Minimax H^∞ Control of Stable Distributed Systems

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Abstract

We study the distributed suboptimal full information H^∞ problem for a stable well-posed linear system with control u, disturbance w, state x, and output y. The problem is to find all suboptimal compensators $u = \mathcal{U}w$, i.e., compensators that make the norm of the closed loop input/output map from w to y less than a given constant γ. Define $Q(x_0, u, w) = \|y\|^2_{L^2(\mathbf{R}^+;Y)} - \gamma^2\|w\|^2_{L^2(\mathbf{R}^+;W)}$. We first choose u to minimize Q and w to maximize $\min_u Q$, and show that the minimizing control and maximizing disturbance can be written in a well-posed feedback form if and only if the input/output map of the system has a (J, S)-inner-outer factorization. We then give a parameterization of all suboptimal compensators. In the generality that we allow it is possible that we have to include a feed-forward term in the compensator. This term is not present in the classical continuous time theory, but it is well-known from the discrete time theory.

1 Introduction

In this note we study the full information H^∞ problem for a distributed parameter system. In our setting the transfer functions need not be rational or meromorphic; they are just plain H^∞ without any extra smoothness. We follow a route based on spectral factorization that is well-known from the theory for the finite dimensional rational H^∞ problem, but there is a lot of details that have to be filled in, and it is not at all obvious how one should proceed at each stage. We believe that the final result is interesting even in the finite dimensional setting, since the point of view that we have adopted is somewhat different from the usual one. In particular, all our proofs are adaptations of standard frequency domain proofs, but they are recast in a state space setting, and some state space ingredients have been added. The key addition is the state space factorization of the Hankel operator induced by the input/output map as the product of the controllability and observability maps. This makes it possible to connect the state space and the frequency domain theories to each other.

The problem that we study is of the following type. We let

$$\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & [\mathcal{B}_1 \; \mathcal{B}_2] \\ \left[\begin{smallmatrix} \mathcal{C} \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ 0 & I \end{smallmatrix}\right] \end{bmatrix} \tag{1}$$

be a *stable well-posed linear system* with control input space U, disturbance input space W, state space H, and output space $Y \times W$ (see [Staffans 1997a, Section 2] or Weiss [1994ab] for reviews of well-posed linear systems.) Here $\mathcal{A}$ is the semigroup around which the system is built, $\mathcal{B}_1$ and $\mathcal{B}_2$ are the two controllability maps corresponding to the two different inputs (they map a past control $u \in L^2(\mathbf{R}^-; U)$ and disturbance $w \in L^2(\mathbf{R}^-; W)$ into the present state $x(0)$), $\mathcal{C}_1$ is the observability map of the first output (it maps the present state $x(0)$ into the future output $y \in L^2(\mathbf{R}^+; Y)$, and $\mathcal{D}_{11}$ and $\mathcal{D}_{12}$ are the corresponding input/output maps. The controlled state at time $t \geq 0$ of Ψ with initial time zero, initial value $x_0 \in H$, control $u \in L^2(\mathbf{R}^+; U)$, and disturbance $w \in L^2(\mathbf{R}^+; W)$ is given by $x(t) = \mathcal{A}(t)x_0 + \mathcal{B}_1\tau(t)\pi_+u + \mathcal{B}_2\tau(t)\pi_+u$ (see the notations at the end of this section), and the first output $y \in L^2(\mathbf{R}^+; Y)$ of Ψ is given by $y = \mathcal{C}_1x_0 + \mathcal{D}_{11}\pi_+u + \mathcal{D}_{12}\pi_+w$. The second output is a copy of the disturbance w; this output has been added in order to give the controller direct access to the disturbance and to simplify the final formulas. We let the diagram drawn in Figure 1 represent these input/state/output relations. Observe the exact locations of the different arrows: an initial value or input acts on the operators located in the corresponding column, and a final state or out-

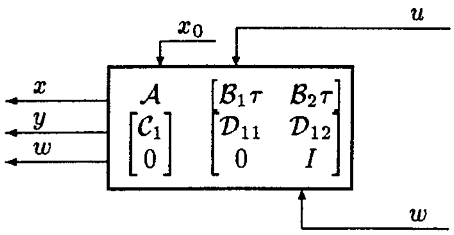

Figure 1: Input-state-output diagram for Ψ

put sums all the contributions to the corresponding row. The goal is to find a simple parameterization of all possible feedback/feed-forward controllers $u = \mathcal{U}w$ that make the norm of the closed loop input/output map from the second input $w \in L^2(\mathbf{R}^+;W)$ to the first output $y \in L^2(\mathbf{R}^+;Y)$ strictly less than a given constant γ. Such a controller is called *suboptimal.*

We use the standard approach, and define the indefinite cost function

$$\begin{aligned} Q(x_0,u,w) &= \|y\|^2_{L^2(\mathbf{R}^+;Y)} - \gamma^2\|w\|^2_{L^2(\mathbf{R}^+;W)} \\ &= \langle \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right], J \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] \rangle_{L^2(\mathbf{R}^+;Y\times W)}, \end{aligned} \qquad (2)$$

where $J = \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$. We first look for the control $u^{\mathrm{crit}} \in L^2(\mathbf{R}^+;U)$ that minimizes Q under the worst possible disturbance $w^{\mathrm{crit}} \in L^2(\mathbf{R}^+;W)$. This problem has a well-defined solution whenever there exists a suboptimal compensator. The critical control/disturbance pair can be written in a well-posed feedback/feed-forward form if and only if the combined input/output operator $\mathcal{D} = \left[\begin{smallmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ 0 & I \end{smallmatrix}\right]$ has a (J,S)*-inner-outer factorization*, where $S = S^*$ is an invertible operator in $U \times W$. With the help of the feedback system that we get in this way we can give a complete parameterization of all suboptimal controllers.

Throughout this note we assume the existence of a (J,S)-inner-outer factorization of $\mathcal{D}$. The existence of such a factorization is necessary only in the sense that without it there does not exist a well-posed suboptimal central controller. In particular, there do exist full information suboptimal H^∞ problems that cannot be solved directly with the method presented here.

In general it is not possible to split a well-posed linear system into a feed-forward part and a strictly causal part. This means that without any further assumptions it is impossible to make any statements about the feedback/feed-forward nature of the central controller that we have obtained. However, if we add an extra regularity assumption, then it becomes possible to investigate the feed-forward part of the optimal solution. Furthermore, under this assumption it is possible to show that the Riccati operator satisfies a nonstandard Riccati equation, and that the feedback operator can be computed from the Riccati operator.

We use the following set of notations.

$\tau(t)$: The time shift operator $\tau(t)u(s) = u(t+s)$.

π_E: $(\pi_E u)(s) = \begin{cases} u(s) & \text{if } s \in E, \\ 0 & \text{if } s \notin E \end{cases}$, here $E \subset \mathbf{R}$ is an interval .

$\pi_+,\ \pi_-$: $\pi_+ = \pi_{\mathbf{R}^+}$ and $\pi_- = \pi_{\mathbf{R}^-}$.

$TI(U;Y)$: The set of bounded linear time-invariant operators from $L^2(\mathbf{R};U)$ into $L^2(\mathbf{R};Y)$.

$TIC(U;Y)$: The set of causal operators in $TI(U;Y)$.

$\mathcal{B},\mathcal{C}$: $\mathcal{B} = [\mathcal{B}_1\ \ \mathcal{B}_2]$, $\mathcal{C} = \left[\begin{smallmatrix} \mathcal{C}_1 \\ 0 \end{smallmatrix}\right]$.

$\mathcal{D}, J$: $\mathcal{D} = \left[\begin{smallmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ 0 & I \end{smallmatrix}\right]$, $J = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$.

$A >> 0$: $A - \epsilon I$ is positive definite for some $\epsilon > 0$.

2 The Minimax Solution

Definition 1 *We call $\mathcal{U} \in TIC(W;U)$ a (uniformly) suboptimal controller for the system Ψ in (1) with the cost function $Q(x_0,u,w)$ defined in (2) iff*

$$\|(\mathcal{D}_{11}\mathcal{U} + \mathcal{D}_{12})\pi_+ w\|^2_{L_2(\mathbf{R}^+;Y)} \leq (\gamma^2 - \epsilon)\|w\|^2_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$ and all $w \in L^2(\mathbf{R}^+;W)$.

Note that $(\mathcal{D}_{11}\mathcal{U} + \mathcal{D}_{12})\pi_+ w$ represents the input/output map from w to y if we take u to be $u = \mathcal{U}\pi_+ w$. Thus, this condition says that the norm of the closed loop input/output map from w to y should be less than γ in the presence of the controller $\mathcal{U}$.

Definition 2 *Let Ψ, Q, and J be defined as above. The system Ψ is called minimax J-coercive iff for each $x_0 \in H$ and $w \in L^2(\mathbf{R}^+;W)$ the function $u \mapsto Q(x_0,u,w)$ is uniformly convex on $L^2(\mathbf{R}^+;U)$, and for each $x_0 \in H$ the function $w \mapsto \inf_{u \in L^2(\mathbf{R}^+;U)} Q(x_0,u,w)$ is uniformly concave on $L^2(\mathbf{R}^+;W)$.*

The following results are almost immediate:

Lemma 3 *Suppose that the function $u \mapsto Q(x_0,u,w)$ is uniformly convex on $L^2(\mathbf{R}^+;U)$. Then a necessary condition for the existence of a suboptimal controller is that Ψ is minimax J-coercive.*

Lemma 4 *Let Ψ be a stable, well-posed, and minimax J-coercive linear system.*

(i) *For each fixed* $x_0 \in H$ *and* $w \in L^2(\mathbf{R}^+; W)$ *there is a unique function* $u^{\min}(x_0, w)$ *that minimizes* $Q(x_0, u, w)$ *with respect to* u. *We denote the corresponding state and output by* $x^{\min}(x_0, w)$ *and* $y^{\min}(x_0, w)$, *respectively.*

(ii) *For each fixed* $x_0 \in H$ *there is a unique function* $w^{\text{crit}}(x_0)$ *that maximizes* $Q(x_0, u^{\min}(x_0, w), w)$ *with respect to* w. *Define* $x^{\text{crit}}(x_0) = x^{\min}(x_0, w^{\text{crit}})$, $u^{\text{crit}}(x_0) = u^{\min}(x_0, w^{\text{crit}})$, *and* $y^{\text{crit}}(x_0) = y^{\min}(x_0, w^{\text{crit}})$.

(iii) *The minimax cost* $Q(x_0, x^{\text{crit}}(x_0), w^{\text{crit}}(x_0))$ *is a nonnegative quadratic function of* x_0, *and it can be written in the form* $Q(x_0, x^{\text{crit}}(x_0), w^{\text{crit}}(x_0)) = \langle x_0, \Pi x_0 \rangle$, *where* $\Pi = \Pi^* \geq 0$. *This operator* Π *is called the Riccati operator of* Ψ.

3 A Feedback/Feed-Forward Representation

The construction in the preceding section gives us a unique minimax control/disturbance pair. In order to get a feedback/feed-forward representation for this pair we need to compute a (J, S)-inner-outer factorization of $\mathcal{D}$:

Definition 5 *Let* $S = S^* \in \mathcal{L}(U \times W)$.

(i) *The operator* $\mathcal{N} \in TIC(U; Y)$ *is* (J, S)*-inner iff* $\mathcal{N}^* J \mathcal{N} = S$.

(ii) *The operator* $\mathcal{X} \in TIC(U; Y)$ *is outer if the image of* $L^2(\mathbf{R}^+; U)$ *under* $\mathcal{X}\pi_+$ *is dense in* $L^2(\mathbf{R}^+; Y)$.

(iii) *The factorization* $\mathcal{D} = \mathcal{N}\mathcal{X}$ *is a* (J, S)*-inner-outer factorization of* $\mathcal{D} \in TIC(U; Y)$ *if* $\mathcal{N} \in TIC(U; Y)$ *is* (J, S)*-inner and* $\mathcal{X} \in TIC(U)$ *is outer.*

Theorem 6 *Let* Ψ *be a stable, well-posed, and minimax* J*-coercive linear system.*

(i) *Suppose that* $\mathcal{D}$ *has a* (J, S)*-inner outer factorization* $\mathcal{N}\mathcal{X}$. *Then* S *is invertible in* $\mathcal{L}(U \times W)$ *and* $\mathcal{X}$ *is invertible in* $TIC(U \times W)$.[1] *Define* $\mathcal{M} = \mathcal{X}^{-1}$ *and* $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix} = \begin{bmatrix} -S^{-1}\pi_+\mathcal{N}^* J \mathcal{C} & (I - \mathcal{X}) \end{bmatrix}$. *Then* $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$ *is an admissible stable state feedback pair for* Ψ, *i.e.,*

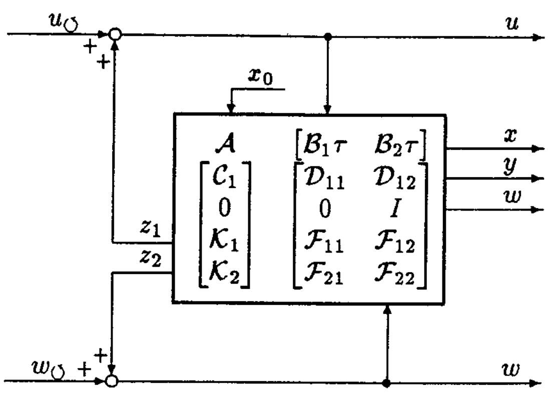

Figure 2: Closed loop feedback connection

the feedback connection drawn in Figure 2 defines a well-posed linear system $\Psi_\circlearrowleft$, *given by*

$$\Psi_\circlearrowleft = \begin{bmatrix} \mathcal{A}_\circlearrowleft & \mathcal{B}_\circlearrowleft \\ \begin{bmatrix} \mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\circlearrowleft \\ \mathcal{F}_\circlearrowleft \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{M}\tau\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix} \mathcal{C} + \mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{N} \\ \mathcal{M} - I \end{bmatrix} \end{bmatrix}.$$

Moreover, the state and outputs of this closed loop system are equal to $x^{\text{crit}}(t, x_0)$, $\begin{bmatrix} y^{\text{crit}}(x_0) \\ w^{\text{crit}}(x_0) \end{bmatrix}$, *and* $\begin{bmatrix} u^{\text{crit}}(x_0) \\ w^{\text{crit}}(x_0) \end{bmatrix}$, *respectively, if we take the two closed loop inputs* $u_\circlearrowleft$ *and* $w_\circlearrowleft$ *to be zero. The Riccati operator* Π *of* Ψ *can be written in the following alternative forms:*

$$\begin{aligned} \Pi &= \mathcal{C}^* J \mathcal{C} - \mathcal{K}^* S \mathcal{K} = \mathcal{C}^* \left(J - J\mathcal{N}S^{-1}\pi_+\mathcal{N}^* J\right)\mathcal{C} \\ &= \mathcal{C}^* J \mathcal{C}_\circlearrowleft = \mathcal{C}_\circlearrowleft{}^* J \mathcal{C}_\circlearrowleft. \end{aligned}$$

(ii) *Conversely, if* $\begin{bmatrix} y^{\text{crit}}(x_0) \\ w^{\text{crit}}(x_0) \\ u^{\text{crit}}(x_0) \\ w^{\text{crit}}(x_0) \end{bmatrix}$ *is equal to the output of some stable state feedback perturbation* $\Psi_\circlearrowleft$ *of* Ψ *with initial value* x_0, *initial time* 0, *zero control, zero disturbance, and some admissible stable state feedback pair* $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$, *then there exists an invertible self-adjoint operator* $S \in \mathcal{L}(U)$ *such that* $\mathcal{N}\mathcal{X}$ *is a* (J, S)*-inner-outer factorization of* $\mathcal{D}$, *where* $\mathcal{M} = (I - \mathcal{F})$ *and* $\mathcal{N} = \mathcal{D}\mathcal{X}^{-1}$. *Moreover,* $\mathcal{K}$ *is given by* $\mathcal{K} = -S^{-1}\pi_+\mathcal{N}^* J \mathcal{C}$.

(iii) *If* $\left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] = \mathcal{C}_\circlearrowleft x_0 + \mathcal{D}_\circlearrowleft \pi_+ \left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right]$ *is the output of the minimax closed loop system* $\Psi_\circlearrowleft$ *with initial state* $x_0 \in H$, *control* $u_\circlearrowleft \in L^2(\mathbf{R}^+; U)$, *and disturbance* $w_\circlearrowleft \in L^2(\mathbf{R}^+; W)$, *then the closed loop cost*

[1] This means that $\mathcal{X}$ is an invertible S-spectral factor of $\mathcal{D}^* J \mathcal{D}$.

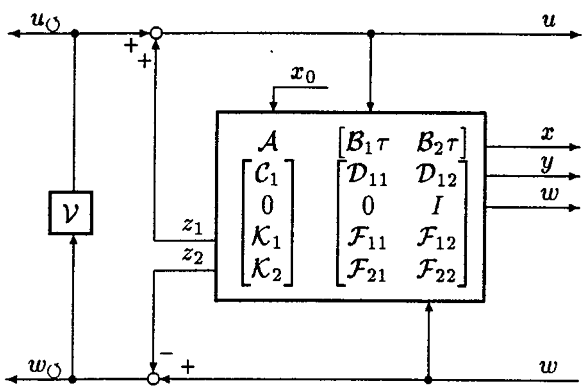

Figure 3: Parameterization of all suboptimal controllers

$Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft)$ is given by

$$\begin{aligned} &Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft) \\ &= \|y\|^2_{L^2(\mathbf{R}^+;Y)} - \gamma^2\|w\|^2_{L^2(\mathbf{R}^+;W)} \\ &= \langle x_0, \Pi x_0\rangle_H \\ &\quad + \left\langle \left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right], S\left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right]\right\rangle_{L^2(\mathbf{R}^+;U\times W)}. \end{aligned}$$

Hypothesis 7 *Throughout the remainder of this note we let Ψ be a stable, well-posed, and minimax J-coercive linear system, and suppose that the two equivalent conditions (i) and (ii) in Theorem 6 hold.*

Because of the final formula for the cost given in part (iii) of Theorem 6, we call S the *sensitivity operator* associated with the given factorization. Observe that this formula rewrites the cost in terms of the initial state x_0 and the two closed loop inputs $u_\circlearrowleft$ and $w_\circlearrowleft$ in Figure 2. This formula plays a key role in the subsequent development.

Lemma 8 *It is possible to choose the factorization in Theorem 6 in such a way that the cross terms in the sensitivity operator S vanish and S is of the type $S = \left[\begin{smallmatrix} S_{11} & 0 \\ 0 & S_{22}\end{smallmatrix}\right]$, where $S_{11} >> 0$ and $S_{22} << 0$.*

4 The Central Controller

In order to get a *central controller* we have to cut the bottom feedback loop in Figure 2, or equivalently, we change the direction of the bottom line to get Figure 3 with $\mathcal{V} = 0$.

Theorem 9 *Suppose that W is finite-dimensional, and that the factorization in Theorem 6 has been chosen in such a way that the conditions on S listed in Lemma 8 hold.*

(i) The connection drawn in Figure 3 with $\mathcal{V} = 0$ defines a well-posed linear system, and the input/output map $\mathcal{U}$ from w to u is a suboptimal controller. We call this controller the central controller induced by the factorization $\mathcal{N}\mathcal{X}$.

(ii) If $\mathcal{V} \in TIC(W;U)$ in Figure 3 satisfies

$$\|S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}\| < 1,$$

then the input/output map $\mathcal{U}$ from w to u is a suboptimal controller.

(iii) Every suboptimal controller $\mathcal{U}$ has a unique representation of the type described in part (ii).

5 Separation of Feed-Forward Terms

In order to talk about feed-forward terms we need the following definition (cf. [Weiss 1994a, Theorem 5.8]):

Definition 10 *(i) An operator $\mathcal{D} \in TIC(V;Y)$ is called regular iff the strong mean*

$$Dv_0 = \lim_{\lambda\to+\infty} \widehat{\mathcal{D}}(\lambda)v_0$$

exists for every $v_0 \in V$; here λ tends to $+\infty$ along the positive real axis and $\widehat{\mathcal{D}}$ is the transfer function (the distribution Laplace transform) of $\mathcal{D}$.

(ii) The operator $D\colon V \to Y$ defined above is called the feed-through (or feed-forward) operator of $\mathcal{D}$.

(iii) A regular operator $\mathcal{D} \in TIC(V;Y)$ is called strictly proper iff its feed-through operator vanishes.

(iv) We say that $\mathcal{D}$ is regular together with its adjoint iff, in addition to (i), the strong mean $\lim_{\lambda\to+\infty} \widehat{\mathcal{D}}^*(\lambda)y_0$ exists for every $y_0 \in Y$. (This limit is equal to D^*y_0 whenever it exists.)

Hypothesis 11 *Throughout the rest of this note we suppose that W is finite-dimensional and that both $\mathcal{D}$ and $\mathcal{X}$ are regular together with their adjoints (for at least one factorization, hence for all factorizations of $\mathcal{D}$).*

Lemma 12 *There is a unique $(J, \widetilde{S})$-inner-outer factorization in Theorem 6 for which $\mathcal{F}$ is strictly proper (i.e., there is "no feed-forward term inside the feedback loop").*

Theorem 13 *Let $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ be the special (strictly proper) $(J, \widetilde{S})$-inner-outer factorization of $\mathcal{D}$ described in Lemma 12.*[2]

[2]This factor will not, in general, be of the type described in Lemma 8.

(i) *The factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ induces a suboptimal central controller if and only if $\widetilde{S}_{22} << 0$. This is the only possible strictly proper suboptimal central controller.*

(ii) *If the feed-through operator D_{11} of $\mathcal{D}_{11}$ satisfies $D_{11}^*D_{11} >> 0$, then $\widetilde{S}_{11} >> 0$, and the factorization for which the feed-through operator of $\mathcal{F}$ is given by $F = \begin{bmatrix} 0 & -\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \\ 0 & 0 \end{bmatrix}$ induces a uniformly suboptimal central controller. The sensitivity operator of this factorization is $\begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \end{bmatrix}$, where $\widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} << 0$. In particular, it is of the type described in Lemma 8, and Theorem 9 applies.*

6 The Riccati Equation

Under the same regularity assumption as in the preceding section it is possible to show that the Riccati operator satisfies a Riccati equation, and that the feedback operator can be computed from the Riccati operator (the latter statement is true without the extra regularity assumption). To formulate this result we need a few more facts about the general theory about well-posed linear systems. More precisely, it is known (see, e.g., Salamon [1987 1989] and Weiss [1994ab]) that, in the case where $u \in W^{1,2}(\mathbf{R}^+;U)$, $w \in W^{1,2}(\mathbf{R}^+;W)$, and $Ax(0)+B_1u(0)+B_2w(0) \in H$ (where A is the generator of $\mathcal{A}$ and B_1 and B_2 are the two control operators; see the formula below), the input-state-output relations of the extended system corresponding to the factorization in part (ii) of Theorem 13 can be written in the form (for all $t \in \mathbf{R}^+$)

$$\begin{aligned} x'(t) &= Ax(t) + B_1u(t) + B_2w(t), \\ y(t) &= \overline{C}_1x(t) + D_{11}u(t) + D_{12}w(t), \\ z_1(t) &= \overline{K}_1x(t) + F_{12}w(t), \\ z_2(t) &= \overline{K}_2x(t), \end{aligned}$$

where $F_{12} = -\widetilde{S}_{11}^{-1}\widetilde{S}_{12}$. The operators $\overline{C}_1$, $\overline{K}_1$, and $\overline{K}_2$ are the Weiss extensions of the observation operators C, K_1, and K_2 defined on dom(A), i.e.,

$$\begin{aligned} \overline{C}_1 &= \lim_{\lambda\to+\infty} \lambda C_1(\lambda I - A)^{-1}, \\ \overline{K}_i &= \lim_{\lambda\to+\infty} \lambda K_i(\lambda I - A)^{-1}, \quad i = 1,2. \end{aligned}$$

The adjoints B_1^* and B_2^* of the operators B_1 and B_2 are defined on dom(A*), and they are extended in a similar way into $\overline{B}_i^* = \lim_{\lambda\to+\infty} \lambda B_i^*(\lambda I - A^*)^{-1}$, $i = 1,2$. Moreover, we define

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \quad \overline{B}^* = \begin{pmatrix} \overline{B}_1^* \\ \overline{B}_2^* \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & D_{12} \\ 0 & I \end{pmatrix},$$

$$C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \end{pmatrix}.$$

Theorem 14 *The Riccati operator Π and the feedback operator K satisfy the following two equations for all $x_0 \in$ dom(A) and $x_1 \in$ dom(A):*

$$\begin{aligned} \langle Ax_0, \Pi x_1\rangle_H + \langle x_0, \Pi A x_1\rangle_H &\\ = -\langle Cx_0, JCx_1\rangle_Y &+ \langle Kx_0, SKx_1\rangle_U, \\ Kx_0 = -S^{-1}(I - F^*)^{-1} &\left(\overline{B}^*\Pi + D^*JC\right)x_0. \end{aligned}$$

It is possible to write out the two components K_1 and K_2 of K explicitly in terms of the data: A substitution into the formula in Theorem 14 gives

$$\begin{aligned} K_1 &= -\widetilde{S}_{11}^{-1}\left(\overline{B}_1^*\Pi + D_{11}^*C_1\right), \\ K_2 &= -\left(\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12}\right)^{-1}\left(\overline{B}_2^*\Pi + D_{12}^*C_1 - \widetilde{S}_{21}K_1\right). \end{aligned}$$

Note that both of these operators appear in the algebraic Riccati equation for Π that we get from Theorem 14, but that only K_1 is used in the actual central control, i.e., in the feedback/feed-forward formula

$$u(t) = \overline{K}_1x(t) + F_{12}w(t)$$

for u. The role of K_2 is to reproduce the "worst possible" disturbance $w(t) = \overline{K}_2x(t)$ in feedback form.

We observe that the Riccati equation that we get differs from the usual one in the sense that there is a new parameter $\widetilde{S}$ that does not normally occur in the continuous time case (although it is standard in the discrete time case).[3] This parameter can be computed from the Riccati operator:

Theorem 15 *The sensitivity operator $\widetilde{S}$ can be computed as the strong limit*

$$\widetilde{S}v_0 = D^*JDv_0 + \lim_{\lambda\to\infty} \overline{B}^*\Pi(\lambda I - A)^{-1}Bv_0$$

for each $v_0 \in U \times W$; here λ tends to $+\infty$ along the positive real axis.

For details, see Staffans [1995 1996ab 1997abc], and, in particular, Staffans [1997de].

References

R. F. Curtain and M. Green. Analytic system problems and J-lossless coprime factorization for infinite-dimensional linear systems. *J. Linear Algebra*, 1997. To appear.

M. Green and D. L. L. Limebeer. *Linear Robust Control.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[3] The reader may compare the formulas for K_1 and K_2 given above to those valid in the discrete case; see, e.g., Green and Limebeer [1995]. It is natural to expect a feed-forward term from w to u in our case, too, since the class of well-posed linear systems that we treat is so large that the class of discrete systems can be imbedded in it. See Staffans [1996a].

D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of the American Mathematical Society*, 300:383–431, 1987.

D. Salamon. Realization theory in Hilbert space. *Mathematical Systems Theory*, 21:147–164, 1989.

O. J. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Mathematics of Control, Signals, and Systems*, 8:167–197, 1995.

O. J. Staffans. On the discrete and continuous time infinite-dimensional Riccati equations. *Systems and Control Letters*, 29:131–138, 1996a.

O. J. Staffans. Quadratic optimal control through coprime and spectral factorizations. Reports on Computer Science & Mathematics 178, Åbo Akademi University, Åbo, Finland, 1996b.

O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. To appear in Transactions of American Mathematical Society, 1997a.

O. J. Staffans. Coprime factorizations and well-posed linear systems. Submitted to SIAM Journal on Control and Optimization, 1997b.

O. J. Staffans. Quadratic optimal control of well-posed linear systems. Submitted to SIAM Journal on Control and Optimization, 1997c.

O. J. Staffans. Feedback representations of critical controls for well-posed linear systems. Submitted, 1997d.

O. J. Staffans. On the distributed stable full information H^∞ minimax problem. Submitted, 1997e.

B. van Keulen. *H_∞-Control for Distributed Parameter Systems: A State Space Approach.* Birkhäuser Verlag, Basel Boston Berlin, 1993.

G. Weiss. Transfer functions of regular linear systems. Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342:827–854, 1994a.

G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994b.

M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. To appear in Mathematics of Control, Signals, and Systems, 1997.