Emergency Control of Unstable Behavior of Nonlinear Systems Induced by Fault

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Abstract

This paper presents the impulse control approach intended to urgently return to the stability basin the system states affected by abrupt changes in certain system coefficients on a short time interval. Because of its short duration, the modeling of both the fault and controller involves δ - functions significantly simplifying analysis and control of fault phenomena. The design of an impulse controller is based on the technique for computing fault-induced jumps of the system states, which is described in the paper. A sample impulse controller instantaneously returning states of a Van-der-Pol system to the stability basin is designed.

1. Introduction

The impulse control technique based on δ - functions as controls was applied to the optimal control problems in the field of spacecraft navigation [1] and heat conduction [2], the filtering problems over discontinuous observations [3], and others. This paper makes an attempt to extend the application domain of impulse control to the problems of fault description and compensation as well as stability control.

An impulse control approach that is applied to a dynamic system in the case of urgent necessity to change back a system state affected by fault is developed. It is assumed that system fault significantly affects the operation of a relatively small subsystem of the initial system on a short time interval and clears up at the end of this interval. Such fault results in pseudoimpulsive behavior of a relatively small number of the system coefficients, which abruptly increase to the peak values and abruptly return to the nominal values, as it occurs in transient stability problems for power systems. The pseudoimpulsive coefficients can be modeled by δ - functions. Thus, the initial system-governing equation becomes the equation in distributions, which describes the system fault. The solution of an equation in distributions is defined as a vibrosolution [4–6], whose jumps occur at points where δ - functions are activated. A number of examples, where jumps of the system state can be computed analytically, are given in the paper. Otherwise, system state jumps are computed through numerical integration of a subsystem, which is significantly reduced in comparison with the initial one.

An impulse control method is designed to urgently return the system states affected by fault to the stability basin. For this purpose, an impulse controller introduces δ - functions into the system equation. This method is applied to a Van-der-Pol system, where the impulse controller generates a jump of the system state into the stability basin, thus preventing the system state from transition to infinity.

The paper is organized as follows. A fault model is described in Section 2. The basic technique for computing jumps in system states affected by fault is given in Section 3. A number of examples where analytic computation of jumps is possible are given in Section 4. The impulse control method is presented in Section 5 and applied to a Van-der-Pol system in Section 6. Properties of the proposed method are discussed in Section 7. Section 8 concludes this study.

2. Fault Modeling

Let us first describe the application of δ - functions to modeling of a system fault on a short time interval.

Consider an equation governing a dynamic system

$$\dot{x}(t) = k_1(t)f_1(x,t) + k_2(t)f_2(x,t) + \ldots + k_n(t)f_n(x,t),$$

$$x(t_0) = x_0, \quad x \in R^N, \quad f_i(x,t) \in R^N,$$

$$k_i(t) \in R^{N\times N}, \quad i = 1, \ldots n \qquad (1)$$

Assume that, due to a short system fault, a relatively small number of the coefficients $k_1(t), k_2(t), \ldots, k_n(t)$, say $k_1(t), \ldots, k_m(t)$, change in a pseudoimpulsive manner on a short time interval $[t_0, t_0+\Delta t]$. Namely, the coefficients $k_1(t), \ldots, k_m(t)$ abruptly increase to their peak values and return to the pre-fault values. This induces abrupt changes in the system state on the interval $[t_0, t_0+\Delta t]$. The problem is to find the post-fault system state $x(t_0+\Delta t)$, or the system state jump $x(t_0+\Delta t) - x(t_0)$, provided that a pre-fault state $x(t_0)$ is given. The determination of the post-fault system state or system state jump is necessary for forming the impulse control.

Suppose that a fault affects coefficients only from a small "fault" subsystem of the initial system. Because of lack of accurate knowledge and observation of the faulted coefficients $k_1(t), \ldots, k_m(t)$, these coefficients are represented as δ - functions with the corresponding intensities, which are assumed the peak values of the faulted coefficients or estimated using a data record of the fault behavior. Such modeling of pseudoimpulsive behavior of the faulted coefficients is physically motivated and simplifies computation of state jumps.

3. Computation of System State Jumps

The coefficients $k_1(t), \ldots, k_m(t)$ of (1) are replaced by δ - functions with intensities $\mu_1, \ldots, \mu_m$. The intensities are measured or computed as $\mu_j = M_j \Delta t$, where $M_j = \sup k_j(t)$, $t \in [t_0, t_0+\Delta t]$. Then, the equation (1) takes the form:

$$\dot{x}(t) = [\mu_1 f_1(x,t) + \ldots + \mu_m f_m(x,t)]\delta(t-t_0) +$$
$$k_{m+1}(t) f_{m+1}(x,t) + \ldots + k_n(t) f_n(x,t),$$
$$x(t_0) = x_0, \tag{2}$$

where $k_{m+1}(t), \ldots, k_n(t)$ are nonimpulsive coefficients. Let us note that the equation (2) is an equation in distributions, whose solution is a discontinuous function of bounded variation. The solution can be defined and its jumps can be computed by virtue of the theory given in [4–6], whose application to the equation (2) yields the following propositions.

Proposition 1. Let 1) the functions $\mu_1 f_1(x,t), \ldots, \mu_m f_m(x,t)$, $k_{m+1}(t) f_{m+1}(x,t)$, $\ldots$, $k_n(t) f_n(x,t)$ be piecewise continuous in x, t and satisfy the one-side Lipschitz condition in x [7],
2) functions $f_1(x,t)$, $\ldots$, $f_m(x,t)$ have piecewise continuous derivatives in x and t: $\partial f_1(x,t)/\partial x$, $\partial f_1(x,t)/\partial t$, $\ldots$, $\partial f_m(x,t)/\partial x$, $\partial f_m(x,t)/\partial t$, and
3) the $N \times N$ - dimensional system in differentials

$$\frac{d\xi(z,\omega,\mu,u,s)}{du} = \mu_1 f_1(\xi,s) + \ldots + \mu_m f_m(\xi,s),$$
$$\xi(\omega) = z, \tag{3}$$

where $\mu = (\mu_1, \ldots, \mu_m)$ is a vector of intensity matrices, be solvable for arbitrary initial values $\omega \in R^N$, $z \in R^N$ inside a cone of positive directions $K = \{u \geq \omega \mid u_i \geq \omega_i, i = 1, \ldots, N\}$ and $s \geq t_0$. *Then:* there exists the only solution $x(t)$ (called a vibrosolution) that is the only limit in the $*-$weak topology of the bounded variation functions space

$$* - \lim x^k(t) = x(t),$$

for all pre-limiting solutions $x^k(t)$ corresponding to absolutely continuous nondecreasing approximations $u^k(t)$ of a Heaviside function $\chi(t-t_0)$, $d\chi(t-t_0)/dt = \delta(t-t_0)$:

$$\dot{x}^k(t) = [\mu_1 f_1(x^k,t) + \ldots + \mu_m f_m(x^k,t)]\dot{u}^k(t) +$$
$$+ k_{m+1}(t) f_{m+1}(x^k,t) + \ldots + k_n(t) f_n(x^k,t),$$
$$x^k(t_0) = x_0.$$

Remark. The $*-$weak convergence in the bounded variation functions space

$$* - \lim x^k(t) = x(t), \quad t \geq t_0,$$

takes place if and only if the following conditions hold
1) $\lim \| x^k(t_0) - x(t_0) \| = 0, \quad t \geq t_0,$
2) $\lim \| x^k(t) - x(t) \| = 0, \quad t \geq t_0$, in all continuity points of the function $x(t)$,
3) $\sup_k Var[t_0,T] x^k(t) < \infty$ for any $T \geq t_0$, where $Var[a,b] f(t)$ denotes variation of a function $f(t)$ on an interval $[a,b]$.

Proposition 2. A vibrosolution is also the only solution of the following equation with a measure

$$dx(t) = (k_{m+1}(t) f_{m+1}(x,t) + \ldots + k_n(t) f_n(x,t))dt +$$
$$+ \sum_{t_j} G(x_{0j}, 0, \mu, 1, t_j) d\chi(t-t_j), \; x(t_0) = x_0, \tag{4}$$

where $G(z,\omega,\mu,u,s) = \xi(z,\omega,\mu,\omega+u,s) - z$, and $\xi(z,\omega,\mu,u,s)$ is a solution of the system in differentials (3); t_i are points where δ - function is active, $\chi(t-t_j)$ is a Heaviside function, x_{0j} is a value of the state $x(t)$ before a jump.

Thus, the equations (3) and (4) enable us to compute the jumps of the equation (1) state $x(t)$, which are induced by pseudoimpulsive behavior of coefficients $k_1(t), \ldots, k_m(t)$. Explicit analytic formulas for a jump $\Delta x(t_0) = G(x_0, 0, \mu, 1, t_0)$ can be obtained in special cases, and numerical simulation of the significantly reduced fault subsystem yields the jumps values in other cases. The stability of a vibrosolution (in particular, a value of its jump) with respect to $*-$weak approximations of a Heaviside function enables us to use any approximation for numerical computation of a jump. For example, a pseudoimpulsive coefficient $k_j(t)$ with intensity $\mu_j \Delta t$ can be represented as a constant μ_j on an interval of length Δt, $2\mu_j$ on an interval of length $\Delta t/2$, or another $*-$ weak approximation of a Heaviside function. All possible approximations yield the same limit, which is equal to a vibrosolution jump as $\Delta t \to 0$. Moreover, it can be proved that if $f_1(x), \dots, f_m(x)$ are time-invariant, $k_j(t)$ remain constant during a time interval with length ΔT, and $k_j \Delta T = \mu_j \Delta t$, then the integral expression $x(t) = \int_{t_0}^{t_0+\Delta T} (k_1(t) f_1(x) + \dots + k_m(t) f_m(x) dt$, which should be used for numerical computation, is the same as for computing a jump by virtue of the equation (4).

The numbers of both coupled equations and terms in each of these equations in the fault subsystem, which is used for computation of a vibrosolution jump, are significantly reduced in comparison with the initial system. This allows fast and, possibly, online numerical computation of the state jumps.

4. Examples of Explicit Computation of State Jumps

In Examples 1 and 2, only the fault subsystems are given. The terms with nonimpulsive coefficients are insignificant for computation of jumps.

1. Let us consider a system

$$\dot{x} = k(t) x^n, \ x(t_0) = x_0, \ x \in R,$$

which will be used later for design of an impulse controller in a Van-der-Pol system. Assuming that the intensity of the coefficient $k(t)$ is equal to μ, we obtain

$$\Delta x = ((1-n)\mu + x_0^{1-n})^{1/(1-n)} - x_0, \quad \text{if} \quad n \neq 1, \quad \text{and}$$

$$\Delta x = x_0(\exp(\mu) - 1), \quad \text{if} \quad n = 1.$$

This result readily follows from the fact that $\xi = ((1-n)\mu u + x_0^{1-n})^{1/(1-n)}$ and $\xi = x_0 \exp(\mu u)$ are the solutions of the systems (3) in these cases, respectively. For $n > 1$, Δx is equal to ∞, if x_0 and μ satisfy the condition

$$x_0^{1-n} = (n-1)\mu.$$

2. Consider another system equation related to the theory of transient stability of power networks

$$\dot{x} = k_1(t) \sin(ax) + k_2(t) \cos(ax), \ x(t_0) = x_0, \ x \in R^N,$$

where μ_1 and μ_2 are intensities of $k_1(t)$ and $k_2(t)$ at a point t_0. Since $\xi = a^{-1}\{2 \arctan[\exp(a(\mu_1^2 + \mu_2^2)^{1/2} u) + \tan((a x_0 + \theta)/2)] - \theta\}$ is the solution of the system (3) in this case, we obtain

$$\Delta x = a^{-1}\{2 \arctan[\exp(a(\mu_1^2 + \mu_2^2)^{1/2}) + \tan((a x_0 + \theta)/2)] - \theta\} - x_0,$$

where $\theta = \mu_1^{-1} \mu_2 e$, and $e = (1, \dots, 1)$ is the unit N-dimensional vector.

3. Finally, consider a Ricatti equation for the estimate variance in the Kalman-Bucy filter

$$\dot{P} = AP + PA^* + GG^* - PC^*HCP,$$

$$P(t_0) = P_0, \quad P \in R^{N \times N},$$

where P is the estimate variance, G and H are variances of Gaussian noises, and C is a transition matrix in an observation equation. If H changes pseudoimpulsively on an interval $[t_0, t_0 + \Delta t]$, then the corresponding jump of the variance matrix P is equal to

$$\Delta P = P_0[I + C^* h C P_0]^{-1} - P_0,$$

where h is the intensity matrix for the matrix H, and I is the $N \times N$-dimensional identity matrix. The function $\xi = P_0[I + C^* h C P_0 u]^{-1}$ is the solution of the system (3) for this example.

Thus, the application of δ - functions to computation of the fault-induced jumps of system states enables us either to obtain explicit analytic formulas or to significantly simplify their numerical computation.

5. Impulse Control Approach

The impulse control approach is applied to a dynamic system in the case of urgent necessity to change back the system states affected by fault. The nominal equilibrium position of the system is considered stable with a compact stability basin, whose boundary can be estimated. Let us assume that the system state leaves the stability basin due to short fault and its further motion produces severe problems in the system operation. The fault modeling via δ - functions, described in Section 2, motivates design of an impulse controller based on δ - functions, which urgently returns the system state to the stability basin and adequately responds to the pseudoimpulsive behavior of the faulted system.

An impulse controlled dynamic system can be written in the form

$$\dot{x}(t) = [\mu_1 f_1(x,t) + \dots + \mu_m f_m(x,t)]\delta(t - t_1)$$
$$+ f_{m+1}(x,t) + \dots + f_n(x,t),$$
$$x(t_1) = x^*, \quad x \in R^N \tag{5}$$

where $\mu_1, \ldots, \mu_m$ are impulse control intensities, t_1 is the point where impulse control is active. If $f_1(x,t) = \ldots = f_{i-1}(x,t) = f_{i+1}(x,t) = \ldots = f_m(x,t) = 0$ and $f_i(x,t) = 1$, the impulse control is additive.

The equation (5), as well as (2), is an equation in distributions. The solution of (5) is defined as a vibrosolution, and its jumps are computed in accordance with Propositions 1 and 2. As noted, jumps of the impulse controlled system state (5) can be analytically computed in special cases. A number of examples are given below, where the impulse control method is applied to a Van-der-Pol system. Even if jumps of the system state (5) cannot be analytically computed, the number of terms necessary for numerical jump computation is reduced in comparison with the total number of terms in (5), from n to m.

Let us note that the faulted system (2) is governed by the same equation as the impulse controlled system (5). Thus, one can readily design the impulse control $\dot{u}(t)$ compensating for the fault action. For example, if the pseudoimpulsive coefficients $k_1(t), \ldots, k_m(t)$ affected by fault are represented as δ - functions with intensities $\nu_1, \ldots, \nu_m$, then the impulse control returning the system to the pre-fault state can be designed by assigning the intensities $-\nu_1, \ldots, -\nu_m$, i.e., $\dot{u}(t) = (-\nu_1\delta(t-t_1), \ldots, -\nu_m\delta(t-t_1))$.

Consider a general method for design of an impulse control $\dot{u}(t) = (\mu_1\delta(t-t_1), \ldots, \mu_m\delta(t-t_1))$ moving a state of the system (5) into its stability basin. The system (5) can be written in the compact form

$$\dot{x}(t) = f(x,t) + b(x,t)\dot{u}(t), \quad x(t_1) = x^*, \qquad (6)$$

where $\dot{u}(t)$ is an impulse control,
$b(x,t) = (f_1(x,t), \ldots, f_m(x,t)) \in R^{N\times m}$, $f(x,t) = f_{m+1}(x,t) + \ldots + f_n(x,t) \in R^N$. The second addition in (6) is equal to 0 everywhere, except for the point t_1 where impulse control is active. Let the initial state x^* be disposed outside the stability basin. Assume that there exists a Lyapunov function $L(x,t)$ such that

$$S(x,t) = dL(x,t)/dt \mid_{\dot{x}(t)=f(x,t)} < 0$$

for $t \geq t_1$, $x \in \omega \subset \omega_0$, and ω_0 is the stability basin of (6). To move a state of (6) into the stability basin, an impulse controller should generate a jump of the state in such a way that the Lyapunov function is negative at the post-jump state x_1

$$S(x,t) \mid_{x=x_1, t=t_1} < 0, \quad x_1 = x^* + \Delta x(t_1). \qquad (7)$$

In accordance with Proposition 2, the jump corresponding to an initial point x^* and an intensity vector $\mu = (\mu_1, \ldots, \mu_m)$ is equal to

$$\Delta x(t_1) = G(x^*, 0, \mu, 1, t_1), \qquad (8)$$

where $G(z, \omega, \mu, u, s) = \xi(z, \omega, \mu, \omega + u, s) - z$, and $\xi(z, \omega, \mu, u, s)$ is a solution of (3). Thus, the expressions (7) and (8) compose a closed system for determination of an intensity vector μ and, therefore, an impulse control $\dot{u}(t)$. The optimal impulse control can be determined from the following system

$$S(x,t) \mid_{x=x_1, t=t_1} \rightarrow \min_{\mu}, \quad x_1 = x^* + \Delta x(t_1),$$

$$\Delta x(t_1) = G(x^*, 0, \mu, 1, t_1),$$

where the function $S(x,t)$ should be minimized over all possible intensities.

6. Impulse Control of a Van-der-Pol system

Let us consider the application of the impulse control method to a Van-der-Pol oscillator, where the control objective is to return the system state to the stability basin, preventing it from transition to infinity.

Consider a system described by the Van-der-Pol equation

$$d^2x/dt^2 + \omega x + \alpha dx/dt - \beta(dx/dt)^3 = 0,$$

$$x(t_0) = x_0, \quad \alpha, \beta > 0. \qquad (9)$$

This system has the stable equilibrium at the origin with the stability basin bounded by the limit cycle $x^2 + (dx/dt)^2 = r^2$, where $r = \sqrt{\alpha/\beta}$. Upon introducing the variable $v = dx/dt$, the equation (9) can be written as the system of first-order equations

$$dx/dt = v, \ dv/dt + \omega x + \alpha v - \beta v^3 = 0,$$

$$x(t_0) = x_0, \quad v(t_0) = dx(t_0)/dt. \qquad (10)$$

Each trajectory outgoing from the interior of the limit cycle approaches zero, i.e., $\|x(t)\| \rightarrow 0$, as $t \rightarrow \infty$, if $x_0^2 + v_0^2 < r^2$, and each trajectory starting from a point outside the cycle tends to infinity, i.e., $\|x(t)\| \rightarrow \infty$, as $t \rightarrow \infty$, if $x_0^2 + v_0^2 > r^2$. Assume that the initial point (x_0, v_0) jumps out of the limit cycle due to fault. The control objective is to urgently return the system state to the stability basin, i.e., the interior of the limit cycle. A number of impulse controllers solving this problem are considered below.

1. Assume that additive impulse control is available. If a fault moves the system state to a position $(0, v_0)$, where $v_0 > r$, then an additive control $\dot{u}(t) = \mu\delta(t - t_0)$ solving the problem is included in the second equation of (10)

$$dv/dt + \omega x + \alpha v - \beta v^3 + \dot{u}(t) = 0.$$

The intensity μ of $\dot{u}(t)$ should belong to the range $v_0 - r < \mu < v_0 + r$.

If a fault moves the system to a position $(x_0, 0)$, where $x_0 > r$, then an additive control is included in the first equation of (10)

$$dx/dt = v - \dot{u}(t).$$

The intensity μ of $\dot{u}(t)$ should belong to the range $x_0 - r < \mu < x_0 + r$.

Both equations of (10) should be controlled, if a fault moves the system state to a position (x_0, v_0), $x_0 \neq 0$, $v_0 \neq 0$, $x_0^2 + v_0^2 > r^2$, which is located beyond the phase plane axes and stability basin.

2. Assume that multiplicative impulse control $\dot{u}(t) = \alpha v - \beta v^3$, where α and β are impulsive coefficients, is available. Let $\alpha = \mu\delta(t - t_0)$ and $\beta = 0$. The value of μ returning the system state to the interior of the limit cycle is determined as follows. Due to Proposition 2, the jump $\Delta v(t_0)$ inspired by the control $\mu\delta(t - t_0)v$ is equal to $\Delta v(t_0) = -v_0(\exp(\mu) - 1)$. Thus, the desired intensity is $\mu = \ln(1 - \Delta v(t_0)/v_0)$, where $\mid \Delta v \mid > \mid v_0 - \sqrt{r^2 - x_0^2} \mid$. This result readily follows from Example 1 of Section 4 for $n = 1$.

Analogously, if $\alpha = 0$ and $\beta = -\mu\delta(t - t_0)$, then the jump $\Delta v(t_0)$ inspired by the control $\mu\delta(t - t_0)v^3$ is equal to $\Delta v(t_0) = v_0 - (-2\mu + v_0^{-2})^{-1/2}$. Thus, the desired intensity is $\mu = (1/2)(v_0^{-2} - (v_0 - \Delta v(t_0))^{-2})$, where $\mid \Delta v \mid > \mid v_0 - \sqrt{r^2 - x_0^2} \mid$. This result readily follows from Example 1 of Section 4 for $n = 3$.

7. Discussion

Using an additive impulse control $\dot{u}(t)$ with an appropriate intensity, it is possible to return a system state to the stability basin from any post-fault state. The resource of additive impulse control can be insufficient to return a system state to the stability basin from any post-fault state (for example, if $Var[t_0, T]u(t) < C = const$, where $\dot{u}(t) = \mu\delta(t - t_0)$, i.e., impulse control intensity $\mu < C$). In this case, several additive controllers $\mu_0 u(t - t_0), \mu_1 u(t - t_1), \ldots, \mu_m u(t - t_m)$ operating subsequently at $t = t_0, t_1, \ldots, t_m$ solve the impulse control problem. If additive impulse control cannot be used, the question whether it is possible to return a system state to the stability basin from any post-fault state is more complicated. However, the investigation is simplified in the case of analytic computation of state jumps.

Modeling of short impulsive behavior of system or controller coefficients by δ - functions is physically motivated and highly simplifies subsequent mathematical analysis. Intensities of an impulse control correspond to the controller objectives. Design of an impulse controller requires only observation of the state jumps in the fault subsystem. The jumps can be measured directly or, as shown in Section 3, can be computed if intensities of the faulted coefficients are estimated.

8. Conclusion

This paper presents the impulse control approach intended to urgently return the state of a dynamic system affected by fault to the stability basin. A method for design of an impulse controller is addressed and applied to a Van-der-Pol system, thus preventing its state from transition to infinity. The technique for computing the fault-induced jumps of the system state is described. The fault-modeling procedure based on this technique is designed.

9. References

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