Robust Diagonal Stabilization and Finite Precision Problem: an LMI approach

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Abstract

Realization of digital filters or implementation of controllers in a digital computer may lead to unexpected instabilities resulting from the finite precision effects. Stability is usually ensured for an idealized discrete-time realization of the system. Nevertheless, as soon as A/D and D/A conversions get involved, the quantization of the state of the system, due to adder overflow, magnitude truncation, finite-wordlength format, may introduce severe nonlinearities responsible for overflow oscillations, limit cycles or chaotic behavior, even under zero input. This paper considers a parameter-dependent, discrete-time system in the companion form. We derive LMI conditions ensuring stability for the uncertain system in spite of the finite precision effect. We also seek an LMI formulation for the synthesis of a static output-feedback controller that guarantees robust stability for the finite precision problem.

1 Introduction

Proving the stability of a discrete-time system of the form

$$x(k+1) = Ax(k), \tag{1}$$

leads to the resolution of a Lyapunov equation, *ie* to find $P = P^T > 0$ such that

$$A^T PA - P = -QQ^T, \tag{2}$$

with (A, Q) observable. Then, it guarantees that the above system is asymptotically stable in a "mathematical" sense, *ie* assuming the system is implemented with infinite precision. In this case, explicit solutions for this equation are given in [2, 1], when A is in the companion form. However, this case represents an idealized behavior of the system. Most control problems involve the issue of finite precision computation. The most widely studied area where this problem arose during the last two decades concerns the fixed-point arithmetic digital filter. Implementing a controller in a digital computer also points out the difficulty to guarantee stability when A/D and D/A conversions are involved. There is an extensive literature on the effects of finite wordlength or quantization in digital control and signal processing. Adder overflow in second-order digital filters are responsible for self-sustaining oscillations, called limit cycles [22]. Overflow oscillations [8] and chaos [12] are also studied as the dramatic effects induced by fixed-point arithmetic for digital filters. Moreover, digital feedback can induce chaotic behavior when ordinary linear feedback is applied. Since magnitude truncation, overflow and underflow result in highly nonlinear systems, the performance and even the stability of such systems can be drastically affected. For all these examples, as explained in [4, 5], quantization effects can not simply be viewed as "white noise" or as approximate measurements. Therefore, it is of great interest to study these effects and find ways of reducing them. [21] gives conditions on the filter coefficients to suppress limit cycles in first order digital filters. For stabilization of systems when measurements of the state are quantized, an optimal realization is proposed in [18]. Extensions of the LQG theory [15] and the Bounded-Real Lemma [19] are also discussed. State and state-estimate feedback stabilization are proposed in [4, 5, 14].

To study the effects of quantization, let define $\mathcal{Q}_n = \{g : \mathbf{R}^n \to \mathbf{R}^n,\ g(x) = [g_1(x_1)\ \dots\ g_n(x_n)]^T,\ g(0) = 0 \mid \forall i \in [1 \dots n],\ \forall x_i \in \mathbf{R},\ |g_i(x_i)| \le |x_i|\}$ and let introduce the state $\tilde{x} = g(x)$, $g \in \mathcal{Q}_n$. The operator g represents the finite precision effects and can specify the following arithmetics (see [6]): *zeroing arithmetic*, *saturation arithmetic* or *decimal-truncation arithmetic*. Then, the quantized system is

$$\begin{aligned} x(k+1) &= A\tilde{x}(k), \\ \tilde{x}(k+1) &= g(x(k+1)). \end{aligned} \tag{3}$$

Let define the *diagonal stability* for the system (1) as the existence of a diagonal solution P to the Lyapunov equation (2). In [16, 9, 20, 13], the following theorem is stated:

Theorem 1.1 *If the discrete-time system (1) is diagonally stable, then the quantized system (3) is stable in spite of the finite precision effects.*

When A is a companion matrix, a simple condition ensuring that (2) admits a diagonal solution is given in [17]. This condition is detailed in §2.2.

The purpose of the paper is to derive some LMI conditions for

1. analysis of robust diagonal stability of a class of uncertain discrete-time systems,
2. synthesis of a controller that stabilizes a class of uncertain discrete-time systems robustly in respect to both the finite precision effects and the uncertainties.

In §2, some LMI-based conditions are derived for the analysis of systems when finite precision problem is involved. The main theorems are stated for diagonal robust output-feedback control in §3.

Our notations are as follows: for a real matrix P, $P > 0$ (resp. $P \geq 0$) means P is symmetric and positive-definite (resp. positive semi-definite) ; then, $P^{1/2}$ denotes its symmetric square root. We also use the notation $\mathbf{diag}(A, B)$ with $A \in \mathbf{R}^{p\times q}$ and $B \in \mathbf{R}^{m\times n}$ to denote the block-diagonal matrix written with A, B as its diagonal blocks. For $r = [r_1, \ldots, r_n]$, $r_i \in \mathbf{N}$, we define the sets

$$\begin{array}{l}\mathcal{D}(r) = \{\Delta = \mathbf{diag}(\delta_1 I_{r_1}, \ldots, \delta_n I_{r_n}),\ |\ \delta_i \in \mathbf{R}\},\\ \mathcal{B}(r) = \left\{B = \mathbf{diag}(B_1, \ldots, B_n),\ \middle|\ B_i \in \mathbf{R}^{r_i\times r_i}\right\},\\ \mathcal{S}(r) = \{S \in \mathcal{B}(r)\ \mid S_i > 0,\ \ i = 1, \ldots, n\},\end{array}$$

Finally, the symbol $\mathbf{Co}$ denotes the convex hull.

2 Analysis of Diagonal Robust Stability

2.1 LFR of the Coefficients

Consider the matrix $A(p)$ given by

$$A(p) = (a_{ji}(p))_{1\leq i\leq n}^{1\leq j\leq n}$$

where $a_{ji}(.)$ is a rational function of the parameters p_k, $k = 1, \ldots N$. We seek a Linear Fractional Representation of these coefficients. A methodology was given in [10] to derive an LFR for parameter-dependent systems when the parameters appear in rational functions. This methodology is based on simple operations such as addition, multiplication, stacking, shuffling and inversion. Then, the coefficients can be written:

$$\begin{array}{l} a_{ji}(p) = \alpha_{ji} + \beta_{ji}\Delta_{ji}(p)(I - \delta_{ji}\Delta_{ji}(p))^{-1}\gamma_{ji},\\ \\ \Delta_{ji}(p) = \mathbf{diag}(p_1 I_{r_1(j,i)}, \ldots p_N I_{r_N(j,i)}) \in \mathcal{D}(r),\end{array} \tag{4}$$

where $\mathcal{N}_{ji} = \sum_{k=1}^{N} r_k(j,i)$, $\alpha_{ji} \in \mathbf{R}$, $\beta_{ji} \in \mathbf{R}^{1\times\mathcal{N}_{ji}}$, $\delta_{ji} \in \mathbf{R}^{\mathcal{N}_{ji}\times\mathcal{N}_{ji}}$ and $\gamma_{ji} \in \mathbf{R}^{\mathcal{N}_{ji}\times 1}$. This is equivalent to the following Linear Fractional Representation for the parameter-dependent coefficient $aji(p)$

$$\begin{array}{rcll} a_{ji}(p) & = & \alpha_{ji} & + \ \beta_{ji}\pi_{ji},\\ \psi_{ji} & = & \gamma_{ji} & + \ \delta_{ji}\pi_{ji},\\ \pi_{ji} & = & \Delta_{ji}(p)\psi_{ji}. & \end{array} \tag{5}$$

2.2 LMI Conditions

Consider the following uncertain, discrete-time system in the companion form

$$x(k+1) = A(p)x(k), \tag{6}$$

where x is the state vector of the system, p is the parameter vector, and

$$A(p) = \begin{bmatrix} -a_1(p) & -a_2(p) & \ldots & -a_{n-1}(p) & -a_n(p)\\ 1 & 0 & \ldots & 0 & 0\\ . & . & \ldots & . & .\\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

The parameters $p_i(t)$, $i \in 1, \ldots n$, are unknown but bounded. Let define $\mathcal{P} = \{p(t) \in \mathbf{C}^N$ s.t. $\forall t \geq 0$, $\forall j \in [1, \ldots N]$, $|p_j(t)| \leq \overline{p}_j\}$. The notation $p = 0$ will be used to signify $\forall t \geq 0$, $\forall j \in [1, \ldots N]$, $p_j(t) = 0$.

We seek the domain of uncertainty where the system is guaranteed to be diagonally stable. In other words, we seek under some constraints the maximum bounds for the parameters such that the diagonal stability of the following quantized system,

$$\begin{array}{l} x(k+1) = A(p)\tilde{x}(k),\\ \tilde{x}(k+1) = g(x(k+1)), \end{array} \tag{7}$$

as described in Figure 1, is guaranteed within these bounds. As proved in [17], the Lyapunov equation for

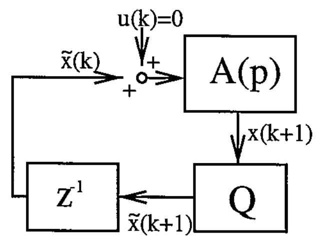

Figure 1: Model of filters with quantizer in the closed-loop

the nominal system, *ie* for (6) with $p = 0$, admits a diagonal solution P if and only if

$$\sum_{i=1}^{n} |a_i| \leq 1. \tag{8}$$

Moreover, the diagonal elements P_i of P must satisfy

$$P_1 \geq P_2 \geq \ldots \geq P_n > 0.$$

Then, the quantized system (3) is stable for every quantizer operator $g \in \mathcal{Q}_n$. These results can be used for the analysis of linear systems with no uncertainty, but the problem is much harder when it concerns uncertain systems. Then, the question is whether the parameter-dependent system (6) is diagonally stable or not robustly with respect to $p \in \mathcal{P}$. From (8), we can state that the system (6) is diagonally stable if

$$\forall\, p \in \mathcal{P}, \qquad \sum_{i=1}^{n} |a_i(p)| \leq 1. \tag{9}$$

The first idea would be to define $\underline{a}_i = \inf\{a_i(p) : p \in \mathcal{P}\}$ and $\overline{a}_i = \sup\{a_i(p) : p \in \mathcal{P}\}$ and then to check that

$$\sum_{i=1}^{n} \|\, [\ |\overline{a}_i| \quad |\underline{a}_i|\ ]\, \|_\infty \leq 1.$$

But this arises a major problem: since $a_i(p)$ may not be convex in p, the computation of the global maximum $\overline{a}_i$ or minimum $\underline{a}_i$ of $a_i(p)$ is an NP-hard problem. Anyway such an approach may be used when the bounded parameter vector p is assumed to appear in the companion matrix in an additive way, *ie* $a_i(p) = \alpha_i + p_i$, $i = 1, \ldots n$ and $p \in \mathcal{P}$ (see [6]) Consider now the parameter-dependent discrete-time system in the companion form described by (6) where the coefficients $a_i(.)$, $i = 1, \ldots n$, of the companion matrix A are rational functions of the parameters p_j, $j = 1, \ldots N$. Since $a_i(p)$ may not be convex in p, solving (9) is now much harder. According to §2.1, we can write the coefficients $a_i(.)$ with the following Linear Fractional Representation:

$$\begin{aligned} a_i(p) &= \alpha_i + \beta_i \Delta_i(p)(I - \delta_i \Delta_i(p))^{-1}\gamma_i, \\ \Delta_i(p) &= \mathbf{diag}(p_1 I_{r_1(i)}, \ldots p_N I_{r_n(i)}) \in \mathcal{D}(r). \end{aligned} \tag{10}$$

Then, we can state the main results of the paper about the state-feedback stabilization through the following theorem:

Theorem 2.1 *The discrete-time, parameter-dependent system in the companion form (6) with $p \in \mathcal{P}$ is diagonally stable if we can find $z_i \in \mathbf{R}^+$, $S_i \in \mathcal{S}(r)$, $T_i \in \mathcal{S}(r)$, $i = 1, \ldots n$, such that*

$$\sum_{i=1}^{n} z_i \leq 1, \tag{11}$$

$$\left[\begin{array}{c|c} \alpha_i - z_i & \frac{1}{2}\beta_i \\ +\gamma_i^T S_i \gamma_i & +\gamma_i^T S_i \delta_i \\ \hline \frac{1}{2}\beta_i^T & \delta_i^T S_i \delta_i \\ +\delta_i^T S_i \gamma_i & -S_i \end{array}\right] < 0,\ i = 1, \ldots n, \tag{12}$$

$$\left[\begin{array}{c|c} -\alpha_i - z_i & -\frac{1}{2}\beta_i \\ +\gamma_i^T T_i \gamma_i & +\gamma_i^T T_i \delta_i \\ \hline -\frac{1}{2}\beta_i^T & \delta_i^T T_i \delta_i \\ +\delta_i^T T_i \gamma_i & -T_i \end{array}\right] < 0,\ i = 1, \ldots n, \tag{13}$$

where α_i, β_i, γ_i and δ_i are the coefficients of the LFR (10) of $a_i(p)$.

This theorem takes the structure of the uncertainty matrix $\Delta(p)$ into account. Nevertheless, it neglects the realness of the parameters p_i, $i = 1, \ldots N$. This can be improved by introducing the skewsymmetric scaling matrices G_j and H_j, $j = 1, \ldots n$. Then, we can state:

Theorem 2.2 *The discrete-time, parameter-dependent system in the companion form (6), with p real and $p \in \mathcal{P}$, is diagonally stable if we can find $z_i \in \mathbf{R}^+$, $S_i \in \mathcal{S}(r)$, $T_i \in \mathcal{S}(r), G_i \in \mathcal{G}(r)$, $H_i \in \mathcal{G}(r)$, $i = 1, \ldots n$, such that*

$$\sum_{i=1}^{n} z_i \leq 1, \tag{14}$$

$$\left[\begin{array}{c|c} \alpha_i - z_i & \frac{1}{2}\beta_i - \gamma_i^T G_i \\ +\gamma_i^T S_i \gamma_i & +\gamma_i^T S_i \delta_i \\ \hline \frac{1}{2}\beta_i^T + G_i \gamma_i & \delta_i^T S_i \delta_i - S_i \\ +\delta_i^T S_i \gamma_i & +G_i \delta_i - \delta_i^T G_i \end{array}\right] < 0,\ i = 1, \ldots n, \tag{15}$$

$$\left[\begin{array}{c|c} -\alpha_i - z_i & -\frac{1}{2}\beta_i - \gamma_i^T H_i \\ +\gamma_i^T T_i \gamma_i & +\gamma_i^T T_i \delta_i \\ \hline H_i \gamma_i - \frac{1}{2}\beta_i^T & \delta_i^T T_i \delta_i + H_i \delta_i \\ +\delta_i^T T_i \gamma_i & -T_i - \delta_i^T H_i \end{array}\right] < 0,\ i = 1, \ldots n, \tag{16}$$

where α_i, β_i, γ_i and δ_i are the coefficients of the LFR (10) of $a_i(p)$.

3 Diagonal Robust Synthesis

Consider the following unstable SIMO, parameter-dependent, discrete-time system in the companion form

$$\begin{array}{rcl} x(k+1) &=& A(p)x(k) + Bu(k), \\ y(k) &=& Cx(k), \\ u(k) &=& Ky(k), \end{array} \tag{17}$$

with A in the companion form, $B = [1\ 0\ \ldots\ 0]^T$, $C = (c_{ji})_{1\leq i\leq n}^{1\leq j\leq m}$ and $p \in \mathcal{P}$. We seek an output-feedback controller diagonally stabilizing it. Then, this controller is guaranteed to diagonally stabilize the following quantized uncertain discrete-time system in the companion form.

$$\begin{array}{rcl} x(k+1) &=& A(p)\tilde{x}(k) + Bu(k), \\ \tilde{x}(k+1) &=& g(x(k+1)), \\ y(k) &=& C\tilde{x}(k), \\ u(k) &=& Ky(k). \end{array} \tag{18}$$

As described in Figure 2, we can also define a more common class of systems for which the measurements

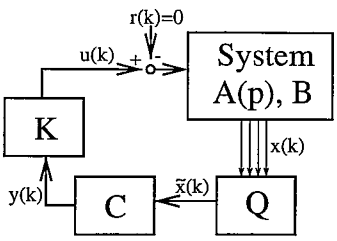

Figure 2: Model of systems with quantized measurements

of the state are quantized [4, 5, 12]. Such systems are given by the following quantized state-representation

$$\begin{array}{rcl} x(k+1) & = & A(p)x(k)+Bu(k), \\ \tilde{x}(k) & = & g(x(k)), \\ \tilde{y}(k) & = & C\tilde{x}(k), \\ u(k) & = & K\tilde{y}(k). \end{array} \tag{19}$$

This is worth noticing that the system (19) can also be written in the quantized state-representation (18). To prove it, consider the quantized, discrete-time, parameter-dependent system described by (19). We have

$$x(k+1) = A(p)x(k) + BKC\tilde{x}(k).$$

Let $A(p) = (a_{ji}(p))_{1\le i\le n}^{1\le j\le n}$ and $BKC = (b_{ji})_{1\le i\le n}^{1\le j\le n}$. Then we can write

$$x(k+1) = \sum_{i=1}^{n}[a_{ji}(p)x_i(k) + b_{ji}g(x_j(k))].$$

This is now easy to derive that

$$\exists\, h \in \mathcal{Q}_n \text{ such that } \forall\, i \in 1,\dots n,$$

$$a_{ji}(p)x_i(k) + b_{ji}g(x_j(k)) = (a_{ji}(p)+b_{ji})h_i(x_j(k)),$$

which means that the system can also be described by (18). This latter representation, although more conservative, will be used for further theorems.

Assume now that the nominal discrete-time system described by (6) is unstable. We seek a constant controller $K = [k_1\ k_2\ \dots\ k_m]$ such that the system described by (17) is diagonally stable. Some theorems are stated in [6] to stabilize the nominal system or a class of uncertain systems, when the uncertain parameters appear in an additive way in the coefficients of the system. The latter theorem remains true for the general case of uncertain systems, but we need to know the bounds on the coefficients (*ie* $\overline{a}_i$ and $\underline{a}_i$). As we explained in 2.2, the computation of these bounds is an NP-hard problem. The main results of the paper for the output-feedback stabilization when the coefficients $a_i(.)$, $i = 1,\dots n$, are rational functions of the parameters p_j, $j = 1,\dots N$, are stated in the following two theorems:

Theorem 3.1 *$\exists\, K = [k_1\ k_2\ \dots\ k_m]$ diagonally stabilizing (17) with $p \in \mathcal{P}$ if we can find $z_i \in \mathbf{R}^+$, $S_i \in \mathcal{S}(r)$, $T_i \in \mathcal{S}(r)$, $i = 1,\dots n$, and $k_j \in \mathbf{R}$, $j = 1,\dots m$, such that*

$$\sum_{i=1}^{n} z_i \le 1, \tag{20}$$

$$\left[\begin{array}{c|c} \sum_{j=1}^{m} c_{ji}k_j - \alpha_i & \\ -\frac{1}{2}\beta_i + \gamma_i^T S_i\gamma_i - z_i & +\gamma_i^T S_i\delta_i \\ \hline -\frac{1}{2}\beta_i^T & \delta_i^T S_i\delta_i \\ +\delta_i^T S_i\gamma_i & -S_i \end{array}\right] < 0,\ i = 1,\dots n, \tag{21}$$

$$\left[\begin{array}{c|c} \alpha_i - \sum_{j=1}^{m} c_{ji}k_j & \frac{1}{2}\beta_i \\ -z_i + \gamma_i^T T_i\gamma_i & +\gamma_i^T T_i\delta_i \\ \hline \frac{1}{2}\beta_i^T & \delta_i^T T_i\delta_i \\ +\delta_i^T T_i\gamma_i & -T_i \end{array}\right] < 0,\ i = 1,\dots n, \tag{22}$$

where α_i, β_i, γ_i and δ_i are the coefficients of the LFR (10) of $a_i(p)$.

As described in §2.2 for the analysis of stability, we can take the realness of the parameters into account by introducing the skewsymmetric scaling matrices G_i and $H_i \in \mathcal{G}(r)$, $i = 1,\dots n$. Then, we can state the following theorem.

Theorem 3.2 *$\exists\, K = [k_1\ k_2\ \dots\ k_m]$ diagonally stabilizing (17), with p real and $p \in \mathcal{P}$, if we can find $z_i \in \mathbf{R}^+$, $S_i \in \mathcal{S}(r)$, $T_i \in \mathcal{S}(r)$, $i = 1,\dots n, G_i \in \mathcal{G}(r)$, $H_i \in \mathcal{G}(r)$, $i = 1,\dots n$, and $k_j \in \mathbf{R}$, $j = 1,\dots m$, such that*

$$\sum_{i=1}^{n} z_i \le 1, \tag{23}$$

$$\left[\begin{array}{c|c} \sum_{j=1}^{m} c_{ji}k_j - \alpha_i & \gamma_i^T S_i\delta_i - \frac{1}{2}\beta_i \\ +\gamma_i^T S_i\gamma_i - z_i & -\gamma_i^T G_i \\ \hline G_i\gamma_i - \frac{1}{2}\beta_i^T & \delta_i^T S_i\delta_i - S_i \\ +\delta_i^T S_i\gamma_i & +G_i\delta_i - \delta_i^T G_i \end{array}\right] < 0, i = 1,\dots n \tag{24}$$

$$\left[\begin{array}{c|c} \alpha_i - \sum_{j=1}^{m} c_{ji}k_j & \frac{1}{2}\beta_i - \gamma_i^T H_i \\ -z_i + \gamma_i^T T_i\gamma_i & +\gamma_i^T T_i\delta_i \\ \hline \frac{1}{2}\beta_i^T + H_i\gamma_i & \delta_i^T T_i\delta_i - T_i \\ +\delta_i^T T_i\gamma_i & +H_i\delta_i - \delta_i^T H_i \end{array}\right] < 0, i = 1,\dots n \tag{25}$$

where α_i, β_i, γ_i and δ_i are the coefficients of the LFR (10) of $a_i(p)$.

We can choose to optimize a linear or quadratic criterion on the variable $K = [k_1\ k_2\ \dots\ k_m]$ such as input or output norm-bounds, saturations... (see [10, 3, 7] for more details). These additional conditions can be translated into LMIs.

4 Concluding Remarks

Linear Matrix Inequality conditions ensuring the output-feedback diagonal stabilization for a parameter-dependent, discrete-time system in the companion form were given. In other words, we extended the notion of robustness for discrete-time systems to the finite precision problem. In spite of the well-known numerical danger inherent in the companion form of systems, such a realization remains current so that the results presented in this paper may find illustrative applications. For instance, the analysis of the robust stability may be used to design digital filters. Moreover our output-feedback controller guarantees the stability of the closed-loop system even when implemented in a digital computer with finite wordlength arithmetic. One of the remaining open problems shall be to incorporate the possible coupling of the parameters $a_i(p)$ via the common variable p.

References

[1] A. Betser, N. Cohen, and E. Zeheb. On solving the Lyapunov and Stein equations for a companion matrix. *Syst. & Contr. Letters*, 25(3):211–218, June 1995.

[2] R. Bitmead. Explicit solutions of the discrete-time Lyapunov matrix equation and Kalman-Yakubovich equations. *IEEE Trans. Aut. Contr.*, 26(6):1291–1294, December 1981.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequality in systems and control theory.* SIAM, 1994.

[4] D. Delchamps. Controlling the flow of information in feedback systems with measurement quantization. In *Proc. IEEE Conf. on Decision & Contr.*, pages 2355–2360, Tampa, FL, December 1989.

[5] D. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Aut. Contr.*, 35(8):916–924, August 1990.

[6] S. Dussy. Robust stabilization of discrete-time parameter-dependent systems: the finite precision problem. In *Proc. IEEE Conf. on Decision & Contr.*, Kobe, Japan, December 1996.

[7] S. Dussy and L. El Ghaoui. Robust gain-scheduled control of a class of nonlinear parameter-dependent systems: application to an uncertain inverted pendulum. In *Proc. Conf. on Contr. & Applications*, pages 516–521, Dearborn, MI, September 1996.

[8] P. Ebert, J. Mazo, and M. Taylor. Overflow oscillations in digital filters. *Bell Syst. Tech. Journ.*, 48:2999–3020, November 1969.

[9] N. El-Agizi and M. Fahmy. Two-dimensional digital filters with no overflow oscillations. *IEEE Trans. Acoustics, speech, & signal proc.*, 27(5):465–469, October 1979.

[10] L. El Ghaoui and G. Scorletti. Control of rational systems using Linear-Fractional Representations and Linear Matrix Inequalities. *Automatica*, 32(9):1273–1284, September 1996.

[11] M. Fan, A. Tits, and J.C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Contr.*, 36(1):25–38, January 1991.

[12] X. Feng and K. Loparo. A study of chaos in discrete time linear systems with quantized state feedback. In *Proc. IEEE Conf. on Decision & Contr.*, pages 2107–2112, Tucson, AZ, December 1992.

[13] E. Kaszkurewicz and A. Bhaya. Robust stability and diagonal Liapunov functions. *SIAM Journ. Matrix Anal. Appl.*, 14(2):508–520, April 1993.

[14] G. Li and M. Gevers. Optimal finite precision implementation of a state-estimate feedback controller. *IEEE Trans. Aut. Contr.*, 37(12):1487–1498, December 1990.

[15] K. Liu and R. Skelton. Optimal controllers for finite wordlenght implementation. *IEEE Trans. Aut. Contr.*, 37(9):1294–1304, September 1992.

[16] W. Mills, C. Mullis, and R. Roberts. Digital filter realizations without overflow oscillations. *IEEE Trans. Acoustics, speech, & signal proc.*, 26(4):334–338, August 1978.

[17] P. Regalia. On finite Lyapunov functions for companion matrices. *IEEE Trans. Aut. Contr.*, 37(10):1640–1644, October 92.

[18] M. Rotea and D. Williamson. Optimal realizations of finite wordlenght digital filters and controllers. *IEEE Trans. Circ. Syst.*, 42(2):61–72, February 1995.

[19] P. Vaidyanathan. The discrete-time Bounded-Real Lemma in digital filtering. *IEEE Trans. Circ. Syst.*, 32(9):918–924, September 1985.

[20] P. Vaidyanathan and V. Liu. An improved sufficient condition for absence of limit cycles in digital filters. *IEEE Trans. Circ. Syst.*, 34(3):319–332, March 1987.

[21] M. Werter. Suppression of limit cycles in the first-order two-dimensional direct form figital filter with a controlled rounding arithmetic. *IEEE Trans. Sign. Proc.*, 40(6):1599–1601, June 1992.

[22] A. Willson. Limit cycles due to adder overflow in digital filters. *IEEE Trans. Circ. Theory*, 19(4):342–346, July 1972.

A Proof of Theorem 2.1

Consider the following parameter-dependent discrete-time system in the companion form (6), where the coefficients $a_i(.)$, $i = 1, \dots n$ are rational functions of the parameters p_j, $j = 1, \dots N$. From (8), we can derive that the system (6) is diagonally stable if

$$\forall p \in \mathcal{P}, \ \sum_{i=1}^{n} |a_i(p)| \leq 1. \tag{26}$$

Let introduce the variables $z_i \geq 0$, $i = 1, \dots n$. Then, the above condition is equivalent to

$$\exists z_i \geq 0, \ 1 \leq i \leq n, \ \text{ such that}$$

$$\sum_{i=1}^{n} z_i \leq 1, \tag{27}$$

$$\forall\, p \in \mathcal{P},\ a_i(p) \leq z_i,\ i = 1, \ldots n, \tag{28}$$

$$\forall\, p \in \mathcal{P},\ a_i(p) \geq -z_i,\ i = 1, \ldots n. \tag{29}$$

We want to translate (28) and (29) into LMIs. The key of the theorem is based on the following idea: we define fictitious signals y_i and u_i and we shall express the previous conditions on $a_i(p)$ through simple LMI-based conditions on these signals. The coefficients $a_i(.)$ are rational functions of the parameters p_j, so that $y_i = (a_i(p) - z_i)u_i$ with $y_i \in \mathbf{R}^*$ and $u_i \in \mathbf{R}^*$, $i = 1, \ldots n$, can be defined by the following LFR, with $\Delta_i(p) \in \mathcal{D}(r)$:

$$\begin{array}{rclcl} y_i & = & (\alpha_i - z_i)u_i & + & \beta_i \pi_i, \\ \psi_i & = & \gamma_i u_i & + & \delta_i \pi_i, \\ \pi_i & = & \Delta_i(p)\psi_i, & & \end{array} \tag{30}$$

The condition $p \in \mathcal{P}$ allows us to bound the uncertainty matrix, so that we can write a "normalized" LFR (30) with $||\Delta_i(p)|| \leq 1$. With this representation of $a_i(p)$, we can infer that (28) holds if and only if

$$\forall\, p \in \mathcal{P},\ y_i u_i \leq 0,\ i = 1, \ldots n, \tag{31}$$

We have $y_i u_i = u_i(\alpha_i - z_i)u_i + \frac{1}{2}\pi_i^T \beta_i^T u_i + \frac{1}{2} u_i \beta_i \pi_i$, $i = 1, \ldots n$. Let introduce the symmetric scaling matrices $S_i \in \mathcal{S}(r)$, $i = 1, \ldots n$, such that

$$\pi_i^T S_i \pi_i \leq \psi_i^T S_i \psi_i = (u_i \gamma_i^T + \pi_i^T \delta_i) S_i (u_i \gamma_i + \delta_i \pi_i).$$

Then, (28) holds if we can find $Si \in \mathcal{S}(r)$, $i = 1, \ldots n$, such that

$$\begin{bmatrix} u_i \\ \pi_i \end{bmatrix}^T \begin{bmatrix} \alpha_i - z_i & \frac{1}{2}\beta_i \\ \frac{1}{2}\beta_i^T & 0 \end{bmatrix} \begin{bmatrix} u_i \\ \pi_i \end{bmatrix} < 0 \text{ holds}, \tag{32}$$

for all $u_i \neq 0$ and π_i satisfying

$$\begin{bmatrix} u_i \\ \pi_i \end{bmatrix}^T \begin{bmatrix} \gamma_i^T S_i \gamma_i & \gamma_i^T S_i \delta_i \\ \delta_i^T S_i \gamma_i & \delta_i^T S_i \delta_i - S_i \end{bmatrix} \begin{bmatrix} u_i \\ \pi_i \end{bmatrix} \geq 0.$$

Using $\mathcal{S}$-procedure, (32) is equivalent to

$$\begin{bmatrix} \alpha_i - z_i + \gamma_i^T S_i \gamma_i & \frac{1}{2}\beta_i + \gamma_i^T S_i \delta_i \\ \frac{1}{2}\beta_i^T + \delta_i^T S_i \gamma_i & \delta_i^T S_i \delta_i - S_i \end{bmatrix} < 0, i = 1, \ldots n. \tag{33}$$

Reiterating the same operation with (29) achieves the proof of theorem 2.1.

B Proof of Theorem 2.2

In Annex A, we took the block-diagonal structure of Δ into account by introducing symmetric scalings S_i and T_i, $i = 1, \ldots n$, commuting with Δ. Additional constraints were found in [11] when Δ is not only block-diagonal but also real. It was stated that, given any square complex matrix M and any $\Delta \in \mathcal{D}(r)$, the equation $\Delta M x = x$ implies the additional constraint $x^H M^H G x = x^H G M x$ for all $G \in \mathcal{G}(r)$. We can adapt this to the case $p = \Delta q$ (see [10]). The realness of the parameters p in (28) implies the following additional constraint

$$\pi_i^T G_i \psi_i - \psi_i^T G_i \pi_i = 0 \text{ for all } G_i \in \mathcal{G}(r),\ i = 1, \ldots n. \tag{34}$$

Then, when the parameters are real, (28) holds if we can find $Si \in \mathcal{S}(r)$, $G_i \in \mathcal{G}(r)$, $i = 1, \ldots n$, such that

$$\begin{bmatrix} u_i \\ \pi_i \end{bmatrix}^T \begin{bmatrix} \alpha_i - z_i & \frac{1}{2}\beta_i \\ \frac{1}{2}\beta_i^T & 0 \end{bmatrix} \begin{bmatrix} u_i \\ \pi_i \end{bmatrix} < 0 \text{ holds}, \tag{35}$$

for all $u_i \neq 0$ and π_i satisfying

$$\begin{bmatrix} u_i \\ \pi_i \end{bmatrix}^T \begin{bmatrix} \gamma_i^T S_i \gamma_i & \gamma_i^T S_i \delta_i \\ \delta_i^T S_i \gamma_i & \delta_i^T S_i \delta_i - S_i \end{bmatrix} \begin{bmatrix} u_i \\ \pi_i \end{bmatrix} \geq 0 \text{ and}$$

$$\begin{bmatrix} u_i \\ \pi_i \end{bmatrix}^T \begin{bmatrix} 0 & -\gamma_i^T G_i \\ G_i \gamma_i & G_i \delta_i - \delta_i^T G_i \end{bmatrix} \begin{bmatrix} u_i \\ \pi_i \end{bmatrix} = 0.$$

Using $\mathcal{S}$-procedure, (28) with $p_j \in \mathbf{R}$, $j = 1, \ldots N$, holds if

$$\left[\begin{array}{c|c} \alpha_i - z_i & \frac{1}{2}\beta_i - \gamma_i^T G_i \\ +\gamma_i^T S_i \gamma_i & +\gamma_i^T S_i \delta_i \\ \hline \frac{1}{2}\beta_i^T + G_i \gamma_i & \delta_i^T S_i \delta_i - S_i \\ +\delta_i^T S_i \gamma_i & +G_i \delta_i - \delta_i^T G_i \end{array}\right] < 0,\ i = 1, \ldots n, \tag{36}$$

Reiterating the same operation with (29) achieves the proof of theorem 2.2.

C Proofs of Theorem 3.1 and Theorem 3.2

Consider the parameter-dependent, discrete-time system in the companion form described by (17), where the coefficients $a_i(.)$ are rational functions of the parameters p_j, $j = 1, \ldots N$. We can readily state that $\exists\, K = [k_1\ k_2\ \ldots\ k_m]$ such that the Lyapunov equation $(A(p) + BKC)^T P (A(p) + BKC) - P < 0$, $p \in \mathcal{P}$, admits a diagonal solution if

$$\forall\, p \in \mathcal{P},\ \sum_{i=1}^{n} | - a_i(p) + \sum_{j=1}^{m} c_{ji} k_j | \leq 1. \tag{37}$$

From this point, the proof of theorem 3.1 is equivalent to the one presented in Annex A with $a_i(p)$ replaced by $(a_i(p) - \sum_{j=1}^{m} k_j c_{ji})$.

Based on the results of [11], the proof of theorem 3.2 is similar as the one of theorem 2.2 in Annex B. It consists in introducing skewsymmetric matrices G_i and $H_i \in \mathcal{G}(r)$, $i = 1, \ldots n$, and in expressing the realness of Δ with the additional constraint (34).