Adaptive Control of Discrete-TimeOutput-Feedback Nonlinear Systems[1]

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Abstract

In adaptive control design for discrete-time output-feedback nonlinear plants, the main obstacle is the lack of effective ways to design estimators for the unmeasured states and the unknown parameters. Since these quantities appear as arguments of arbitrary nonlinear functions, traditional estimation methods can not be used. To resolve this problem, we propose a new systematic methodology by which one can recover all the necessary information about the unknown part of the system in finite time, so that control schemes for global stabilization and tracking can be designed for such plants.

1 Introduction

In recent years, a great deal of progress has been made in the area of adaptive control of continuous-time nonlinear systems [1, 2]. For their discrete-time counterparts, on the other hand, very few results exist. With the exception of our recent solution [8] for the strict-feedback problem, the other existing results [3, 4, 5, 7] either require restrictive growth conditions on the nonlinearities or only deal with a scalar nonlinear system which contains a single unknown parameter.

In discrete-time, backstepping amounts to simply "looking ahead" and choosing the control law which forces the states to acquire their desired values after a finite number of time steps. In the presence of unknown parameters, however, it is impossible to calculate these "look-ahead" values of the states. Furthermore, since these calculations involve the unknown parameters as arguments of arbitrary nonlinear functions, no known parameter estimation method is applicable, as all of them require a linear parameterization to guarantee global results. The main contribution of [8] is the introduction of a novel uncertainty identification scheme which, in a finite number of time steps, computes the projections of the unknown vector parameter along the basis of a subspace generated by the nonlinear vector fields of the plant. Once these projections are known, the control law becomes a straightforward "look-ahead" design. However, the computation of these projections relies on the measurement of all the state variables. If some of these variables are not measured, the corresponding projection information is lost.

In this paper we introduce a new systematic method for recovering the lost projection information. The available projections are computed through simple linear operations from a set of measured vectors which are first decomposed into orthogonal subspaces. To ensure that all the necessary information is obtained, we use the control input to drive the measured output to values which provide measurements of the projection of θ along linearly independent directions.

In Section 2 we first examine the difficulties raised from the design of a "look-ahead" controller for discrete-time output-feedback nonlinear systems. To resolve these difficulties, we then introduce the projection recovery technique in Section 3.

2 Problem Formulation

In this section, we will first describe the problem in detail and then analyze the obstacles that must be overcome in order to solve it.

Let us consider the following second-order discrete-time nonlinear system

$$\begin{aligned} x_{t+1} &= y_t + \theta^{\mathrm{T}}\varphi(x_t) && (2.1)\\ y_{t+1} &= u_t + \theta^{\mathrm{T}}\psi(x_t)\,, && (2.2) \end{aligned}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters and only the state x_t is available for measurement. We denote $\varphi_t = \varphi(x_t)$ and $\psi_t = \psi(x_t)$, with $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}^p$ and $\psi(\cdot) : \mathbb{R} \to \mathbb{R}^p$ known functions. This system is in output-feedback form, because its nonlinearities depend only on the measured output x_t.

At each time t, our objective is to pre-compute the following projections of system (2.1)–(2.2):

$$\begin{matrix} \theta^{\mathrm{T}}\varphi(x_t) & \theta^{\mathrm{T}}\varphi(x_{t+1}) \\ \theta^{\mathrm{T}}\psi(x_t) & \end{matrix} \qquad (2.3)$$

As we stated in [8], one of the applications of these pre-computed projections is to implement a "look-ahead" controller which stabilizes the system and regulates x_t to zero. This is because if all the terms of (2.3) were known, then we could choose:

$$\begin{aligned} u_t &= -\theta^{\mathrm{T}}\psi(x_t) - \theta^{\mathrm{T}}\varphi(x_{t+1}) \\ &= -\theta^{\mathrm{T}}\left[\psi(x_t) + \varphi(y_t + \theta^{\mathrm{T}}\varphi(x_t))\right], \quad (2.4) \end{aligned}$$

from which we obtain

$$\begin{aligned} x_{t+2} &= y_{t+1} + \theta^{\mathrm{T}}\varphi(x_{t+1}) \\ &= u_t + \theta^{\mathrm{T}}\left[\psi(x_t) + \varphi(y_t + \theta^{\mathrm{T}}\varphi(x_t))\right] \\ &= 0\,. \end{aligned} \tag{2.5}$$

Let us now review the method proposed in [8] for pre-computing the projections of (2.3) at time t, and examine the difficulties in achieving this goal.

The main idea of [8] can be summarized as follows:

- First, using the fact that φ and ψ are known, we compute off-line a basis for the span of these vector fields over **all** values of x and y. The dimension of this basis is at most p, where p is the number of unknown parameters. In order to pre-compute the projections given in (2.3), we use the control input to drive the state vector to points which correspond to the basis vectors. Since we can measure the states x_t , y_t, we can compute the terms $x_t - y_{t-1} = \theta^{\mathrm{T}}\varphi_{t-1}$ and $y_t - u_{t-1} = \theta^{\mathrm{T}}\psi_{t-1}$ at various times t; by driving the state to values which render φ_{t-1} and ψ_{t-1} basis vectors, we obtain the projections of θ along the corresponding basis. This phase is finite in duration for any finite-dimensional system with a finite number of unknown parameters.

- Once all the projections of θ along the above basis vectors are collected, we use them to pre-compute all the terms appearing in (2.3) by expressing $\varphi(x)$ and $\psi(x,y)$ as linear combinations of these basis vectors and then using the same coefficients to compute $\theta^{\mathrm{T}}\psi(x)$ and $\theta^{\mathrm{T}}\varphi(x,y)$ as linear combinations of the projections.

This two-stage process depends critically on the following fact:

> Contrary to their continuous-time counterparts, **discrete-time nonlinear systems can not exhibit the finite escape time phenomenon.** This implies that we can afford to postpone closing the loop with a controller for a finite time period.

This approach is very different from traditional certainty equivalence approaches, which replace the unknown θ with an estimate $\hat{\theta}$. Equation (2.4) shows that any such attempt would be stifled by the fact that θ appears inside the nonlinear function φ; this becomes a nonlinear state and parameter estimation problem, for which no global methods are available.

Returning to the output-feedback case where y_t is no longer measurable, we have the additional difficulty that neither $\theta^{\mathrm{T}}\varphi_{t-1} = x_t - y_{t-1}$ nor $\theta^{\mathrm{T}}\psi_{t-1} = y_t - u_{t-1}$ can be computed individually. Instead, at time t we can only obtain

$$x_t - u_{t-2} \;=\; \theta^{\mathrm{T}}(\varphi_{t-1} + \psi_{t-2})\,. \tag{2.6}$$

To pre-compute each term in (2.3), however, we still need to be able to compute the projections of θ along φ and ψ individually. Hence, if it so happens that the subspace spanned by $\varphi(x)+\psi(\tilde{x})$ is of lower dimension than the subspace spanned by φ and ψ, then measuring $x_t - u_{t-2}$ can only give us all the projections of θ along a basis of the subspace spanned by $\varphi(x)+\psi(\tilde{x})$. But to pre-compute each term in (2.3), we need all the projections of θ along a basis of the subspace spanned by φ and ψ. To recover those missing projections, one might try reconstructing the unmeasurable state y_t. However, equation (2.4) shows that replacing y_t with an estimate $\hat{y}_t$ would again raise the nonlinear estimation problem which we discussed in the context of replacing θ by an estimate $\hat{\theta}$.

To overcome this seemingly insurmountable problem, during the identification phase we will select u in such a way that if ψ_t is linearly dependent on a set of (known) linearly independent vectors from the set of past values of ψ, i.e., from the set $\{\psi_0, \psi_1, \ldots, \psi_{t-1}\}$, then its decomposition along those vectors does not coincide (i.e., does not have the same coefficients) with the decomposition of φ_{t+1} along the corresponding vectors from the set $\{\varphi_1, \varphi_2, \ldots, \varphi_t\}$. Achieving this "independence" will allow us to recover $\theta^{\mathrm{T}}\varphi_{t+1}$ and $\theta^{\mathrm{T}}\psi_t$ from $\theta^{\mathrm{T}}(\varphi_{t+1}+\psi_t)$ by simple linear operations.

3 Projection Recovery

As we have seen in the above discussion, pre-computing each term in (2.3) requires knowledge of the projections of θ along the vectors which constitute a basis of the subspace spanned by φ and ψ. On the other hand, (2.6) shows that measuring x_t can only provide us with the projections of θ along the vectors which constitute a basis of the subspace spanned by $\varphi(x) + \psi(\tilde{x})$. Therefore, in this section we develop a systematic method to recover those missing projections.

Notations: For brevity, we define

$$S_{\varphi,\psi} \;=\; \operatorname*{span}_{x\in\mathbb{R}}\{\varphi(x)\} \bigcap \operatorname*{span}_{x\in\mathbb{R}}\{\psi(x)\}, \tag{3.1}$$

which implies that we have the direct sum decompositions

$$\operatorname*{span}_{x\in\mathbb{R}}\{\varphi(x)\} \;=\; S_{\varphi} \bigoplus S_{\varphi,\psi} \tag{3.2}$$

$$\operatorname*{span}_{x\in\mathbb{R}}\{\psi(x)\} \;=\; S_{\psi} \bigoplus S_{\varphi,\psi}\,, \tag{3.3}$$

where S_φ and S_ψ are the corresponding orthogonal complements of $S_{\varphi,\psi}$ with respect to the subspaces $\mathrm{span}_{x\in\mathbb{R}}\{\varphi(x)\}$ and $\mathrm{span}_{x\in\mathbb{R}}\{\psi(x)\}$.

Using these direct sum decompositions, we then decompose

$$\begin{aligned} S &= \underset{x,\tilde{x}\in\mathbb{R}}{\mathrm{span}}\{\varphi(x),\psi(\tilde{x})\} \\ &= S_\varphi \bigoplus S_\psi \bigoplus S_{\varphi,\psi}\,. \end{aligned} \tag{3.4}$$

As explained in the above section, the pre-computation of each term in (2.3) requires all the projections of θ along the directions which constitute a basis for $S = \mathrm{span}_{x,\tilde{x}\in\mathbb{R}}\{\varphi(x),\psi(\tilde{x})\}$. For clarity and without loss of generality, we assume $S_{\varphi,\psi} = \{0\}$. The general case of $S_{\varphi,\psi} \neq \{0\}$ requires some straightforward but tedious modifications of the following procedure, which would further complicate the task of introducing it to the reader.

3.1 Decomposition procedure

We start by describing the decomposition procedure, which is to be implemented at every step of the identification phase.

Measure the state x_t and then evaluate φ_t, ψ_t at each step. Using (2.1)–(2.2), all the measurements can be expressed as follows (here we assume for simplicity that $u_0 = \cdots = u_{t-2} = 0$):

$$\begin{bmatrix} \theta^{\mathrm{T}}(\varphi_1+\psi_0) \\ \vdots \\ \theta^{\mathrm{T}}(\varphi_{t-1}+\psi_{t-2}) \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_t \end{bmatrix}. \tag{3.5}$$

The objective now is to rewrite these measurements in a way that can be used for choosing u_t. To this end, we employ the following procedure:

1.
   - Find a basis for $S_{\Phi_{t-1}} = \mathrm{span}\{\varphi_1,\dots,\varphi_{t-1}\}$. This basis should consist of linearly independent vectors from the set $\Phi_{t-1} = \{\varphi_1,\dots,\varphi_{t-1}\}$, and its dimension is denoted by k_2+1 with $k_2 \le t-2$.
   - Express each of the remaining $t-k_2-2$ vectors from the set Φ_{t-1} as linear combinations of this basis, say $\varphi_j = c_{j,1}\varphi_1 + \cdots + c_{j,k_2+1}\varphi_{k_2+1}$. Here, without loss of generality, we assume that $\varphi_1,\dots,\varphi_{k_2+1}$ form a basis for $S_{\Phi_{t-1}}$.
   - For each of these $t-k_2-2$ vectors, subtract from the corresponding measured projection $\theta^{\mathrm{T}}(\varphi_j+\psi_{j-1})$ with $1 \le j \le k_2+1$ the same linear combination of the projections corresponding to the basis vectors of $S_{\Phi_{t-1}}$, to obtain a new computed projection:

   $$\begin{aligned} \theta^{\mathrm{T}}\tilde{\psi}_{j-1} &\triangleq \theta^{\mathrm{T}}\left[\psi_{j-1} - c_{j,1}\psi_0 - \cdots - c_{j,k_2+1}\psi_{k_2}\right] \\ &= +\theta^{\mathrm{T}}(\varphi_j+\psi_{j-1}) - \left[c_{j,1}\theta^{\mathrm{T}}(\varphi_1+\psi_0)\right. \\ &\quad \left. + \cdots + c_{j,k_2+1}\theta^{\mathrm{T}}(\varphi_{k_2+1}+\psi_{k_2})\right] \\ &= x_{j+1} - \left[c_{j,1}x_2 + \cdots + c_{j,k_2+1}x_{k_2+2}\right] \\ &\triangleq \tilde{x}_{j+1}\,. \end{aligned} \tag{3.6}$$

   - These operations transform (3.5) into the following set of projections:

   $$\begin{bmatrix} \theta^{\mathrm{T}}(\varphi_1+\psi_0) \\ \vdots \\ \theta^{\mathrm{T}}(\varphi_{k_2+1}+\psi_{k_2}) \\ \theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} \\ \vdots \\ \theta^{\mathrm{T}}\tilde{\psi}_{t-2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_{k_2+2} \\ \tilde{x}_{k_2+3} \\ \vdots \\ \tilde{x}_t \end{bmatrix}. \tag{3.7}$$

   - From the set $\{\tilde{\psi}_{k_2+1},\dots,\tilde{\psi}_{t-2}\}$ we select a basis for the subspace spanned by $\tilde{\psi}_{k_2+1}, \dots, \tilde{\psi}_{t-2}$. Without loss of generality, we assume that $\{\tilde{\psi}_{k_2+1},\dots,\tilde{\psi}_{k_3}\}$ with $k_3 \le t-2$ is such a basis.

2. Repeat the operations of Step 1 with respect to $\Psi_{k_3+1} = \{\psi_0,\dots,\psi_{k_2},\tilde{\psi}_{k_2+1},\dots,\tilde{\psi}_{k_3}\}$. First, find a basis of dimension k_3+1-k_1. Here, we require this basis to include all the vectors of $\tilde{\psi}_{k_2+1},\dots,\tilde{\psi}_{k_3}$. Without loss of generality, we assume that $\{\psi_{k_1},\dots,\psi_{k_2},\tilde{\psi}_{k_2+1},\dots,\tilde{\psi}_{k_3}\}$ constitute this basis. Then, following 1, express the remaining k_1 vectors as linear combinations of these basis vectors, and subtract the corresponding projections.

3. The assumption $S_{\varphi,\psi} = \{0\}$ and the linear independence of $\{\varphi_1,\dots,\varphi_{k_2+1}\}$ guarantee that any nonzero vector formed by arbitrary linear combinations of $\{\varphi_1,\dots,\varphi_{k_2+1}\}$ does not belong to S_ψ. Hence, we can arrange the resulting new projections, along with the remaining k_2-k_1 of the original projections which were not affected by Steps 1 or 2 in the following way:

   $$\begin{bmatrix} \theta^{\mathrm{T}}\tilde{\varphi}_1 \\ \vdots \\ \theta^{\mathrm{T}}\tilde{\varphi}_{k_1} \\ \theta^{\mathrm{T}}(\varphi_{k_1+1}+\psi_{k_1}) \\ \vdots \\ \theta^{\mathrm{T}}(\varphi_{k_2+1}+\psi_{k_2}) \\ \theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} \\ \vdots \\ \theta^{\mathrm{T}}\tilde{\psi}_{k_3} \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \vdots \\ \tilde{x}_{k_1+1} \\ x_{k_1+2} \\ \vdots \\ x_{k_2+2} \\ \tilde{x}_{k_2+3} \\ \vdots \\ \tilde{x}_{k_3+2} \end{bmatrix}. \tag{3.8}$$

From the above procedure, it should be clear that the following properties are true:

$$\begin{aligned} & \operatorname{rank}\{\psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\} \\ = \; & \operatorname{rank}\{\psi_0, \ldots, \psi_{t-2}\} \qquad (3.9) \\ & \operatorname{rank}\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}, \varphi_{k_1+1}, \ldots, \varphi_{k_2+1}\} \\ = \; & \operatorname{rank}\{\varphi_1, \ldots, \varphi_{t-1}\}. \qquad (3.10) \end{aligned}$$

Furthermore, $\{\psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\}$ and $\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}, \varphi_{k_1+1}, \ldots, \varphi_{k_2+1}\}$ are both sets of linearly independent vectors. This completes the decomposition procedure.

3.2 Basis identification

From the above procedure we see that, at each time t, $\tilde{\varphi}_j \in S_{\Phi_{t-1}} \subset S_\varphi$ and $\tilde{\psi}_j \in S_{\Psi_{t-1}} \subset S_\psi$. Hence, (3.8) shows that in order to complete the task of computing the projections of θ along the vectors which constitute bases for the subspaces $S_{\Psi_{t-1}}$ and $S_{\Phi_{t-1}}$, we still need to find a way to compute the projections $\theta^{\mathrm{T}}\varphi_{k_1+1}, \ldots, \theta^{\mathrm{T}}\varphi_{k_2+1}, \theta^{\mathrm{T}}\psi_{k_1}, \ldots, \theta^{\mathrm{T}}\psi_{k_2}$. This is because all the other projections in (3.8) are already either in S_φ or in S_ψ. To obtain these remaining projections, we note that the subspaces S_φ and S_ψ are invariant with respect to time and are finite dimensional. Hence, if we keep $u_t = 0$ for each t, measure the state x_{t+2} and evaluate ψ_{t+2} and φ_{t+2}, then there must exist $t_1 > 0$ such that $\psi_{t_1+1} \in S_{\Psi_{t_1-1}}$; in fact, we will have $0 < t_1 < \dim S_\psi$.

At this time ($t = t_1 + 1$), we perform decomposition procedure of the previous subsection and obtain equation (3.8). Since $\psi_{t_1+1} \in S_{\Psi_{t_1-1}}$, we can find constants $c_{t_1+1,k_1}, \ldots, c_{t_1+1,k_3}$ to express ψ_{t_1+1} as

$$\begin{aligned} \psi_{t_1+1} = \; & c_{t_1+1,k_1}\psi_{k_1} + \cdots + c_{t_1+1,k_2}\psi_{k_2} \\ & c_{t_1+1,k_2+1}\tilde{\psi}_{k_2+1} + \cdots + c_{t_1+1,k_3}\tilde{\psi}_{k_3}. \qquad (3.11) \end{aligned}$$

From the equation (3.8) the projections of θ along the vectors $\tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}$ are already known. Therefore, at time $t = t_1 + 1$ we can select the control input

$$\begin{aligned} u_{t_1+1} &= -c_{t_1+1,k_2+1}\theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} - \cdots - c_{t_1+1,k_3}\theta^{\mathrm{T}}\tilde{\psi}_{k_3} \\ &= -c_{t_1+1,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+1,k_3}\tilde{x}_{k_3+2} \qquad (3.12) \end{aligned}$$

to eliminate the projections of θ along those known directions at the next step. Then, at time $t = t_1 + 2$ we measure x_{t_1+2} and use it to evaluate $\psi_{t_1+2} = \psi(x_{t_1+2})$ and $\varphi_{t_1+2} = \varphi(x_{t_1+2})$. Now we have to distinguish between the following two cases:

Case 1 $\boxed{\psi_{t_1+2} \notin S_{\Psi_{t_1-1}} \text{ or } \varphi_{t_1+2} \notin S_{\Phi_{t_1-1}}}$: Then, at least one of the following equalities must be valid:

$$\begin{aligned} & \operatorname{rank}\{\psi_{t_1+2}, \psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\} \\ = \; & \operatorname{rank}\{\psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\} + 1 \qquad (3.13) \\ & \operatorname{rank}\{\varphi_{t_1+2}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}, \varphi_{k_1+1}, \ldots, \varphi_{k_2+1}\} \\ = \; & \operatorname{rank}\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}, \varphi_{k_1+1}, \ldots, \varphi_{k_2+1}\} + 1. \qquad (3.14) \end{aligned}$$

In this case we just apply the decomposition procedure again. Recall that the subspaces S_φ and S_ψ are invariant with respect to time and finite dimensional. Therefore, after a finite number of steps (at most p) both $\psi_{t+2} \in S_{\Psi_t}$ and $\varphi_{t+2} \in S_{\Phi_t}$ must be valid.

Case 2 $\boxed{\psi_{t_1+2} \in S_{\Psi_{t_1-1}} \text{ and } \varphi_{t_1+2} \in S_{\Phi_{t_1-1}}}$: Then, we can find constants $c_{t_1+2,k_1}, \ldots, c_{t_1+2,k_3}, d_{t_1+2,1}, \ldots, d_{t_1+2,k_2+1}$ to express

$$\begin{aligned} \psi_{t_1+2} &= c_{t_1+2,k_1}\psi_{k_1} + \cdots + c_{t_1+2,k_2}\psi_{k_2} \\ &\quad + c_{t_1+2,k_2+1}\tilde{\psi}_{k_2+1} + \cdots + c_{t_1+2,k_3}\tilde{\psi}_{k_3} \qquad (3.15) \\ \varphi_{t_1+2} &= d_{t_1+2,1}\tilde{\varphi}_1 + \cdots + d_{t_1+2,k_1}\tilde{\varphi}_{k_1} \\ &\quad + d_{t_1+2,k_1+1}\varphi_{k_1+1} \\ &\quad + \cdots + d_{t_1+2,k_2+1}\varphi_{k_2+1}. \qquad (3.16) \end{aligned}$$

Therefore, at $t = t_1 + 2$ we can compute the value of

$$\begin{aligned} e_{t_1+2} &= |d_{t_1+2,k_1+1} - c_{t_1+1,k_1}| \\ &\quad + \cdots + |d_{t_1+2,k_2+1} - c_{t_1+1,k_2}|, \qquad (3.17) \end{aligned}$$

which indicates whether the decompositions of ψ_{t_1+1} along $\{\psi_{k_1}, \ldots, \psi_{k_2}\}$ and of φ_{t_1+2} along $\{\varphi_{k_1+1}, \ldots, \varphi_{k_2+1}\}$ have the same coefficients. Depending on the value of e_{t_1+2}, we further divide the procedure into the following two subcases:

Subcase 2.1 $\boxed{e_{t_1+2} \neq 0}$: In this subcase, using (3.11), (3.12) and (3.16), at $t = t_1 + 3$ we can measure

$$\begin{aligned} x_{t_1+3} &= u_{t_1+1} + \theta^{\mathrm{T}}(\varphi_{t_1+2} + \psi_{t_1+1}) \\ &= -c_{t_1+1,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+1,k_3}\tilde{x}_{k_3+2} \\ &\quad + \theta^{\mathrm{T}}(\varphi_{t_1+2} + \psi_{t_1+1}) \\ &= -c_{t_1+1,k_2+1}\theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} - \cdots - c_{t_1+1,k_3}\theta^{\mathrm{T}}\tilde{\psi}_{k_3} \\ &\quad + d_{t_1+2,1}\theta^{\mathrm{T}}\tilde{\varphi}_1 + \cdots + d_{t_1+2,k_1}\theta^{\mathrm{T}}\tilde{\varphi}_{k_1} \\ &\quad + d_{t_1+2,k_1+1}\theta^{\mathrm{T}}\varphi_{k_1+1} + \cdots + d_{t_1+2,k_2+1}\theta^{\mathrm{T}}\varphi_{k_2+1} \\ &\quad + c_{t_1+1,k_1}\theta^{\mathrm{T}}\psi_{k_1} + \cdots + c_{t_1+1,k_2}\theta^{\mathrm{T}}\psi_{k_2} \\ &\quad + c_{t_1+1,k_2+1}\theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} + \cdots + c_{t_1+1,k_3}\theta^{\mathrm{T}}\tilde{\psi}_{k_3} \\ &= (d_{t_1+2,k_1+1} - c_{t_1+1,k_1})\theta^{\mathrm{T}}\varphi_{k_1+1} \\ &\quad + \cdots + (d_{t_1+2,k_2+1} - c_{t_1+1,k_2})\theta^{\mathrm{T}}\varphi_{k_2+1} \\ &\quad + c_{t_1+1,k_1}\theta^{\mathrm{T}}(\psi_{k_1} + \varphi_{k_1+1}) \\ &\quad + \cdots + c_{t_1+1,k_2}\theta^{\mathrm{T}}(\psi_{k_2} + \varphi_{k_2+1}) \\ &\quad + d_{t_1+2,1}\theta^{\mathrm{T}}\tilde{\varphi}_1 + \cdots + d_{t_1+2,k_1}\theta^{\mathrm{T}}\tilde{\varphi}_{k_1}. \qquad (3.18) \end{aligned}$$

Hence, we have

$$\begin{aligned} & (d_{t_1+2,k_1+1} - c_{t_1+1,k_1})\theta^{\mathrm{T}}\varphi_{k_1+1} \\ & + \cdots + (d_{t_1+2,k_2+1} - c_{t_1+1,k_2})\theta^{\mathrm{T}}\varphi_{k_2+1} \\ = \; & x_{t_1+3} - c_{t_1+1,k_1}\theta^{\mathrm{T}}(\psi_{k_1} + \varphi_{k_1+1}) \\ & - \cdots - c_{t_1+1,k_2}\theta^{\mathrm{T}}(\psi_{k_2} + \varphi_{k_2+1}) \\ & - d_{t_1+2,1}\theta^{\mathrm{T}}\tilde{\varphi}_1 - \cdots - d_{t_1+2,k_1}\theta^{\mathrm{T}}\tilde{\varphi}_{k_1} \\ = \; & x_{t_1+3} - c_{t_1+1,k_1}x_{k_1+2} - \cdots - c_{t_1+1,k_2}x_{k_2+2} \\ & - d_{t_1+2,1}\tilde{x}_2 - \cdots - d_{t_1+2,k_1}\tilde{x}_{k_1+1}. \qquad (3.19) \end{aligned}$$

Therefore, we obtain a projection of θ along the vector

$$v \triangleq (d_{t_1+2,k_1+1} - c_{t_1+1,k_1})\varphi_{k_1+1} + \cdots + (d_{t_1+2,k_2+1} - c_{t_1+1,k_2})\varphi_{k_2+1} \in S_\varphi \quad (3.20)$$

which is linearly independent of $\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}\}$ since $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_1}, \varphi_{k_1+1}, \ldots, \varphi_{k_2+1}$ are linearly independent.

Using the fact that $e_{t_1+2} \neq 0$ implies that $|d_{t_1+2,j} - c_{t_1+1,j}| \neq 0$ for some $k_1 + 1 \leq j \leq k_2$, we define the vector

$$\begin{aligned} w = \psi_{j-1} &+ \frac{(d_{t_1+2,k_1+1} - c_{t_1+1,k_1})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{k_1} \\ &+ \cdots + \frac{(d_{t_1+2,j-1} - c_{t_1+1,j-2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{j-2} \\ &+ \frac{(d_{t_1+2,j+1} - c_{t_1+1,j})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_j \\ &+ \cdots + \frac{(d_{t_1+2,k_2+1} - c_{t_1+1,k_2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{k_2} \quad (3.21) \end{aligned}$$

which is linearly independent of $\{\tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\}$ since $\psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}$ are linearly independent, and rewrite (3.20) as

$$\begin{aligned} \varphi_j = &\frac{1}{(d_{t_1+2,j} - c_{t_1+1,j})}v - \frac{(d_{t_1+2,k_1+1} - c_{t_1+1,k_1})}{(d_{t_1+2,j} - c_{t_1+1,j})}\varphi_{k_1+1} \\ &- \cdots - \frac{(d_{t_1+2,j-1} - c_{t_1+1,j-2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\varphi_{j-1} \\ &- \frac{(d_{t_1+2,j+1} - c_{t_1+1,j})}{(d_{t_1+2,j} - c_{t_1+1,j})}\varphi_{j+1} \\ &- \cdots - \frac{(d_{t_1+2,k_2+1} - c_{t_1+1,k_2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\varphi_{k_2+1}\,. \quad (3.22) \end{aligned}$$

Combining (3.19), (3.21), (3.20) and (3.22) we compute

$$\begin{aligned} \theta^{\mathrm{T}} w \triangleq\ &\theta^{\mathrm{T}}\Bigg[\psi_{j-1} + \frac{(d_{t_1+2,k_1+1} - c_{t_1+1,k_1})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{k_1} \\ &+ \cdots + \frac{(d_{t_1+2,j-1} - c_{t_1+1,j-2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{j-2} \\ &+ \frac{(d_{t_1+2,j+1} - c_{t_1+1,j})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_j \\ &+ \cdots + \frac{(d_{t_1+2,k_2+1} - c_{t_1+1,k_2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\psi_{k_2}\Bigg] \\ =\ &\theta^{\mathrm{T}}(\psi_{j-1} + \varphi_j) - \frac{1}{(d_{t_1+2,j} - c_{t_1+1,j})}\theta^{\mathrm{T}} v \\ &+ \frac{(d_{t_1+2,k_1+1} - c_{t_1+1,k_1})}{(d_{t_1+2,j} - c_{t_1+1,j})}\theta^{\mathrm{T}}(\psi_{k_1} + \varphi_{k_1+1}) \\ &+ \cdots + \frac{(d_{t_1+2,j-1} - c_{t_1+1,j-2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\theta^{\mathrm{T}}(\psi_{j-2} + \varphi_{j-1}) \\ &+ \frac{(d_{t_1+2,j+1} - c_{t_1+1,j})}{(d_{t_1+2,j} - c_{t_1+1,j})}\theta^{\mathrm{T}}(\psi_j + \varphi_{j+1}) \\ &+ \cdots + \frac{(d_{t_1+2,k_2+1} - c_{t_1+1,k_2})}{(d_{t_1+2,j} - c_{t_1+1,j})}\theta^{\mathrm{T}}(\psi_{k_2} + \varphi_{k_2+1}) \\ =\ &x_{j+1} - \frac{1}{(d_{t_1+2,j} - c_{t_1+1,j})}\{x_{t_1+3} - c_{t_1+1,k_1}x_{k_1+2} \\ &- \cdots - c_{t_1+1,k_2}x_{k_2+2} - d_{t_1+2,1}\tilde{x}_2 \\ &- \cdots - d_{t_1+2,k_1}\tilde{x}_{k_1+1}\} \\ &+ \frac{(d_{t_1+2,k_1+1} - c_{t_1+1,k_1})}{(d_{t_1+2,j} - c_{t_1+1,j})}x_{k_1+2} \\ &+ \cdots + \frac{(d_{t_1+2,j-1} - c_{t_1+1,j-2})}{(d_{t_1+2,j} - c_{t_1+1,j})}x_j \\ &+ \frac{(d_{t_1+2,j+1} - c_{t_1+1,j})}{(d_{t_1+2,j} - c_{t_1+1,j})}x_{j+2} \\ &+ \cdots + \frac{(d_{t_1+2,k_2+1} - c_{t_1+1,k_2})}{(d_{t_1+2,j} - c_{t_1+1,j})}x_{k_2+2}\,. \quad (3.23) \end{aligned}$$

is linearly independent of $\{\tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}\}$ since $\psi_{k_1}, \ldots, \psi_{k_2}, \tilde{\psi}_{k_2+1}, \ldots, \tilde{\psi}_{k_3}$ are linearly independent.

Therefore, we have recovered the projections of θ along $v \in S_\psi$ and $w \in S_\varphi$. Hence, we have reduced the number of the projections needed to be recovered in both subspaces S_φ and S_ψ.

Subcase 2.2 $\boxed{e_{t_1+2} = 0}$: If we have already collected all the projections of θ along the vectors which constitute a basis for the subspace S, then we have obtained enough information for pre-computing the terms in (2.3). Otherwise, since $e_{t_1+2} = 0$, at $t = t_1 + 2$ we can pre-compute the state x_{t_1+3} as (compare to (3.18) with $e_{t_1+2} = 0$)

$$\begin{aligned} x_{t_1+3} &= c_{t_1+1,k_1}\theta^{\mathrm{T}}(\psi_{k_1} + \varphi_{k_1+1}) \\ &\quad + \cdots + c_{t_1+1,k_2}\theta^{\mathrm{T}}(\psi_{k_2} + \varphi_{k_2+1}) \\ &\quad + d_{t_1+2,1}\theta^{\mathrm{T}}\tilde{\varphi}_1 + \cdots + d_{t_1+2,k_1}\theta^{\mathrm{T}}\tilde{\varphi}_{k_1} \\ &= c_{t_1+1,k_1}x_{k_1+2} + \cdots + c_{t_1+1,k_2}x_{k_2+2} \\ &\quad + d_{t_1+2,1}\tilde{x}_2 + \cdots + d_{t_1+2,k_1}\tilde{x}_{k_1+1}\,, \quad (3.24) \end{aligned}$$

and hence we can pre-compute the values of ψ_{t_1+3} and φ_{t_1+3}. If $\psi_{t_1+3} \notin S_{\Psi_{t_1}}$ or $\varphi_{t_1+3} \notin S_{\Phi_{t_1}}$, then we are back to Case 1. If $\psi_{t_1+3} \in S_{\Psi_{t_1}}$ and $\varphi_{t_1+3} \in S_{\Phi_{t_1}}$, then at $t = t_1 + 2$ we can find constants $c_{t_1+3,k_1}, \ldots, c_{t_1+3,k_3}, d_{t_1+3,1}, \ldots, d_{t_1+3,k_2+1}$ such that

$$\begin{aligned} \psi_{t_1+3} &= c_{t_1+3,k_1}\psi_{k_1} + \cdots + c_{t_1+3,k_2}\psi_{k_2} \\ &\quad + c_{t_1+3,k_2+1}\tilde{\psi}_{k_2+1} + \cdots + c_{t_1+3,k_3}\tilde{\psi}_{k_3} \quad (3.25) \\ \varphi_{t_1+3} &= d_{t_1+3,1}\tilde{\varphi}_1 + \cdots + d_{t_1+3,k_1}\tilde{\varphi}_{k_1} \\ &\quad + d_{t_1+3,k_1+1}\varphi_{k_1+1} \\ &\quad + \cdots + d_{t_1+3,k_2+1}\varphi_{k_2+1}\,, \quad (3.26) \end{aligned}$$

from which at $t = t_1 + 2$ we pre-compute

$$\begin{aligned} e_{t_1+3} &= |d_{t_1+3,k_1+1} - c_{t_1+2,k_1}| \\ &\quad + \cdots + |d_{t_1+3,k_2+1} - c_{t_1+2,k_2}|\,. \quad (3.27) \end{aligned}$$

If $e_{t_1+3} \neq 0$, then we are back in Subcase 2.1. Otherwise, at $t = t_1 + 2$ we choose

$$\begin{aligned} u_{t_1+2} &= \alpha - c_{t_1+2,k_1} x_{k_1+2} - \cdots - c_{t_1+2,k_2} x_{k_2+2} \\ &\quad -d_{t_1+3,1}\tilde{x}_2 - \cdots - d_{t_1+3,k_1}\tilde{x}_{k_1+1} \\ &\quad -c_{t_1+2,k_2+1}\theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} - \cdots - c_{t_1+2,k_3}\theta^{\mathrm{T}}\tilde{\psi}_{k_3} \\ &= \alpha - c_{t_1+2,k_1} x_{k_1+2} - \cdots - c_{t_1+2,k_2} x_{k_2+2} \\ &\quad -d_{t_1+3,1}\tilde{x}_2 - \cdots - d_{t_1+3,k_1}\tilde{x}_{k_1+1} \\ &\quad -c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+2,k_3}\tilde{x}_{k_3+2} \end{aligned} \tag{3.28}$$

with α to be determined next. Therefore, at $t = t_1 + 2$ using (3.15) and (3.26) we can pre-compute

$$\begin{aligned} x_{t_1+4} &= u_{t_1+2} + \theta^{\mathrm{T}}(\varphi_{t_1+3} + \psi_{t_1+2}) \\ &= \alpha - c_{t_1+2,k_1} x_{k_1+2} - \cdots - c_{t_1+2,k_2} x_{k_2+2} \\ &\quad -d_{t_1+3,1}\tilde{x}_2 - \cdots - d_{t_1+3,k_1}\tilde{x}_{k_1+1} \\ &\quad -c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+2,k_3}\tilde{x}_{k_3+2} \\ &\quad +d_{t_1+3,1}\theta^{\mathrm{T}}\tilde{\varphi}_1 + \cdots + d_{t_1+3,k_1}\theta^{\mathrm{T}}\tilde{\varphi}_{k_1} \\ &\quad +d_{t_1+3,k_1+1}\theta^{\mathrm{T}}\varphi_{k_1+1} + \cdots + d_{t_1+3,k_2+1}\theta^{\mathrm{T}}\varphi_{k_2+1} \\ &\quad +c_{t_1+2,k_1}\theta^{\mathrm{T}}\psi_{k_1} + \cdots + c_{t_1+2,k_2}\theta^{\mathrm{T}}\psi_{k_2} \\ &\quad +c_{t_1+2,k_2+1}\theta^{\mathrm{T}}\tilde{\psi}_{k_2+1} + \cdots + c_{t_1+2,k_3}\theta^{\mathrm{T}}\tilde{\psi}_{k_3} \\ &= \alpha - c_{t_1+2,k_1} x_{k_1+2} - \cdots - c_{t_1+2,k_2} x_{k_2+2} \\ &\quad -d_{t_1+3,1}\tilde{x}_2 - \cdots - d_{t_1+3,k_1}\tilde{x}_{k_1+1} \\ &\quad -c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+2,k_3}\tilde{x}_{k_3+2} \\ &\quad +d_{t_1+3,1}\tilde{x}_2 + \cdots + d_{t_1+3,k_1}\tilde{x}_{k_1+1} \\ &\quad +(d_{t_1+3,k_1+1} - c_{t_1+2,k_1})\theta^{\mathrm{T}}\varphi_{k_1+1} \\ &\quad +\cdots + (d_{t_1+3,k_2+1} - c_{t_1+2,k_2})\theta^{\mathrm{T}}\varphi_{k_2+1} \\ &\quad +c_{t_1+2,k_1}\theta^{\mathrm{T}}(\varphi_{k_1+1} + \psi_{k_1}) \\ &\quad +\cdots + c_{t_1+2,k_2}\theta^{\mathrm{T}}(\varphi_{k_2+1} + \psi_{k_2}) \\ &\quad +c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} + \cdots + c_{t_1+2,k_3}\tilde{x}_{k_3+2} \\ &= \alpha - c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} - \cdots - c_{t_1+2,k_3}\tilde{x}_{k_3+2} \\ &\quad +c_{t_1+2,k_2+1}\tilde{x}_{k_2+3} + \cdots + c_{t_1+2,k_3}\tilde{x}_{k_3+2} \\ &= \alpha\,. \end{aligned} \tag{3.29}$$

This means that α can be chosen to yield

$$\begin{aligned} &|d_{t_1+4,k_1+1} - c_{t_1+3,k_1}| \\ &+\cdots + |d_{t_1+4,k_2+1} - c_{t_1+3,k_2}| \neq 0\,, \end{aligned} \tag{3.30}$$

with $d_{t_1+4,k_1+1}, \ldots, d_{t_1+4,k_2+1}$ being determined by

$$\begin{aligned} \varphi_{t_1+4} &= d_{t_1+4,1}\tilde{\varphi}_1 + \cdots + d_{t_1+4,k_1}\tilde{\varphi}_{k_1} \\ &\quad +d_{t_1+4,k_1+1}\varphi_{k_1+1} \\ &\quad +\cdots + d_{t_1+4,k_2+1}\varphi_{k_2+1}\,. \end{aligned} \tag{3.31}$$

It is always possible to satisfy (3.30), because we can drive x_{t_1+4} to any value. This brings us back to Subcase 2.1.

4 Concluding Remarks

The procedure given in the previous section will continue until all the projections of θ along a basis of S_φ and S_ψ are recovered. Once we have this information, we can implement any controller which requires the computation of $\theta^{\mathrm{T}}\varphi$ and $\theta^{\mathrm{T}}\psi$. One particular example is the "look-ahead" controller described in Section 2, but the same projection information can be used with any other control design.

The procedure we presented here for the second-order nonlinear system (2.1)–(2.2) can be generalized to output-feedback systems (i.e., systems in which the nonlinearities depend only on the measured output) of arbitrary order. Obviously, the expressions for the general case are quite complicated, but the basic idea remains the same: the subspace decomposition must be applied at each step, and the control input must be chosen so that new directions are added until all the necessary projections are collected.

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