Finite-Dimensionality of Information States in Optimal Control of Stochastic Systems: A Lie Algebraic Approach

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Abstract

In this paper we introduce the sufficient statistic algebra which is responsible for propagating the sufficient statistic, or information state, in the optimal control of stochastic systems. Certain Lie algebraic methods widely used in nonlinear control theory, are then employed to derive finite-dimensional controllers. The sufficient statistic algebra enables us to determine á priori whether there exist finite-dimensional controllers; it also enables us to classify all finite-dimensional controllers.

1 Introduction

The DMZ equation of nonlinear filtering of diffusion processes is a linear, stochastic, partial differential equation (PDE) which describes in a recursive manner the evolution of the unnormalized conditional distribution of the state process, $\{x(t); t \geq 0\}$, given the observations, $\{y(t); t \geq 0\}$. If this distribution has a density function, say, $\{\pi(x,t); t \geq 0\}$, then

$$\frac{d}{dt}\pi(x,t) = L_0\pi(x,t) + h(x)\pi(x,t) \circ \frac{d}{dt}y(t). \tag{1.1}$$

Consequently, $\{\pi(x,s); 0 \leq s \leq t\}$ evolves forward in time with initial condition $\pi(x,0)$. Here, L_0 is a certain second-order differential operator related to the drift and diffusion coefficients of the state process, the Kolmogorov forward operator, and $h(x)$ is a zero-order differential operator related to the signal in the observations.

Brockett and Clark [1], proposed that due to the analogy between (1.1) and the control system $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, the Lie algebraic methods might be applicable to (1.1) as well. In particular, they proposed that the finite-dimensionality of solutions to (1.1) can be deduced from the Lie algebra generated by the operators L_0, h. Moreover Ocone [2], noted that if the Lie algebra generated by the operators L_0, h, is finite-dimensional, then (under certain conditions) the Wei-Norman method can be used to derive the structure of the recursive filters, (see [2, 3, 4, 5]. Recently, gauge transformations have been introduced in [6, 7, 8], to identify nonlinear control problems with finite-dimensional controllers.

In the present paper we point out how the Lie algebraic methods can be used to address the question of finite-dimensionality of optimal controllers in problems of optimal control of partially observed stochastic systems. Note that in the absence of control optimality, this framework can be used to address the question of finite-dimensionality of optimal (in least-squares sense) observers for nonlinear stochastic control systems. This framework would enable us to investigate the question of classification and finite-dimensionality of optimal controls a priori, by investigating the Lie algebra of certain operators associated with the model at hand. The Lie algebra method yields new classes of nonlinear systems which are not a subset of our earlier classes in [6, 7, 8].

In particular, the observation that leads to these developments is that for optimal control problems (with usual integral cost function) affine in the control inputs, the information state satisfies a controlled version of the DMZ equation, namely,

$$\begin{aligned}\frac{d}{dt}\pi^u(x,t) &= L_0\pi^u(x,t) + L\pi^u(x,t)u(t,y) \\ &+ h(x)\pi^u(x,t) \circ \frac{d}{dt}y(t),\end{aligned} \tag{1.2}$$

where $u(\cdot)$ is the control input and L is certain first-order differential operator. Therefore, by analogy with finite-dimensional nonlinear affine control systems, we view (1.2) as a bilinear equation with control inputs $u(\cdot)$, $\frac{d}{dt}y(\cdot)$. This gives rise to the investigation of the Lie algebra generated by the operators L_0, L, h, which we call sufficient statistic algebra. In fact, from certain results of realization theory, we deduce that if the sufficient statistic algebra, $\mathcal{L}_S \doteq \{L_0, L, h\}_{L.A.}$, is finite-dimensional, then (under certain conditions) the optimal controller is finite-dimensional.

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2 Mathematical Constructs

Consider the Ito stochastic differential system

$$\begin{aligned} dx(t) &= f(x(t))dt + \textstyle\sum_{j=1}^{\ell} g_j(x(t))u_j(t,y)dt \\ &+ \textstyle\sum_{j=1}^{m} \sigma_j(x(t))dw_j(t), \qquad x(0) \in \Re^n, \end{aligned} \tag{2.3}$$

$$dy_j(t) = h_j(x(t))dt + db_j(t), \qquad y_j(0) = 0 \in \Re, \tag{2.4}$$

$1 \leq j \leq d$. Here $\{w_i(t); t \in [0,T]\}$ and $\{b_j(t); t \in [0,T]\}$, are mutually independent standard Brownian motion processes, for all $1 \leq i \leq m, 1 \leq j \leq d$, which are also independent of the random variable $x(0)$. $u(\cdot) = [u_1, u_2, ..., u_\ell]'(\cdot)$ is a vector of control processes. All stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, P^u)$ equipped with a complete filtration, $\{\mathcal{F}_{0,t}; t \in [0,T]\}$, and a finite-time interval, $[0,T]$.

The usual optimal control problem addresses the minimization over the controls $u(\cdot) \in \mathcal{U}_{ad}$, (see Definition 2.3), of the integral cost criterion $J(u)$:

$$J(u) = E^u \left\{ \int_0^T \ell(x(t), u(t,y))dt + \varphi(x(T)) \right\}. \tag{2.5}$$

Notation 2.1

"$'$" denotes transposition of a matrix, I_k denotes $k \times k$ identity matrices, $\{\alpha_j\}_{j=1}^n, \{\alpha_{i,j}\}_{i,j=1}^n$ denote finite sequences in $\Re$;

$C^\infty(M)$ denotes the vector space of all infinite differentiable real-valued functions defined on an $n-$dimensional differentiable manifold M;

$g(x) = [g_1(x), g_2(x), \ldots, g_\ell](x), [g]_{i,j}(x) = g_{i,j}(x), h(x) = [h_1, h_2, \ldots, h_d]'(x), y(t) = [y_1, y_2, \ldots y_d]'(t)$;

$\Phi : \Re^n \rightarrow \Re$ is C^2 with compact support;

$\{\mathcal{F}_{0,t}^y; t \in [0,T]\}$ denotes the complete filtration generated by the observations $\sigma-$algebra, $\sigma\{y(s); 0 \leq s \leq t\}$, E^u, E denote expectations w.r.t. measures P^u, P, respectively.

Assumptions 2.2

U is a compact subset of $\Re^\ell$;

$f : \Re^n \rightarrow \Re^n, g_i : \Re^n \rightarrow \Re^n, \sigma_j : \Re^n \rightarrow \Re^n$, $1 \leq i \leq \ell, 1 \leq j \leq m$, are $C^\infty(\Re^n)$ vector fields, $h_j : \Re^n \rightarrow \Re$, $1 \leq j \leq d$, are $C^\infty(\Re^n)$ functions, and

$$|f| + |\sigma_j| + |g_i| + |h_k| \leq k_1 (1 + |x|), \quad \forall i,j,k;$$

$\ell : \Re^n \times U \rightarrow \Re, \varphi : \Re^n \rightarrow \Re, \ell \geq 0, \varphi \geq 0$, and

$$|\ell(x,u)| \leq k_2 (1 + |x| + |u|)^{k_3}, \ |\varphi(x)| \leq k_4 (1 + |x|)^{k_5};$$

The random variable $x(0)$ has distribution $\Pi_0(dx) = \pi_0(x)dx$, with $\pi_0(\cdot) \in L^2(\Re^n)$.

Definition 2.3 *The set of admissible controls denoted by $\mathcal{U}_{ad}$ is defined by $\mathcal{U}_{ad} \doteq \{u(\cdot); u(\cdot) \in L_y^2([0,T]; \Re^\ell), u(t,y) \in U \subset \Re^\ell, a.e.t, P-a.s.\}$.*

2.1 Sufficient Statistic

Let $\Pi_t(\Phi) \doteq E^u \left[\Phi(x(t)) | \mathcal{F}_{0,t}^y\right]$; let

$$\Lambda_{0,t} = \exp\left(\sum_{j=1}^d \int_0^t h_j(x(s))dy_j(s) - \frac{1}{2}\sum_{j=1}^d \int_0^t h_j^2(x(s))ds \right).$$

Introduce the Radon-Nikodym derivative, (see [9, 8]), $\frac{dP^u}{dP}|_{\mathcal{F}_{0,T}} = \Lambda_{0,T}$. By a version of Bayes formula we have:

$$\Pi_t(\Phi) = \frac{E\left[\Phi(x(t))\Lambda_{0,t} | \mathcal{F}_{0,t}^y\right]}{E\left[\Lambda_{0,t} | \mathcal{F}_{0,t}^y\right]} \doteq \frac{\pi_t(\Phi)}{\pi_t(1)}. \tag{2.6}$$

Here $\Pi_t(\cdot)$ and $\pi_t(\cdot)$ are measure-valued processes; the latter is the unnormalized version of the former.

Theorem 2.4 *[9, 8] Let $\Phi \in C^2(\Re^n)$ and suppose $\pi_t(\cdot)$ has a density function $\pi : \Re^n \times \Omega \times [0,T] \rightarrow \Re$. Then*

$$\pi_t(\Phi) = E\left[\Phi(x(t))\Lambda_{0,t} | \mathcal{F}_{0,t}^y\right] = \int_{\Re^n} \Phi(z)\pi(z,t)dz, \tag{2.7}$$

where $\pi(\cdot)$ is a solution of the controlled version of the DMZ equation (Fisk-Stratonovich form):

$$\begin{aligned} \pi(x,t) &= \pi(x,0) + \textstyle\int_0^t L_0\pi(x,s)ds \\ &+ \textstyle\sum_{j=1}^{\ell} \int_0^t L_j\pi(x,s)u_j(s,y)ds \\ &+ \textstyle\sum_{j=1}^{d} \int_0^t h_j(x)\pi(x,s) \circ dy_j(s), \end{aligned} \tag{2.8}$$

$$\begin{aligned} A(\Phi)(x) &\doteq \tfrac{1}{2}\textstyle\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \left([\sigma\sigma']_{i,j}\, \Phi\right)(x) \\ &+ \textstyle\sum_{j=1}^n \left(f_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(f_j)\right)(\Phi)(x), \\ L_j(\Phi)(x) &\doteq -\textstyle\sum_{i=1}^n \left(g_{i,j}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}(g_{i,j})\right)(\Phi)(x), \\ 1 \leq j \leq \ell, \quad & L_0(\Phi)(x) = \left(A - \tfrac{1}{2}\textstyle\sum_{j=1}^d h_j^2\right)(\Phi)(x). \end{aligned} \tag{2.9}$$

Moreover, for $u \in \mathcal{U}_{ad}$ the cost function (2.5) has the representation

$$\begin{aligned} J_{0,T}(u) = E\Big\{ &\textstyle\int_0^T \int_{\Re^n} \ell(z, u(t,y))\pi(z,t)dzdt \\ &+ \textstyle\int_{\Re^n} \varphi(z)\pi(z,T)dz \Big\}. \end{aligned} \tag{2.10}$$

In the formulation of Theorem 2.4, the conditional density is an information state, or a sufficient statistic. Therefore, by construction (2.8) propagates the information available to the controller. In the sequel we assume the measure-valued process $\pi_t(\cdot)$ has a unique density $\pi(\cdot)$ satisfying (2.8).

Definition 2.5 *Let $X, Y : C^\infty(M) \rightarrow C^\infty(M)$, be differential operators with C^∞ coefficients. The vector space of all differential operators (with C^∞ coefficients) is a Lie algebra with the Lie bracket of X, Y defined by*

$$[X,Y](\Phi) \doteq X(Y(\Phi)) - Y(X(\Phi)), \ \forall \Phi \in C^\infty(M).$$

Definition 2.6 *The estimation algebra $\mathcal{L}_E$ of the filtering problem (2.3), (2.4) (with $u_j = 0, 1 \le j \le \ell$), is the Lie algebra generated by, $\{L_0, h_1, h_2, \ldots, h_d\}$, defined by*

$$\mathcal{L}_E \doteq \{L_0, h_1, h_2, \ldots, h_d\}_{L.A.}. \tag{2.11}$$

The sufficient statistic algebra $\mathcal{L}_S$ of the control problem (2.3)-(2.5) is the Lie algebra generated by, $\{L_0, L_1, L_2, \ldots, L_\ell, h_1, h_2, \ldots, h_\ell\}$, defined by

$$\mathcal{L}_S \doteq \{L_0, L_1, L_2, \ldots, L_\ell, h_1, h_2, \ldots, h_d\}_{L.A.}. \tag{2.12}$$

3 Sufficient Statistic Algebras

Assumptions 3.1 *Assumption 2.2 hold, $m = n$, and $[\sigma_1, \sigma_2, \ldots, \sigma_n][\sigma_1, \sigma_2, \ldots, \sigma_n]'(x) = I_n$, that is, $\sigma(x)$ is orthogonal; in the scalar case it is assumed that $\sigma = 1$.*

Define

$$\begin{aligned} D_i &\doteq \tfrac{\partial}{\partial x_i} - f_i, \quad 1 \le i \le n, \\ \eta &\doteq \textstyle\sum_{i=1}^n \tfrac{\partial}{\partial x_i} f_i + \sum_{i=1}^n f_i^2 + \sum_{i=1}^d h_i^2. \end{aligned} \tag{3.13}$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \tag{3.14}$$

We shall need the following calculations.

Lemma 3.2 *Let*

$$w_{i,j}(x) = \frac{\partial}{\partial x_i} f_j(x) - \frac{\partial}{\partial x_j} f_i(x), \quad 1 \le i, j \le n.$$

Then

$$[D_i, D_j] = w_{j,i}, \quad 1 \le i, j \le n;$$

$$[D_i, h_j] = \tfrac{\partial}{\partial x_i} h_j, \quad 1 \le i \le n, 1 \le j \le d;$$

$$\begin{aligned} \left[D_i^2, h_j\right] &= \tfrac{\partial^2}{\partial x_i^2}(h_j) + 2\tfrac{\partial}{\partial x_i}(h_j) D_i, \\ &1 \le i \le n, 1 \le j \le d; \end{aligned}$$

$$[L_i, h_j] = -\textstyle\sum_{k=1}^n g_{k,i} \tfrac{\partial}{\partial x_k}(h_j), \ 1 \le i \le \ell, 1 \le j \le d;$$

$$\left[D_i^2, D_j\right] = 2w_{j,i} D_i + \tfrac{\partial}{\partial x_i}(w_{j,i}), \quad 1 \le i, j \le n;$$

$$\begin{aligned} [L_0, D_j] &= \tfrac{1}{2}\textstyle\sum_{i=1}^n \left(2w_{j,i} D_i + \tfrac{\partial}{\partial x_i}(w_{j,i})\right) \\ &+ \tfrac{1}{2}\tfrac{\partial}{\partial x_j}(\eta), \quad 1 \le j \le n; \end{aligned}$$

$$\begin{aligned} [D_i, L_j] = -\textstyle\sum_{k=1}^n &\left(\tfrac{\partial^2}{\partial x_i \partial x_k}(g_{k,j}) - g_{k,j}\tfrac{\partial}{\partial x_k}(f_i)\right. \\ &\left. + \tfrac{\partial}{\partial x_i}(g_{k,j})\tfrac{\partial}{\partial x_k}\right), \quad 1 \le i \le n, 1 \le j \le \ell; \end{aligned}$$

$$\begin{aligned} \left[D_i^2, L_j\right] &= -\textstyle\sum_{k=1}^n \left\{\tfrac{\partial^3}{\partial x_i^2 \partial x_k}(g_{k,j})\right. \\ &+ 2\tfrac{\partial^2}{\partial x_i \partial x_k}(g_{k,j})\left[\tfrac{\partial}{\partial x_i} - f_i\right] + \tfrac{\partial^2}{\partial x_i^2}(g_{k,j})\tfrac{\partial}{\partial x_k} \\ &+ 2\tfrac{\partial}{\partial x_i}(g_{k,j})\tfrac{\partial^2}{\partial x_i \partial x_k} - 2f_i\tfrac{\partial}{\partial x_i}(g_{k,j})\tfrac{\partial}{\partial x_k} \\ &\left. + g_{k,j}\tfrac{\partial^2}{\partial x_i \partial x_k}(f_i) + 2g_{k,j}\tfrac{\partial}{\partial x_k}(f_i)\left[\tfrac{\partial}{\partial x_i} - f_i\right]\right\}, \\ &1 \le i \le n, 1 \le j \le \ell; \end{aligned}$$

$$\begin{aligned} [L_i, L_j] = \textstyle\sum_{m=1}^n \sum_{k=1}^n &\left\{g_{m,i}\tfrac{\partial^2}{\partial x_m \partial x_k}(g_{k,j})\right. \\ &- g_{k,j}\tfrac{\partial^2}{\partial x_k \partial x_m}(g_{m,i}) + g_{m,j}\tfrac{\partial}{\partial x_m}(g_{k,j})\tfrac{\partial}{\partial x_k} \\ &\left. - g_{k,j}\tfrac{\partial}{\partial x_k}(g_{m,j})\tfrac{\partial}{\partial x_m}\right\}, \quad 1 \le i, j \le \ell; \end{aligned}$$

3.1 The Linear Case

Here we analyze the linear control system

$$\begin{aligned} dx(t) &= Fx(t)dt + \textstyle\sum_{j=1}^\ell B_j u_j(t, y) \\ &+ \textstyle\sum_{j=1}^n G_j dw_j(t), \\ dy_j(t) &= \textstyle\sum_{i=1}^n H_{j,i} x_i(t) dt + db_j(t), \quad 1 \le j \le d. \end{aligned} \tag{3.15}$$

Lemma 3.3 *(Scalar case). Suppose $n = \ell = d = m = 1$. The sufficient statistic algebra has dimension 4 with basis*

$$\mathcal{L}_S = Span\left\{L_0 = \frac{1}{2}(D^2 - \eta), x, D = \frac{\partial}{\partial x} - Fx, 1\right\} \tag{3.16}$$

The non-zero commutative relations are

$$[L_0, x] = D, \qquad [L_0, D] = D + \frac{1}{2}\frac{\partial}{\partial x}(\eta), \qquad [D, x] = 1.$$

Moreover, $\mathcal{L}_S = \mathcal{L}_E$.

Proof. See Theorem 3.4.

Theorem 3.4 *(Multidimensional case). The sufficient statistic algebra has dimension at most $2n + 2$ with basis*

$$\begin{aligned} \mathcal{L}_S = Span &\left\{L_0 = \tfrac{1}{2}\left(\textstyle\sum_{i=1}^n D_i^2 - \eta\right),\right. \\ &\left. x_1, x_2, \ldots, x_n, D_1, D_2, \ldots, D_n, 1\right\}. \end{aligned} \tag{3.17}$$

The non-zero commutative relations are

$$\begin{aligned} &[L_0, x_j] = D_j, \quad [L_0, D_j] = \textstyle\sum_{i=1}^n (F_{i,j} - F_{j,i}) D_i \\ &+ \tfrac{1}{2}\tfrac{\partial}{\partial x_j}(\eta), \quad 1 \le j \le n; \quad [D_i, x_j] = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases} \end{aligned}$$

Moreover, $\mathcal{L}_E = \mathcal{L}_S$.

Proof.

$$\begin{aligned} Y_j &\doteq [L_0, h_j] = \tfrac{1}{2}\textstyle\sum_{i=1}^n \left[D_i^2, h_j\right] = \sum_{i=1}^n H_{j,i} D_i, \\ &1 \le j \le d; X_{j,i} \doteq [Y_j, h_i] = \textstyle\sum_{k=1}^n \sum_{\ell=1}^n H_{j,k} H_{i,\ell} \\ &.[D_k, x_\ell] = \textstyle\sum_{k=1}^n H_{j,k} H_{i,k}, \quad 1 \le i, j \le d; \end{aligned}$$

Therefore, $D_1, D_2, \ldots, D_n$ and 1 are elements of $\mathcal{L}_S$. Also, from the computation

$$Z_j \doteq [L_0, Y_j] = \sum_{i=1}^n [L_0, H_{j,i} D_i] = \sum_{i=1}^n \sum_{k=1}^n H_{j,k} \\ (F_{k,i} - F_{i,k}) D_k + \frac{1}{2} \sum_{i=1}^n H_{j,i} \frac{\partial}{\partial x_i}(\eta), \quad 1 \le j \le d,$$

we deduce that $x_1, x_2, \ldots, x_n$ are also elements of $\mathcal{L}_S$. Now,

$$Y_{j,k} \doteq [Y_j, Y_k] = \sum_{i=1}^n \sum_{\ell=1}^n [H_{j,i} D_i, H_{k,\ell} D_\ell] \\ = \sum_{i=1}^n \sum_{\ell=1}^n H_{j,i} H_{k,\ell} w_{\ell,i}, \quad 1 \le j, k \le d.$$

Proceeding we calculate

$$L_{0,j} \doteq [L_0, L_j] = \frac{1}{2} \left[\sum_{i=1}^n D_i^2 - \eta, - \sum_{k=1}^n B_{k,j} \frac{\partial}{\partial x_k} \right] \\ = - \sum_{i=1}^n \sum_{k=1}^n \left\{ B_{k,j} \frac{\partial}{\partial x_k}(f_i) \left(\frac{\partial}{\partial x_i} - f_i \right) \right\} \\ - \frac{1}{2} \sum_{k=1}^n B_{k,j} \frac{\partial}{\partial x_k}(\eta), \quad 1 \le j \le \ell.$$

Hence, $L_{0,j}$ is a linear combination of elements $D_1, D_2, \ldots, D_n, x_1, x_2, \ldots x_n, 1$. In addition,

$$[L_j, h_i] = - \sum_{k=1}^n B_{k,j} H_{i,k}, \quad 1 \le j \le \ell, 1 \le i \le d; \\ [L_j, D_i] = - \sum_{k=1}^n B_{k,j} F_{i,k}, \quad 1 \le j \le \ell, 1 \le i \le d; \\ [L_j, Y_i] = - \sum_{k=1}^n B_{k,j} H_{i,k}, \quad 1 \le j \le \ell, 1 \le i \le d.$$

Therefore, we deduce that $\mathcal{L}_S$ is finite-dimensional with basis as specified, and that $\mathcal{L}_E$ and $\mathcal{L}_S$ generate the same algebra.

3.2 The Nonlinear Drift Case

Here we investigate the nonlinear control system

$$dx(t) = f(x(t))dt + \sum_{j=1}^\ell B_j u_j(t, y) dt \\ + \sum_{j=1}^n \sigma_j(x(t)) dw_j(t), \tag{3.18}$$

$$dy_j(t) = \sum_{i=1}^n H_{j,i} x_i dt + db_j(t), \qquad 1 \le j \le d.$$

Lemma 3.5 *(The two-dimensional case). Suppose $n = 2, m = 2, \ell = d = 1$, and*

$$f_1 = \sum_{j=1}^n F_{1,j} x_j, \quad f_2 = f_2(x_1, x_2), \quad B_{2,1} = 0, \\ w_{i,j} = \text{constant}, \ \text{for}\, i \ne j. \tag{3.19}$$

1. If

$$\eta = \sum_{i=1}^2 \frac{\partial}{\partial x_i} f_i + \sum_{i=1}^2 f_i^2 + h_1^2 \\ = \text{Quadratic function of } (x_1, x_2) \ge 0, \tag{3.20}$$

then the sufficient statistic algebra has dimension at most 6 with basis

$$\mathcal{L}_S = \mathit{Span} \left\{ L_0, x_1, x_2, \frac{\partial}{\partial x_1}, D_2 = \frac{\partial}{\partial x_2} - f_2, 1 \right\}. \tag{3.21}$$

2. If $h_1 = H_{1,1} x_1$ and $\eta =$ A nonnegative quadratic function of $(x_1, x_2) + \gamma(x_2)$ for some $\gamma \in C^\infty(\Re)$, then the sufficient statistic algebra is given by (3.21).

Moreover, $\mathcal{L}_E = \mathcal{L}_S$.

The non-zero commutative relations are

$$[L_0, x_i] = D_i, \ 1 \le i \le 2; \ \left[L_0, \frac{\partial}{\partial x_1}\right] = w_{1,2} D_2 \\ + \frac{1}{2} \frac{\partial}{\partial x_1}(\eta) + \sum_{i=1}^2 F_{1,i} D_i; \ [L_0, D_2] = w_{2,1} D_1 + \frac{1}{2} \frac{\partial}{\partial x_2}(\eta); \\ \left[\frac{\partial}{\partial x_1}, D_2\right] = - \left(\frac{\partial}{\partial x_2}(f_1) + w_{1,2} \right); \\ \left[\frac{\partial}{\partial x_1}, x_i\right] = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{if } i = 2. \end{cases}$$

Proof. 1. From Theorem 3.4 we have $Y_1 \doteq [L_0, h_1] = \sum_{i=1}^2 H_{1,i} D_i, X_{1,1} \doteq [Y_1, h_1] = \sum_{k=1}^2 H_{1,k}^2$; Hence, $\frac{\partial}{\partial x_1}, x_1, x_2, D_2, 1$ are elements of $\mathcal{L}_S$. Also,

$$Z_1 \doteq [L_0, Y_1] = \sum_{\ell=1}^2 \sum_{i=1}^2 H_{1,\ell} \left(w_{\ell,i} D_i + \frac{\partial}{\partial x_i}(w_{\ell,i}) \right) \\ + \frac{1}{2} \sum_{\ell=1}^2 H_{1,\ell} \frac{\partial}{\partial x_\ell}(\eta).$$

Since $w_{i,j}$ are constants and η is a quadratic function of (x_1, x_2), we conclude that $Z_1 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \frac{\partial}{\partial x_1} + \alpha_4 D_2 + \alpha_5.1$. Proceeding we calculate

$$L_{0,1} \doteq [L_0, L_1] = - \frac{1}{2} \sum_{i=1}^2 \sum_{k=1}^2 \left\{ 2 B_{k,1} \frac{\partial}{\partial x_k} \right. \\ \left. .(f_i) \left(\frac{\partial}{\partial x_i} - f_i \right) + B_{k,1} \frac{\partial^2}{\partial x_i \partial x_k}(f_i) \right\} - \frac{1}{2} [\eta, L_1].$$

Since $B_{k,1} = 0$, for $k = 2$ and $f_1 = \sum_{j=1}^2 F_{1,j} x_j$ we have

$$L_{0,1} = - \frac{1}{2} \left\{ 2 B_{1,1} F_{1,1} \left(\frac{\partial}{\partial x_1} - f_1 \right) + 2 B_{1,1} \frac{\partial}{\partial x_1}(f_2) \right. \\ \left. \left(\frac{\partial}{\partial x_2} - f_2 \right) + B_{1,1} \frac{\partial^2}{\partial x_2 \partial x_1}(f_2) \right\} - \frac{1}{2} B_{1,1} \frac{\partial}{\partial x_1}(\eta).$$

If we now substitute $\frac{\partial}{\partial x_1} f_2 = \frac{\partial}{\partial x_2} f_1 + w_{1,2}$, then $L_{0,1}$ is a linear combination of $x_1, x_2, \frac{\partial}{\partial x_1}, D_2, 1$; also, $[L_{0,1}, h_1], [L_{0,1}, L_1], [L_{0,1}, Y_1]$, are linear combinations of these elements as well. Hence, we deduce that $\mathcal{L}_S$ contains elements $L_0, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} - f_2, x_1, x_2, 1$.
2. If we now let $h_1 = H_{1,1} x_1$ we have

$$Y_1 = h_{1,1} D_1 \ X_{1,1} = H_{1,1}^2 \ Z_1 = H_{1,1} w_{1,2} D_2 + \frac{1}{2} H_{1,1} \frac{\partial}{\partial x_1}(\eta).$$

If $\eta = (x_1, x_2) Q (x_1, x_2)' + 2\sigma(x_1, x_2)' + \delta + \gamma(x_2)$, for some $Q \ge 0, \sigma \in (\Re^2)', \delta \in \Re, \gamma \in C^\infty(\Re)$, then Z_1 is a linear combination of elements $x_1, x_2, \frac{\partial}{\partial x_1}, D_2, 1$. Moreover, $L_{0,1} = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \frac{\partial}{\partial x_1} + \alpha_4 D_2 + \alpha_5.1$. In this case, tracing our earlier steps we deduce that $\mathcal{L}_S$ contains the elements $L_0, \frac{\partial}{\partial x_1}, D_2, x_1, x_2, 1$ which are its basis elements.

Example 3.6 *The following stochastic control problem has a finite-dimensional sufficient statistic algebra.*

$$dx_1(t) = (F_{1,1} x_1(t) + F_{1,2} x_2(t))\, dt \\ + B_{1,1} u(t, y) dt + dw_1(t),$$

$$dx_2(t) = f_2(x_2(t)) dt + dw_2(t),$$

$$dy(t) = H_{1,1} x_1(t) dt + db(t).$$

To verify the claim, notice that $w_{1,2} = \frac{\partial}{\partial x_1} f_2 - \frac{\partial}{\partial x_2} f_1 = -F_{1,2}$ *=constant, and*

$$\begin{aligned}\eta &= (F_{1,1}x_1)^2 + (F_{1,2}x_2)^2 + 2F_{1,1}F_{1,2}x_1x_2 \\ &+F_{1,1} + \tfrac{\partial}{\partial x_2} f_2(x_2) + f_2(x_2)^2 + (H_{1,1}x_1)^2 \\ &= (x_1, x_2)Q(x_1, x_2)' + \gamma(x_2),\end{aligned}$$

where

$$Q = \begin{bmatrix} F_{1,1}^2 + H_{1,1}^2 & F_{1,1}F_{1,2} \\ F_{1,1}F_{1,2} & F_{1,2}^2 \end{bmatrix},$$
$$\gamma(x_2) = f_2(x_2)^2 + \tfrac{\partial}{\partial x_2} f_2(x_2) + F_{1,1}.$$

Hence, when $Q \geq 0$ *and* $\gamma \in C^\infty(\Re)$, *statement 2, of Lemma 3.5 applies.*

Theorem 3.7 *(Multidimensional case). Suppose* $n = m$, ℓ, d *are arbitrary, and*

$$f_i = \textstyle\sum_{j=1}^n F_{i,j}x_j, \quad 1 \leq i \leq k,$$
$$f_{k+1} = f_{k+1}(x_1, x_2, \ldots, x_k),$$
$$\vdots$$
$$f_n = f_1(x_1, x_2, \ldots, x_n),$$

$$B_{i,j} = 0, \quad \forall i > k, \quad 1 \leq j \leq \ell,$$

$$w_{i,j} = constant, \quad \forall 1 \leq i \leq k, \; k+1 \leq j \leq n.$$

1. If

$$\begin{aligned}\eta &= \textstyle\sum_{i=1}^n \frac{\partial}{\partial x_i} f_i + \sum_{i=1}^n f_i^2 + \sum_{i=1}^d h_i^2 \\ &= \text{Quadratic function of } (x_1, x_2, \ldots, x_n) \geq 0,\end{aligned} \tag{3.22}$$

then the sufficient statistic algebra has dimension at most $2n+2$ *with basis*

$$\begin{aligned}\mathcal{L}_S = Span\{&L_0, x_1, x_2, \ldots, x_n, \\ &\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_k}, D_{k+1}, D_{k+2}, \ldots, D_n, 1\}.\end{aligned} \tag{3.23}$$

2. If $h_i = \sum_{j=1}^k H_{i,j}x_j$, $1 \leq i \leq d$, $w_{i,j} = constant$, $\forall 1 \leq i \leq k$, $k+1 \leq j \leq n$, *and*

$$\begin{aligned}\eta = &\text{ A nonnegative quadratic function of} \\ &(x_1, x_2, \ldots, x_n) + \gamma(x_{k+1}, x_{k+2}, \ldots, x_n),\end{aligned} \tag{3.24}$$

for some $\gamma \in C^\infty(\Re^{n-k})$, *then the sufficient statistic algebra has dimension at most* $2n+2$ *and is given by (3.23).*

Proof. Follow the derivation of Lemma 3.5.

3.3 The Nonlinear Drift and Observations Case

Next we investigate the correlated nonlinear control system

$$\begin{aligned}dx(t) &= f(x(t))dt + \textstyle\sum_{j=1}^\ell g_j(x(t))u_j(t,y)dt \\ &+ \textstyle\sum_{j=1}^n G_j dw_j(t), \\ dy_j(t) &= h_j(x(t))dt + \textstyle\sum_{i=1}^n a_{j,i}dw_i(t) \\ &+ \textstyle\sum_{i=1}^d N_{j,i}^{\frac{1}{2}} db_i(t), \quad 1 \leq j \leq d.\end{aligned} \tag{3.25}$$

Let

$$\begin{aligned}L_0 &= A - \tfrac{1}{2}\textstyle\sum_{k=1}^d M_k^2, \quad M_k = \sum_{i=1}^d h_i[C^{-1}]_{i,k} + Y_k, \\ Y_k &= -\textstyle\sum_{i=1}^n [Ga'C^{-1}]_{i,k}\frac{\partial}{\partial x_i}, \quad C = aa' + N, \\ \hat{A} &= \tfrac{1}{2}\textstyle\sum_{i,j}^n [GG']_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n \left([Fx]_i \frac{\partial}{\partial x_i} + F_{i,i}\right), \\ \hat{M}_k &= \textstyle\sum_{i=1}^n [Hx]_i[C^{-1}]_{i,k} - \sum_{i=1}^n [Ga'C^{-1}]_{i,k}\frac{\partial}{\partial x_i},\end{aligned}$$

$1 \leq k \leq d$, where A, L_j are defined earlier. The sufficient statistic and estimation algebras are given by

$$\begin{aligned}\mathcal{L}_S &= \{L_0, L_1, L_2, \ldots, L_\ell, M_1, M_2, \ldots, M_d\}_{L.A.}, \\ \mathcal{L}_E &= \{L_0, M_1, M_2, \ldots, M_d\}_{L.A.}.\end{aligned}$$

Let $\phi \in C^\infty(\Re^n)$ and set

$$\begin{aligned}f_i &= \textstyle\sum_{j=1}^n \left(F_{i,j}x_j + [GG']_{i,j}\frac{\partial}{\partial x_j}\phi\right), \quad 1 \leq i \leq n, \\ h_i &= \textstyle\sum_{j=1}^n \left(H_{i,j}x_j + [aG']_{i,j}\frac{\partial}{\partial x_j}\phi\right), \quad 1 \leq i \leq d.\end{aligned} \tag{3.26}$$

Theorem 3.8 *[8]. Suppose (3.26) holds and*

$$g_j = B_j, \quad 1 \leq j \leq \ell, \quad \text{(i.e., independent of } x\text{)}. \tag{3.27}$$

1. If $\phi \in C^\infty(\Re^n)$ *is a solution of*

$$\begin{aligned}&\tfrac{1}{2}\textstyle\sum_{i,j=1}^n \Big([GG']_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(\phi) + [GG']_{i,j}\frac{\partial}{\partial x_i}(\phi)\frac{\partial}{\partial x_j}(\phi) \\ &+2F_{i,j}x_j\tfrac{\partial}{\partial x_i}(\phi)\Big) + \textstyle\sum_{i=1}^n\sum_{j=1}^\ell B_{i,j}u_j\frac{\partial}{\partial x_i}(\phi) \\ &= \tfrac{1}{2}\left(x'Q(u)x + 2m(u)x + \delta(u)\right),\end{aligned}$$

for some $Q(u) = Q'(u) \geq 0, m(u), \delta(u)$, *then* $\mathcal{L}_S$ *is isomorphic to the Lie algebra*

$$\begin{aligned}\hat{\mathcal{L}}_S = \Big\{&\hat{A}_0 - \tfrac{1}{2}\textstyle\sum_{k=1}^d \hat{M}_k^2 - \frac{1}{2}\left(x'Q(u)x + 2m(u)x\right. \\ &\left.+\delta(u)\right), L_1, L_2, \ldots, L_\ell, \hat{M}_1, \hat{M}_2, \ldots, \hat{M}_d\Big\}_{L.A.}.\end{aligned}$$

Moreover, if $Q(u), m(u), \delta(u)$ *are independent of the control* u *then* $\hat{\mathcal{L}}_S$ *is finite-dimensional with basis*

$$\begin{aligned}\hat{\mathcal{L}}_S = Span\Big\{&\hat{A}_0 - \tfrac{1}{2}\textstyle\sum_{k=1}^d \hat{M}_k^2 - \frac{1}{2}\left(x'Qx + 2mx\right. \\ &\left.+\ \delta\right), \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_n}, x_1, x_2, \ldots, x_n, 1\Big\}.\end{aligned} \tag{3.28}$$

2. If $\phi \in C^\infty(\Re^n)$ *is a solution of*

$$\begin{aligned}&\tfrac{1}{2}\textstyle\sum_{i,j=1}^n \Big([GG']_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(\phi) + [GG']_{i,j}\frac{\partial}{\partial x_i}(\phi)\frac{\partial}{\partial x_j}(\phi) \\ &+2F_{i,j}x_j\tfrac{\partial}{\partial x_i}(\phi)\Big) = \tfrac{1}{2}\left(x'Qx + 2mx + \delta\right),\end{aligned}$$

for some $Q(u) = Q' \geq 0, m, \delta$, *then* $\mathcal{L}_E$ *is finite-dimensional isomorphic to the Lie algebra*

$$\begin{aligned}\hat{\mathcal{L}}_E = \Big\{&\hat{A}_0 - \tfrac{1}{2}\textstyle\sum_{k=1}^d \hat{M}_k^2 - \frac{1}{2}\left(x'Qx + 2mx + \delta\right), \\ &\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_d\Big\}_{L.A.},\end{aligned}$$

with basis given by (3.28).

3.4 The Linear Affine Control Case

Theorem 3.9 *(Multidimensional Case). Consider the control system (2.3), (2.4), with*

$$f = Fx, \quad g_j = B_j x \quad 1 \leq j \leq \ell, \quad h(x) = Hx, \quad \sigma(x) = I_n.$$

Then

$$\mathcal{L}_S = Span \left\{ \{\frac{\partial^2}{\partial x_i \partial x_j}\}_{i,j=1}^n, \{x_i \frac{\partial}{\partial x_j}\}_{i,j=1}^n, \right.$$
$$\left. \{\frac{\partial}{\partial x_i}\}_{i=1}^n, \{x_i x_j\}_{i,j=1}^n, 1 \right\}.$$

The non-zero commutative relations are

$$\left[\tfrac{\partial^2}{\partial x_i x_j}, x_k \tfrac{\partial}{\partial x_m}\right] = \delta_{k,j} \tfrac{\partial^2}{\partial x_i \partial x_m} + \delta_{k,i} \tfrac{\partial^2}{\partial x_j \partial x_m},$$
$$\left[\tfrac{\partial^2}{\partial x_i \partial x_j}, x_k x_m\right] = (\delta_{k,j}\delta_{m,i} + \delta_{k,i}\delta_{m,j})$$
$$+ (\delta_{k,j} + x_m + \delta_{m,j} x_k) \tfrac{\partial}{\partial x_i} + (\delta_{k,i} x_m + \delta_{m,i} x_k) \tfrac{\partial}{\partial x_j},$$
$$\left[x_i \tfrac{\partial}{\partial x_j}, \tfrac{\partial}{\partial x_m}\right] = -\delta_{i,k} \tfrac{\partial}{\partial x_j}, \ \left[x_i \tfrac{\partial}{\partial x_j}, x_k x_m\right] = \delta_{k,j} x_i x_m$$
$$+\delta_{m,j} x_i x_k, \ \left[\tfrac{\partial}{\partial x_i}, x_k x_m\right] = \delta_{k,i} x_m + \delta_{m,i} x_k,$$

where $1 \leq i, j, k, m \leq n, \delta_{i,j} = 1$ if $i = j$ and zero otherwise.

Proof. Follows from the commutative relations.

4 Additional Generalizations

Consider the nonlinear control system (2.3), (2.4). Here we are interested in minimizing (over $u(\cdot) \in \mathcal{U}_{ad}$) the exponential-of-integral cost function $J^\theta(u)$:

$$J^\theta(u) = E^u \left\{ \exp\left(\theta \int_0^T \ell(x(t), u(t,y))dt + \theta\varphi(x(T)) \right) \right\},$$

where $\theta > 0$. Similar to Theorem 2.4, the information state approach to this control problem yields:

$$J^\theta(u^*) = \inf_{u \in \mathcal{U}_{ad}} E \left\{ \int_{\Re^n} \exp(\theta\varphi(x)) \pi^\theta(x,T)dx \right\}. \quad (4.29)$$

Here, $\{\pi^\theta(x,s); 0 \leq s \leq t\}$, is an information state; it is a solution of a certain controlled Feynman-Kac stochastic PDE. In particular, when

$$\ell(x,u) = \ell_0(x) + \sum_{j=1}^{\ell} \ell_j(x) u_j^2, \quad (4.30)$$

we have

$$\begin{aligned} \pi^\theta(x,t) &= \pi(x,0) + \textstyle\int_0^t (L_0 + \theta\ell_0) \pi^\theta(x,s)ds \\ &+ \textstyle\sum_{j=1}^{\ell} \int_0^t L_j \pi^\theta(x,s) u_j(s,y) ds \\ &+ \textstyle\sum_{j=1}^{\ell} \int_0^t \theta\ell_j \pi^\theta(x,s) u_j^2(s,y) ds \\ &+ \textstyle\sum_{j=1}^{d} \int_0^t h_j \pi^\theta(x,s) \circ dy_j(s). \end{aligned} \quad (4.31)$$

The sufficient statistic algebra is

$$\begin{aligned} \mathcal{L}_S^\theta \doteq \{ &L_0^\theta, L_1, L_2, \ldots, L_\ell, \theta\ell_1, \theta\ell_2, \ldots, \theta\ell_\ell, \\ &h_1, h_2, \ldots, h_d\}_{L.A.}, \end{aligned} \quad (4.32)$$

where $L_0^\theta = L_0 + \theta\ell_0$. Clearly, $\mathcal{L}_S^\theta$, can be used to classify nonlinear systems with finite-dimensional controllers. An important observation announced in [6], is that we can solve the so-called inverse control problem, by choosing the zeroth order differential operators, $\ell_0, \ell_1, \ell_2, \ell_\ell$, to force $\mathcal{L}_S^\theta$, to be finite-dimensional. When ℓ_0 =polynomial in $(x_1, x_2, \ldots, x_n)$ of degree at most two, and ℓ_j =Constant, $1 \leq j \leq \ell$, we obtain finite-dimensional controllers for the classes of nonlinear systems discussed in earlier sections.

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