Transfer Function Equivalence of Feedback/Feedforward Compensators *

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Abstract

Equivalence of several feedback and/or feedforward compensation schemes in linear systems is investigated. The classes of compensators that are realizable using static or dynamic state feedback are characterized. Stability of the compensated system is studied. Applications to model matching are included.

1 Introduction

This is a tutorial which presents a study of equivalence, from the transfer function point of view, of several commonly used feedback and/or feedforward compensation schemes. It is shown that a cascade compensator is equivalent to a two–degree–of–freedom compensator as well as to a static state feedback applied to a dynamic extension of the system.

The subclasses of these compensators that are equivalent to a standard static or dynamic state feedback are identified. The proofs are constructive and provide simple design procedures.

Of course that two transfer–function equivalent compensators can have different internal properties. That is why a result on the stability of the overall closed–loop system is included.

These results are important *per se* in linear system theory. They are also useful in applications. A typical application area is the model matching problem. The results presented allow splitting the problem in two linear subproblems: first a cascade compensator is determined to achieve the match and then realized in terms of the configuration desired.

2 Classes of Compensators

We shall study several common feedback and/or feedforward configurations with an eye on the equivalence of various compensation schemes.

Consider a linear system governed by the equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $u \in R^m$ is the input and $x \in R^n$ is the state. The system gives rise to the tranfer function

$$T(s) = (sI - A)^{-1}B \tag{2}$$

which is a rational, strictly proper $n \times m$ matrix.

A common compensation scheme used to modify (1) is the *static state feedback* defined by

$$u(s) = Fx(s) + Gv(s) \tag{3}$$

where $v \in R^m$ is an external input and F, G are constant matrices.

A more general compensator is one which involves a *dynamic state feedback* according to the equation

$$u(s) = F(s)x(s) + Gv(s) \tag{4}$$

where F is a proper rational matrix and G is constant.

Generalizing further, one can define a compensator of the form

$$u(s) = F(s)x(s) + G(s)v(s) \tag{5}$$

which makes explicit the presence of a dynamic state feedback as well as a dynamic feedforward, the so–called *two–degree–of–freedom* compensator. Here F and G are proper rational matrices of appropriate sizes.

The equation

$$u(s) = K(s)v(s) \tag{6}$$

where K is a proper rational matrix, defines a pure feedforward dynamic compensator, or *cascade compensator*, which is frequently used in the classical control theory.

Finally, a set of p integrators

$$\dot{x}'(t) = u'(t)$$

can be adjoined to system (1) to give an extended system. A *static state feedback applied to the extended system* according to the equations

$$\begin{aligned} u(s) &= F_{11}x(s) + F_{12}x'(s) + G_1v(s) \\ u'(s) &= F_{21}x(s) + F_{22}x'(s) + G_2v(s) \end{aligned} \tag{7}$$

will result in a dynamic feedback and feedforward relative to the original systems (1).

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3 Transfer Function Equivalence

Consider the classes of compensators defined by (3) – (7). Each class is obtained by allowing F and G to vary within the specified limits.

Two compensator classes are said to be *transfer function equivalent* if, for any compensator of one class, one can find a compensator in the other class such that their application to the given system (1) will result in overall systems having the same transfer functions.

This kind of equivalence reflects just the ability of two compensators to produce the same input–output behaviour. In particular this equivalence says nothing about dynamical order, stability, and other properties of systems which depend on a particular realization.

Our first goal is to investigate which classes are transfer function equivalent.

Theorem 1 [4], [6] The compensator classes (5), (6), and (7) are transfer function equivalent.

Proof: We shall establish the following chain of implications.

We show that each compensator (5) can be represented in the form (6). To see this, we apply (5) to equation (1) in the transfer function form,

$$x(s) = T(s)u(s)$$

and calculate the transfer function from v to u. Comparing with (6), one obtains

$$K(s) = [I - F(s)T(s)]^{-1}\, G(s).$$

Since T is strictly proper, $I - FT$ is bi–proper. Hence K is proper.

We now show that any compensator (6) can be realized in the form (7). Given a proper rational K, let

$$K(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

for some state–space ralization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. Then

$$\begin{aligned} F_{11} &= 0 \quad F_{12} = \bar{C} \quad G_1 = \bar{D} \\ F_{21} &= 0 \quad F_{22} = \bar{A} \quad G_2 = \bar{B} \end{aligned}$$

define a state feedback of the form (7).

Finally let us show that each compensator (7) can be represented in the form (5). To this end we apply (7) to the extended system to obtain the overall system equations

$$\begin{aligned} \dot{x}(s) &= (A + BF_{11})x(s) + BF_{12}x'(s) + G_1 v(s) \\ \dot{x}'(s) &= F_{21}x(s) + F_{22}x'(s) + G_2 v(s) \\ u(s) &= F_{11}x(s) + F_{12}x'(s) + G_1 v(s) \end{aligned}$$

and calculate the transfer functions from x and v to u. On identifying with (5), one obtains

$$\begin{aligned} F(s) &= F_{11} + F_{12}(sI - F_{22})^{-1}F_{21} \\ G(s) &= G_1 + F_{12}(sI - F_{22})^{-1}G_2. \end{aligned}$$

Since $sI - F_{22}$ has a strictly proper inverse, both F and G are proper rational matrices. □

In view of this equivalence, the simplest configuration (6), namely a cascade compensator, will be used to represent any of the above feedback/feedforward compensators.

The class of static/dynamic state feedback compensators (3) and (4) is less general than (6) and will be studied in the sections to follow.

4 Dynamic State Feedback

Dynamic state feedback (4) is a special case of (5), hence of (6). It is interesting to identify the subclass of cascade compensators K which are transfer function equivalent to dynamic state feedback.

These compensators satisfy

$$K(s) = [I - F(s)T(s)]^{-1}G. \tag{8}$$

We impose a technical assumption that G is non–singular; this will greatly simplify the analysis [3].

Theorem 2 [1], [6]. Given a proper rational $m \times m$ matrix K, there exist a proper rational F and a constant non–singular G such that (8) holds if and only if K is bi–proper.

Proof: Since T is strictly proper, $I - FT$ is bi–proper. Since G is non–singular, K is bi–proper as well.

Conversely, suppose that K is bi–proper. Let G be defined by

$$G = K(\infty).$$

Then $V(s) = K^{-1}(s) - G^{-1}$ is a strictly proper rational matrix. The equation

$$V(s) = X(s)T(s) \tag{9}$$

has a proper rational solution X if and only if the infinite zero structure of T coincides with that of $\begin{bmatrix} T \\ V \end{bmatrix}$. The infinite zero structure of T is given by $(s^{-1}, \ldots, s^{-1})$. Since V is strictly proper, the solvability condition is verified and a proper rational X exists that satisfy (9). Let F be defined by

$$F(s) = -GX(s).$$

Then

$$K^{-1}(s) = G^{-1} - G^{-1}F(s)T(s)$$

and (8) holds. □

5 Static State Feedback

This is a further specialization in which both F and G are constant. Which cascade compensators $K(s)$ are transfer function equivalent to static state feedback (3)? Those which satisfy

$$K(s) = [I - FT(s)]^{-1}G. \qquad (10)$$

We again assume that G is non–singular and write T in the form

$$T(s) = N(s)D^{-1}(s) \qquad (11)$$

where N and D are right coprime polynomial matrices.

Theorem 3 [2], [6]. Given a proper rational $m \times m$ matrix K, there exist constant matrices F and G with G non–singular, such that (10) holds if and only if

(a) K is bi–proper

(b) $K^{-1}D$ is polynomial.

Proof: Condition (a) follows from Theorem 2. Then

$$K^{-1}(s)D(s) = G^{-1}D(s) - G^{-1}F\,N(s)$$

is a polynomial matrix, which is (b).

Conversely, let K satisfy (a) and define G by

$$G = K(\infty).$$

Then $V(s) = K^{-1}(s) - G^{-1}$ is a strictly proper rational matrix. Furthermore, let K satisfy (b). Then

$$V(s) = M(s)D^{-1}(s)$$

for a polynomial matrix M. Polynomial row vectors $w(s)$ such that $w(s)D^{-1}(s)$ is strictly proper form an R–linear space $\mathcal{V}$. Using (11), we have

$$T(s) = N(s)D^{-1}(s)$$

and note that the rows of N span $\mathcal{V}$. Therefore the equation

$$V(s) = XT(s)$$

has a constant solution X and

$$F = -GX$$

makes (10) hold. □

If system (1) is controllable, then the rows of N form a basis for $\mathcal{V}$ and the matrices F, G that realize K are unique.

6 Stability

Transfer function equivalent compensators can have different internal properties, those which depend on a particular realization.

Stability is the most important design specification of this sort. That is why it is natural to ask when a compensator, which is transfer function equivalent to a cascade compensator (6), stabilizes the system.

The requirement of stability will mean that the states of the system and of the compensator go to zero from all initial values. A necessary requisite is of course that system (1) is stabilizable.

Theorem 4 . Suppose that a cascade compensator (6) is transfer function equivalent to a compensator of the form (3), (4), (5) or (7). Suppose that system (1) with transfer function (11) is stabilizable. Then the compensator, no matter whether (3), (4), (5) or (7), will stabilize the system if and only if the rational matrix $D^{-1}K$ is stable (i.e., analytic in $\mathrm{Re}\, s \geq 0$).

Proof: For the system, use (11) to write

$$T(s) = N(s)D^{-1}(s)$$

where N and D are right coprime polynomial matrices. For the compensator, write

$$F(s) = -P^{-1}(s)\,Q(s)$$
$$G(s) = P^{-1}(s)\,R(s)$$

where P, Q and R is a triple of left coprime polynomial matrices. Of course F and G can be constant in special cases. The compensator will stabilize the sytem if and only if the rational matrix $(PD+QN)^{-1}$ exists and is stable [8].

Thus,

$$D^{-1}(s)K(s) = (PD + QN)^{-1}(s)R(s)$$

is stable whenever $PD + QN$ verifies the condition. On the other hand, write

$$D^{-1}(s)K(s) = X^{-1}(s)Y(s)$$

for some left coprime polynomial matrices X and Y. Define polynomial matrices P, Q and R by

$$P(s)D(s) + Q(s)N(s) = X(s)$$
$$R(s) = Y(s)$$

and the requirement that $Q(s)(sI - A)^{-1}$ is strictly proper. The resulting matrices F and G are proper rational and define a stabilizing compensator. □

It is important to note that we can ascertain whether an equivalent compensator will be stabilizing before it is actuallly calculated. On the other hand, there is no freedom to try and design a stabilizing compensator; one can only check whether or not the resulting system will be stable.

If the given compensator K can be realized using static state feedback (3), the condition of Theorem 4 can be given a simple interpretation [5]. In fact, $K^{-1}D$ is a polynomial matrix in this case and its determinant is the characteristic polynomial of the closed–loop system.

7 Model Matching

A typical application of the above results is the problem of *model matching* [6], [7], [9]. Given a plant

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with a strictly proper, rational $m \times m$ transfer function matrix T_P and a model transfer function matrix T_M, which is assumed to be also strictly proper, rational, and of size $m \times m$. We seek to find a compensator, specified in one of the forms (3) – (7), such that the closed–loop system is stable and has transfer matrix T_M.

The model matching equation

$$T_P(s)[I - F(s)T(s)]^{-1}G(s) = T_M(s)$$

immediately suggests the following two–step solution: determine a matching compensator K from the equation

$$T_P(s)K(s) = T_M(s) \tag{12}$$

and then realize K in one of the forms (3), (4), (5) or (7) desired,

$$K(s) = [I - F(s)T(s)]^{-1}G(s)$$

where F and G are either proper rational or constant matrices.

The matching equation (12) has a proper rational solution K if and only if the matrices $[T_P \;\; T_M]$ and T_P have identical infinite zero structure [6]. In the scalar case, this means that the relative degree of T_P does not exceed that of T_M.

Using the equivalence result provided by Theorem 1, the above condition is necessary and sufficient to achieve the match via any of the two–degree–of–freedom compensation schemes (5) or (7).

Suppose we want to implement dynamic state feedback (4). Theorem 2 requires that K be bi–proper. Thus the equation

$$T_M(s)K^{-1}(s) = T_P(s)$$

should have a proper rational sotion $K^{-1}(s)$. This is the case if and only if the matrices $[T_P \;\; T_M]$ and T_M have identical infinite zero structure [6]. Combining the two conditions, a match via (4) is possible if and only if T_P and T_M have identical infinite zero structure. This reduces to identical relative degrees in the scalar case.

Finally, let us realize the match using static state feedback (3). Theorem 3 imposes a further condition that $K^{-1}D$ be polynomial. Writing T_P and T_M in terms of their right coprime polynomial factorizations,

$$T_P(s) = N_P(s)D^{-1}(s)$$
$$T_M(s) = N_M(s)E^{-1}(s)$$

and using (12), we observe that

$$K^{-1}(s)D(s) = E(s)\, N_M^{-1}(s)\, N_P(s)$$

is a polynomial matrix if and only if N_M divides N_P on the left. This means that the equation

$$N_M(s)X(s) = N_P(s)$$

must be solvable for a polynomial matrix X. A necessary and sufficient condition is that the matrices $[T_P \;\; T_M]$ and T_M have identical finite zeros structure [6].

Having achieved the match desired, we can check for stability of the closed–loop system. Theorem 4 requires that $D^{-1}K$ be stable, which means that the equation

$$N_P(s)Y(s) = N_M(s)$$

is to have a stable rational solution Y. Thus a stable match can be achieved if and only if the matrices $[T_P \;\; T_M]$ and T_P have identical finite unstable zeros structure [6]. In the scalar case, this amounts to the requirement that all non–minimum–phase zeros of T_P must be included in T_M.

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