precisely, its upper bound is K. In the Petri nets setting, the internal stability is guaranteed if the graph is strongly connected.
Initial internal stock: Under the assumption of $u = \varepsilon = \gamma^{+\infty}\delta^{-\infty}$, *i.e.*, an infinity of parts in the upstream stock since an infinite time, the kanban method allows obtaining initially K parts in the output buffer.
In fig. 4, we present the transfer relation between y_c and y when $K = n$ and $K = n + K'$ respectively. The K' supplementary kanbans do not increase the

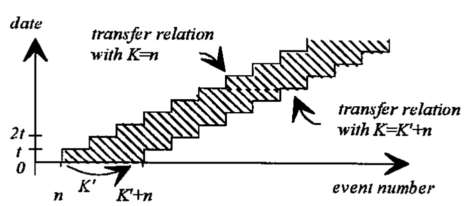

Figure 4: Transfer relation between yc and y of kanban stage with $K = n$ and $K = K' + n$.

production rate of the stage (according to remark in 3.1) and induce a margin of K' parts (see shaded area in *fig. 4*). The beneficial influence of this margin of parts is to allow satisfying more constraining reference inputs or to loose "accidentaly" some tokens (due to exogenous events). The counterpart of this margin is an increase in the internal stock (WIP and FP), since K parts are always present in the stage. This K' number selection is closely linked to the usual trade-off between achievable reference input and stock value.

3.4 Production Line with Multiple Kanban Stages

In [4] the single kanban stage model is extended to a general production line with p kanban stages. The general transfer relation can be described by

$$y = \alpha u \oplus \beta y_c \qquad (6)$$

where y_c is the reference input applied at the last stage, y is the output of the last stage, u is the input transition of the first stage and α, β are periodic series. Indeed, using notation of *fig. 2*, each kanban stage may be described by

$$x_i = \alpha_i u \oplus \beta_i x_{i+1}$$

and the global transfer is given by

$$y = \alpha_{p+1} u \oplus \beta_{p+1} y_c$$

where, α_i and β_i verify the next recursive relations: $\alpha_1 = e$, $\beta_1 = \gamma^{K_1}$, for $1 \leq i \leq p-1$:

$$\begin{cases} \alpha_{i+1} = (\beta_i \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*)^* \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^* \alpha_i \\ \beta_{i+1} = (\beta_i \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*)^* \gamma^{K_{i+1}} \end{cases}$$

$$\alpha_{p+1} = (\beta_p \delta^{t_p}(\gamma^{n_p}\delta^{t_p})^*)^* \delta^{t_p}(\gamma^{n_p}\delta^{t_p})^* \alpha_p,$$
$$\beta_{p+1} = (\beta_p \delta^{t_p}(\gamma^{n_p}\delta^{t_p})^*)^*$$

4 Control Strategy of the Kanban Stage

Under the assumption of an initial stock $u = \varepsilon = \gamma^{+\infty}\delta^{-\infty}$(full stock), we have seen in the previous section that the kanban stage is able to provide $K = (n + K')$ parts initially, i.e., to induce an initial number of parts in the output buffer. This margin can be used :
- to answer to a customer's demand unachievable without this margin;
- to compensate eventual loss of tokens (e.g., fault parts).

4.1 Control to Keep Output Behavior

4.1.1 Control Law

In this first approach, we propose the control which allows to both achieving the same output trajectory y as a classical kanban stage and reducing as far as possible the parts in the stage (*i.e.*, internal stock). This can be formally written as follows:
find the greatest (latest) u such that :

$$y \ = \ [e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] y_c$$

which corresponds to the greatest u verifying:

$$[e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] y_c \oplus \delta^t (\gamma^n \delta^t)^* u = [e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] y_c$$

i.e, owing to *Def. 2*

$$[e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] y_c \succeq \delta^t (\gamma^n \delta^t)^* u \qquad (7)$$

Therefore, according to residuation theory (*Def. 5*), the optimal control is given by

$$u_{opt} = [\delta^t (\gamma^n \delta^t)^*] \setminus [(e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*) y_c] \quad (8)$$

If y_c is completely known, the effective computation of the control u_{opt} can be achieved by considering result given in [1](section 5.6), which is the so-called Backward equations. To avoid this assumption, it is possible to consider the following control which still verifies (7):

$$u_s = [\delta^t (\gamma^n \delta^t)^* \setminus (e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*)] y_c$$

According to property 2.1.1 it is clear that $u_s \preceq u_{opt}$. This transfer relation between u_s and y_c may be analytically expressed as

$$u_s = \delta^{-k_0 t} [e \oplus \gamma^{K - k_0 n + n} \delta^t (\gamma^n \delta^t)^*] y_c \qquad (9)$$

where k_0 is the least integer such as $k_0 n \geq K$.
Proof: The analytical expression (9) can be obtained by considering the two results:

- $\forall k'$ as $k'n \geq K$ we have :

$$[e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] \wedge [\gamma^{-k'n} \delta^{-k't} \oplus \gamma^{K-k'n} \delta^{t-k't} (\gamma^n \delta^t)^*]$$
$$= [\delta^{-(k_0-1)t} \oplus \gamma^{K-(k_0-1)n} \delta^{-(k_0-2)t} (\gamma^n \delta^t)^*]$$

with $k_0 \in \mathbb{N}$ such as $k_0 n \geq K \geq (k_0 - 1)n$.
This first result is obtained by considering respectively the distribution property between $\oplus$ and $\wedge$, the simplification rules: $\gamma^n \delta^t \wedge \gamma^{n'} \delta^{t'} = \gamma^{max(n,n')} \delta^{min(t,t')}$, the rules (2) and $[(\gamma^n \delta^t)^* \preceq \gamma^{-in} \delta^{-it} (\gamma^n \delta^t)^*]$.

- Moreover, $\forall k''$ as $0 \leq k'' \leq k_0$ we have :

$$[\delta^{-(k_0-1)t} \oplus \gamma^{K-(k_0-1)n} \delta^{-(k_0-2)t} (\gamma^n \delta^t)^*]$$
$$\preceq [\gamma^{-k''n} \delta^{-k''t} \oplus \gamma^{K-k''n} \delta^{-(k''-1)t} (\gamma^n \delta^t)^*]$$

Indeed, by considering $j = k_0 - 1 - k''$ this inequality can be written

$$[\delta^{-k_0 t + t} \oplus \cdots \oplus \gamma^{K - k_0 n + jn} \delta^{t - k_0 t + jt}$$
$$\oplus \gamma^{K-k''n} \delta^{t-k''t} (\gamma^n \delta^t)^*]$$
$$\preceq [\gamma^{-k''n} \delta^{-k''t} \oplus \gamma^{K-k''n} \delta^{t-k''t} (\gamma^n \delta^t)^*]$$

and we have

$$[\delta^{-k_0 t + t} \oplus \cdots \oplus \gamma^{K - k_0 n + jn} \delta^{t - k_0 t + jt}]$$
$$= [\delta^{-k_0 t + t} \oplus \cdots \oplus \gamma^{K - k''n - n} \delta^{-k''t}]$$
$$\preceq [\gamma^{-k''n} \delta^{-k''t}]$$

since $K \geq n$. These two results induce that :

$$\delta^{-t} [(\gamma^n \delta^t)^* \setminus (e \oplus \gamma^K (\gamma^n \delta^t)^*)]$$
$$= \delta^{-t} \bigwedge_{i=0}^{k_0} \gamma^{-ni} \delta^{-ti} (e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*)$$
$$= \delta^{-k_0 t} \oplus \gamma^{K - k_0 n + n} \delta^{-k_0 t + t} (\gamma^n \delta^t)^*$$

which ends the proof.
Remarks: The term $\delta^{-k_0 t}$ indicates that the firing dates of y_c must be known over a future horizon of $k_0 t$ time units to compute u_s.

4.1.2 Properties

In this section, we compare the behavior of a classical kanban stage with the stage controlled by u_s. In this way, we compare $x1_{u\varepsilon}$, $x3_{u\varepsilon}$, $y_{u\varepsilon}$ (i.e., when $u = \varepsilon$) with $x1_{us}$, $x3_{us}$ and y_{us} (i.e., when $u = u_s$).

Property 4.1.1 *With control u_s, the internal transitions of the stage are fired later than in a classical kanban stage with $u = \varepsilon$, or formally*

$$x1_{us} \succeq x1_{u\varepsilon} \text{ and } x3_{us} \succeq x3_{u\varepsilon} \qquad (10)$$

This property means that the firing of $x1$ will be delayed by control u_s.

Proof(10): It is obvious by considering (4) and (5), since $u_s \succeq \varepsilon$.
Remarks: By replacing u_s in (4), it can be shown that $x1_{us} = u_s \succeq x1_{u\varepsilon}$.

Property 4.1.2 *Control law u_s allows reducing the internal stock $\mathcal{S}_{x1,y}(t)$(WIP+FP).*

Proof: First, the control construction yields $y_{us} = y_{u\varepsilon} = [e \oplus \gamma^K \delta^t (\gamma^n \delta^t)^*] y_c$, then

$$y_{us} = y_{u\varepsilon} \iff \forall t, \mathcal{C}_{y_{us}}(t) = \mathcal{C}_{y_{u\varepsilon}}(t)$$

Moreover, the former property yields:

$$x1_{us} \succeq x1_{u\varepsilon} \iff \forall t, \mathcal{C}_{x1_{us}}(t) \leq \mathcal{C}_{x1_{u\varepsilon}}(t)$$

where $\leq$ is the usual order. Then, $\forall t, \mathcal{S}_{x1_{u\varepsilon}, y_{u\varepsilon}}(t) \geq \mathcal{S}_{x1_{us}, y_{us}}(t)$.

4.1.3 Extension to Production Line with Multiple Kanban Stages

The former control can be easily extended to production lines with many kanban stages. According to results concerning section 3.4, the input-output transfer relation is expressed as (6). Then, the greater control u which allows keeping an identical behavior, i.e., $y = \beta y_c$ is

$$u_{opt} = \alpha \setminus (\beta y_c)$$

To avoid the assumption of initial knowledge of all reference input y_c, it is possible to consider the following control (see 4.1.1)

$$u_s = (\alpha \setminus \beta) y_c$$

where α and β are two periodic series.
In this general case, the algorithm given in [5] gives the control law and, in particular, the future horizon over which the reference trajectory needs to be known.

4.2 Control to Keep Stock of Processed Parts

This second approach objective is to find the *latest* firing dates of u in order to keep the same stock evolution in the output buffer (stock FP) as in classical kanban stage with $u = \varepsilon$, *i.e.*, to have at each instant the same margin of processed parts in the stage. This objective induces that the system will behave like a classical kanban stage in regard to accidental loss of parts and/or machine failure.

4.2.1 Control Law

Formally, this problem can be expressed as finding the greatest u such that $x3$ behaves like when $u = \varepsilon$, i.e.,

$$x3 = x3_{u\varepsilon}$$

The greatest control verifying previous equality is

$$u_{op} = \gamma^K(\gamma^n\delta^t)^* y_c \tag{11}$$

Proof(11): The objective statement is equivalent to finding the greatest u such that

$$\gamma^K\delta^t(\gamma^n\delta^t)^* y_c \oplus \delta^t(\gamma^n\delta^t)^* u = \gamma^K\delta^t(\gamma^n\delta^t)^* y_c$$

i.e., owing to *Def. 2*,

$$\gamma^K\delta^t(\gamma^n\delta^t)^* y_c \succeq \delta^t(\gamma^n\delta^t)^* u$$

The optimal solution is given by Residuation theory, in this case u_{op} may be written as

$$u_{op} = \delta^{-t}[(\gamma^n\delta^t)^* \setminus [\gamma^K\delta^t(\gamma^n\delta^t)^* y_c]$$

which leads to relation (11) thanks to $a^* \setminus (a^*x) = (a^*x)\ \forall x$ (see [1], section 4.5.2.).
Remark: The control law (11) is causal (see *Def. 10*) and the compensator $\gamma^K(\gamma^n\delta^t)^*$ admits a TEG interpretation with positive temporization.

4.2.2 Properties

Property 4.2.1 *The output stage behavior with u_{op} matches the output with $u = \varepsilon$, i.e.,*

$$y_{uop} = y_{u\varepsilon}$$

Proof: By replacing u_{op} in transfer relation (3), and by considering that $a^*a^* = a^*$, we have immediately

$$\begin{aligned} y_{uop} &= (e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)y_c \oplus \delta^t(\gamma^n\delta^t)^*[\gamma^K(\gamma^n\delta^t)^*]y_c \\ &= (e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)y_c = y_{u\varepsilon} \end{aligned}$$

Property 4.2.2 *With control u_{op}, the input dates of unprocessed parts in the kanban stage are greater than with control $u = \varepsilon$, i.e.,*

$$x1_{uop} \succeq x1_{u\varepsilon}$$

Proof: It is obvious by considering (4), since $u_s \succeq \varepsilon$.

Property 4.2.3 *The stock FP is identical with control u_{op} and with $u = \varepsilon$, or formally:*

$$\mathcal{S}_{x3_{u\varepsilon},y_{u\varepsilon}}(t) = \mathcal{S}_{x3_{uop},y_{uop}}(t), \forall t.$$

Proof: By construction, u_{op} leads to $x3_{uop} = x3_{u\varepsilon}$. Moreover, *Property 4.2.1* yields $y_{uop} = y_{u\varepsilon}$. Obviously, the FP evolution is identical.

Property 4.2.4 *The WIP of the kanban stage with control u_{op} is lower than with control $u = \varepsilon$.*

Proof: $x1_{u\varepsilon} \preceq x1_{uop}$ (property 4.2.2) and $x3_{u\varepsilon} = x3_{uop}$ (property 4.2.1), obviously, $\mathcal{C}_{x1_{u\varepsilon}}(t) - \mathcal{C}_{x3_{u\varepsilon}}(t) \geq \mathcal{C}_{x1_{uop}}(t) - \mathcal{C}_{x3_{uop}}(t), \forall t$.

Property 4.2.5 *Control u_{op} is the greatest control such that $\gamma^K\delta^t(\gamma^n\delta^t)^* y_c = \delta^t(\gamma^n\delta^t)^* u$. This property means that the optimal control allows matching the objective.*

Proof: It is obvious by replacing u by u_{op} in $x3$ and by considering that $a^*a^* = a^*$.
Remark: This result can be seen as an exact inversion problem, i.e.,

$$\delta^t(\gamma^n\delta^t)^*[\delta^t(\gamma^n\delta^t)^* \setminus (\gamma^K\delta^t(\gamma^n\delta^t)^* y_c)] = \gamma^K\delta^t(\gamma^n\delta^t)^* y_c$$

Property 4.2.6 *The transition behavior is such that,*

$$u_{op} = x1_{uop} = x2_{uop} = \gamma^K(\gamma^n\delta^t)^* y_c$$

Proof: By replacing u_{op} in $x1$ and $x2$ we have

$$\begin{aligned} x1_{uop} &= \gamma^K(e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)y_c \\ &\quad \oplus(e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)\gamma^K(\gamma^n\delta^t)^* y_c \\ &= (e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)(\gamma^K \oplus \gamma^K(\gamma^n\delta^t)^*)y_c \end{aligned}$$

since $a^*a^* = a^*$ and $K \geq n$, this yields

$$\begin{aligned} x1_{uop} &= \gamma^K(\gamma^n\delta^t)^*)y_c \\ x2_{uop} &= \gamma^K(\gamma^n\delta^t)^* y_c \oplus (\gamma^n\delta^t)^*)\gamma^K(\gamma^n\delta^t)^* y_c \\ &= \gamma^K(\gamma^n\delta^t)^* y_c \end{aligned}$$

Remark: Property 4.2.6 involves that the stock between u and $x2$ is zero, *i.e.*,

$$\mathcal{S}_{u_\varepsilon, x2_{u\varepsilon}} = \mathcal{C}_{u_{op}}(t) - \mathcal{C}_{x2_{uop}}(t) = 0, \forall t.$$

This is a consequence of the exact inversion seen in property 4.2.5 and means that u_{op} provides a part exactly when it is necessary and never before.

Property 4.2.7 *Feedback control* $u'_{op} = \gamma^K(\gamma^n\delta^t)^*y$ *yields the same control as* u_{op}.

Proof: By considering the transfer relation :

$$x3 = \delta^t(\gamma^n\delta^t)^*u \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*y$$

and by introducing u'_{op} one obtains

$$x3 = \gamma^K\delta^t(\gamma^n\delta^t)^*y = \gamma^K\delta^t(\gamma^n\delta^t)^*(x3 \oplus y_c)$$

recalling that $b^*(ab^*)^* = (a \oplus b)^*$ and $K \geq n$ this yields

$$x3 = \gamma^K\delta^t(\gamma^n\delta^t)^*y_c = x3_{u\varepsilon}$$

furthermore, since $K \geq n$

$$\begin{aligned} u'_{op} &= \gamma^K(\gamma^n\delta^t)^*y = \gamma^K(\gamma^n\delta^t)^*(y_c \oplus x3) \\ &= \gamma^K(\gamma^n\delta^t)^*(e \oplus \gamma^K\delta^t(\gamma^n\delta^t)^*)y_c \\ &= \gamma^K(\gamma^n\delta^t)^*y_c = u_{op} \end{aligned}$$

Remark: Associated with 4.2.6, the former properties mean that the optimal control u_{op} is equivalent to replace the feedback on input transition, $\gamma^K y$, by feedback $\gamma^K(\gamma^n\delta^t)^*y$.

4.2.3 Multiple Kanban Stages Extension

By analogy we can expand the previous principle which consists in finding the greatest feedback $\zeta_i x_{i+1}$ on the stage input x_i such as x''_i behaves like in classical kanban stage, *i.e.*,

$$x''_i = \gamma^{K_i}\delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*x_{i+1} \oplus \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*x''_{i-1}$$

By considering that $x''_i = \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*x_i$ and by assuming in a first step that $x''_{i-1} = \varepsilon$, the greatest feedback can be expressed as

$$\begin{aligned} \zeta_{i_{op}}x_{i+1} &= \delta^{t_i}(\gamma^{n_i}\delta^{t_i})^* \setminus (\gamma^{K_i}\delta^{t_i}(\gamma^{n_i}\delta^{t_i})^*x_{i+1}) \\ &= \gamma^{K_i}(\gamma^{n_i}\delta^{t_i})^*x_{i+1} \end{aligned}$$

which is the greatest solution of our problem according to property 4.2.5.

5 Conclusion

In this paper we have proposed two control strategies of manufacturing system organized with the kanban method. The first control allows both reducing (FP+WIP) and keeping the same output behavior as a system with an infinite stock of unprocessed parts. It is shown that the reference input (customer's demand) must be known over a temporal horizon to satisfy our objective. The second control law allows both reducing WIP and keeping for each kanban stage the same number of processed parts (FP) as in a classical kanban system. The WIP reduction is lower than the ones obtained with the former control, nevertheless the keeping of the processed parts stock FP allows to satisfy the same behavior as a classical kanban stage in regard to exogenous events. It is shown that the law is causal and can be seen as a feedback of the output on the input transition.

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