CONSTRAINED CONTROLLABILITY OF DYNAMICAL SYSTEMS

J e r z y K L A M K A

Institute of Automation, Technical University
ul. Akademicka 16 , 44-100 GLIWICE , POLAND
e-mail :jklamka @ia.polsl.gliwice.pl
phone/fax : (32) 37-21-27

Abstract. The present paper is devoted to a study of constrained controllability and controllability for linear dynamical systems if the controls are taken to be nonnegative. In analogy to the usual definition of controllability it is possible to introduce the concept of positive controllability. We shall concentrate on appro-ximate positive controllability for linear infinite-dimensio-nal dynamical systems when the values of controls are ta-ken from a positive closed convex cone and the operator of the system is normal and has pure discrete point spectrum. The special attention is paid for positive infinite-dimensional linear dynamical systems. General approxi-mate constrained controllability results are then applied for distributed parameter dynamical systems described by linear partial differential equations of parabolic type with different kinds of boundary conditions. Several remarks and comments on the relationships between different concepts of controllability are given. Finally, simple numerical illustrative example is also presented.

1. Introduction

Controllability is one of the fundamental concept in mathematical control theory [1], [3], [6]. Roughly speaking, controllability generally means , that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability which depend on class of dynamical system [1], [3], [9], [12], [14], [16]. Problems of controllability for linear control systems defined in infinite-dimensional Banach spaces, have attracted a good deal of interest over the past 20 years. For infinite dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [1], [3], [6], [7] [12], [13], [14], [15] and [16]. It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces which are not closed. Most of the literature in this direction so far has been concerned however, with unconstrained controllability, and little is known for the case when the control is restricted to take on values in a given subset of the control space. Until now, scare attention has been paid to the important case where the control of a system are nonnegative. In this case controllability is possible only if the system is oscillating in some sense. Therefore, the most difficult case for constrained controllability is for dynamical systems with real eigenvalues [11]. The present paper is devoted to a study of constrained approximate controllability [8], [11] for linear normal infinite-dimensional dynamical systems if the controls are taken to be nonnegative. In analogy to the usual definition of controllability it is possible to introduce the concept of approximate positive controllability [9]. For such dynamical systems direct verification of constrained ap-proximate controllability is rather difficult and complica-ted [8]. Therefore, we generally assume that the values of controls are taken from a positive closed convex cone [11] and the operator of the system is normal and has pure dis-crete point spectrum [12], [14]. The special attention is paid for positive infinite-dimensional linear dynamical systems i.e., for dynamical systems preserving positivity [9]. General constrained approximate controllability re-sults then are applied for general distributed parameter dynamical systems described by linear partial differential equations of parabolic type with different kinds of boun-dary conditions. Finally, as a numerical illustrative exam-ple constrained approximate controllability of one dimen-sional heat equation with homogeneous Dirichlet bounda-ry conditions and scalar nonnegative control is considered.

2. Notations and system description

In this section we introduce some basic notations and definitions which will be used in the parts of the paper. Throughout this paper we use X to denote infinite dimensional separable real Hilbert space. By $L^p([0,t],R^m)$, $1\leq p\leq\infty$ we denote the space of all p-integrable functions on [0,t] with values in R^m, and $L^p_\infty([0,\infty),R^m)$ the space of all locally p-integrable functions on $[0,\infty)$ with values in R^m. Following [9] and [10] we define an order $\leq$ in the space X such that $(X,\leq)$ is a lattice and the ordering is compatible with the structure of X, i.e. X is an ordered vector space. This imply that the set $X^+=\{x\in X : x\geq 0\}$ is a convex positive cone with vertex at zero. Moreover, it follows that $x_1 \leq x_2$ if and only if $x_2 - x_1\in X^+$. An element $x\in X^+$ is called positive, and we write $x>0$ if x is positive and different from zero. Moreover, an element $x^*\in X^+$ is called strictly positive, and we write $x^*>>0$ if $\langle x^*,x\rangle_X > 0$ for all $x>0$. An ordered vector space X is called a vector lattice if any two elements x_1, x_2 in X have a supremum and an infimum denoted by $\sup\{x_1,x_2\}$, respectively, $\inf\{x_1,x_2\}$. For an element x of vector lattice we write $|x|=\sup\{x,-x\}$ and call it the absolute value of x. We call two elements x_1,x_2 of vector lattice X orthogonal, if $\inf\{|x_1|,|x_2|\}=0$. Linear form $w\in X$ is called positive $(w\geq 0)$ if $\langle w,x\rangle_X\geq 0$ for all $x\geq 0$ and strictly positive $(w>>0)$ if $\langle w,x\rangle_X>0$ for all $x>0$. Relevant examples of vector lattices with a strictly positive linear form are given by the following spaces of practical interest: R^n and $L^2(\Omega,R)$, where Ω is a measurable subset of R^n. Linear bounded operator F from a vector lattice X into a vector lattice V is called positive i.e. $F\geq 0$, if $Fx\geq 0$ for $x\geq 0$. Therefore, positive operator F maps positive cone X^+ into positive cone V^+. Let $S(t):X\rightarrow X$, $t\geq 0$, be a strongly continuous semigroup of bounded linear operators. We call the semigroup positive i.e. $S\geq 0$, if X is a vector lattice and S(t) is a positive operator for every $t\geq 0$. If set $M\subseteq X$, we define the polar cone by $M^o=\{w\in X,\ \langle w,x\rangle_X \leq 0$ for all $x\in M\}$. The closure, the convex hull and the interior are denoted respectively, by cl M, co M and int M.

Let us consider linear infinite-dimensional time-invariant control system of the following form

$$x'(t) = Ax(t) + Bu(t) \tag{2.1}$$

Here $x(t)\in X$ is infinite-dimensional separable Hilbert space which is a vector lattice with a strictly positive linear form. B is a linear bounded operator from R^m into X. Therefore operator $B=[b_1,b_2,\ldots,b_j,\ldots,b_m]$ and

$$Bu(t) = \sum_{j=1}^{j=m} b_j u_j(t)$$

where elements $b_j \in X$ for $j=1,2,...,m$, and $u(t)=[u_1(t),u_2(t),...,u_j(t),...,u_m(t)]^{tr}$. We would like to emphasize that the assumption that linear operator B is bounded, rules out the application of our theory to boundary control problems, because in this situation B is typically unbounded. $A: X \supset D(A) \to X$ is normal generally unbounded linear operator with compact resolvent $R(s,A)$ for all s, in the resolvent set $\rho(A)$. Then operator A has the following properties [1], [14], [16]:

1) Operator A has only pure discrete point spectrum $\sigma_p(A)$ consisting entirely with isolated eigenvalues s_i, $i=1,2,3,...$ Moreover, each eigenvalue s_i has finite multiplicity $n_i < \infty$, $i=1,2,3,...$ equal to the dimensionality of the corresponding eigenmanifold.

2) The eigenvectors $x_{ik} \in D(A)$, $i=1,2,3,...$ $k=1,2,3,...,n_i$, form a complete orthonormal set in the separable Hilbert space X.

3) Operator A generates an analytic semigroup of linear bounded operators $S(t): X \to X$, for $t \geq 0$.

Let $U^+ \subset R^m$ be a positive cone in the space R^m, i.e. $U^+ = \{u \in R^m : u_j \geq 0 \text{ for } j=1,2,...,m\}$. We define the set of admissible nonnegative controls U_{ad} as follows $U_{ad} = \{u \in L^p_{loc}([0,\infty),R^m) ;\ u(t) \in U^+ \text{ a.e. on } [0,\infty)\}$. It is well known (see e.g. [1], [3] or [16]), that for each $u \in U_{ad}$ and $x(0) \in X$ there exists unique so called mild solution $x(t,x(0),u) \in D(A)$, $t \geq 0$ of the equation (2.1) given by

$$x(t,x(0),u) = S(t)x(0) + \int_0^t S(t-s)Bu(s)ds$$

We say that dynamical system (2.1) is positive if the semigroup S and operator B are positive [9]. In this case the solution $x(t,x(0),u)$ for initial condition $x(0) \in X^+$ and admissible control $u \in U_{ad}$ remains in X^+ for all $t \geq 0$. We define the attainable or reachable set in time T (from the origin) by

$$K_T(U^+) = \{\int_0^T S(T-s)Bu(s)ds \quad : \quad u \in U_{ad}$$

The set $K_\infty(U^+) = \bigcup_{T>0} K_T(U^+)$ is called the attainable or reachable set in finite time.

Using the concept of attainable set we may define different kinds of controllability for dynamical system (2.1). Generally, for infinite dimensional dynamical system it is necessary to introduce two fundamental notions of controllability, namely exact (strong) controllability and approximate (weak) controllability. However, since our dynamical system has infinite dimensional state space X and finite dimensional control space R^m, then by [13] and [15] it is never exactly controllable in any sense. Therefore, in the sequel we shall concentrate only on approximate controllability with positive controls for dynamical system (2.1).

Definition 2.1 . [1], [3], [6]. Dynamical system (2.1) is said to be approximately controllable with nonnegative controls if $\text{cl } K_\infty(U^+) = X$

In the unconstrained case, i.e. when the controls values are taken from the whole space R^m, we say simply about approximate controllability of dynamical system (2.1). The above notion of approximate controllability is defined in the sense that we want to reach a dense subspace of the entire state space. However, in many instances for positive systems with nonnegative controls, it is known that all states are contained in a closed positive cone X^+ of the state space. In this case approximate controllability in the sense of the above definition is impossible but it is interesting to know conditions under which the reachable states are dense in X^+. This observation leads to the concept of so-called positive approximate controllability.

Definition 2.2. [9] Dynamical system (2.1) is said to be approximately positive controllable if $\text{cl } K_\infty(U^+) = X^+$.

Remark 2.1. From the above two definitions directly follows, that approximate controllability with nonnegative controls always implies approximate positive controllability. However, the converse statement is not generally true.

Finally, we shall recall some fundamental theorems concerning unconstrained and constrained approximate controllability of dynamical system (2.1).Using eigenvectors x_{ik}, $i=1,2,3,...$ $k=1,2,3,...,n_i$ we introduce for the operator B the following notation [6], [14].

$$B_i = \begin{vmatrix} \langle b_1,x_{i1}\rangle_X & \langle b_2,x_{i1}\rangle_X & ...... & \langle b_j,x_{i1}\rangle_X & ...... & \langle b_m,x_{i1}\rangle_X \\ \langle b_1,x_{i2}\rangle_X & \langle b_2,x_{i2}\rangle_X & ...... & \langle b_j,x_{i2}\rangle_X & ...... & \langle b_m,x_{i2}\rangle_X \\ ...... & ...... & ...... & ...... & ...... & ...... \\ \langle b_1,x_{ik}\rangle_X & \langle b_2,x_{ik}\rangle_X & ...... & \langle b_j,x_{ik}\rangle_X & ...... & \langle b_m,x_{ik}\rangle_X \\ ...... & ...... & ...... & ...... & ...... & ...... \\ \langle b_1,x_{in_i}\rangle_X & \langle b_2,x_{in_i}\rangle_X & ...... & \langle b_j,x_{in_i}\rangle_X & ...... & \langle b_m,x_{in_i}\rangle_X \end{vmatrix}$$

B_i, for $i = 1,2,3,...$ are $n_i \times m$-dimensional constant matrices which play an important role in controllability investigations [3], [6], [11], [14], [16]. For the case when eigenvalues s_i are simple, i.e. $n_i = 1$, for $i = 1,2,3,...$, B_i are m-dimensional row vectors

$$b^i = \left[\langle b_1,x_i\rangle_X, \langle b_2,x_i\rangle_X, ..., \langle b_j,x_i\rangle_X, ..., \langle b_m,x_i\rangle_X\right] \quad for\ i=1,2,3,...$$

For simplicity of notation let us denote $b_{ikj} = \langle b_j,x_{ik}\rangle_X$ for $i=1,2,3,...$ $k=1,2,...,n_i$, and $j=1,2,...,m$. Therefore, we may express matrices B_i and vectors b^i as follows

$$B_i = \begin{vmatrix} b_{i11} & b_{i12} & ... & b_{i1j} & ... & b_{i1m} \\ b_{i21} & b_{i22} & ... & b_{i2j} & ... & b_{i2m} \\ ... & ... & ... & ... & ... & ... \\ b_{ik1} & b_{ik2} & ... & b_{ikj} & ... & b_{ikm} \\ ... & ... & ... & ... & ... & ... \\ b_{in_i1} & b_{in_i2} & ... & b_{in_ij} & ... & b_{in_im} \end{vmatrix} \qquad for\quad i=1,2,3,....$$

$b^i = [b_{i1}, b_{i2}, ... ,b_{ij}, ... ,b_{im}]$ for $i=1,2,3,...$

Since the operator A is selfadjoint, then using the above notations it is possible to express the solution $x(t,x(0),u)$ as follows

$$x(t,x(0),u) = \sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i} v^0_{ik}(t)x_{ik} + \sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i} v^u_{ik}(t)x_{ik} \quad (2.2)$$

where

$$v^0_{ik}(t) = \exp(s_i t)\langle x(0), x_{ik}\rangle$$
$$\text{for } i = 1,2,3,\ldots \quad \text{and} \quad k = 1,2,\ldots,n_i$$
$$v^u_{ik}(t) = \int_0^t \exp s_i(t-\tau)\left(\sum_{j=1}^{j=m} b_{ikj} u_j(\tau)\right)d\tau$$
$$\text{for } i = 1,2,3,\ldots \quad \text{and} \quad k = 1,2,\ldots,n_i$$

We start with the well known (see e.g. [3], [6], [14] or [16] for details) necessary and sufficient conditions for approximate controllability with unconstrained controls.

Theorem 2.1 [14] Dynamical system (2.1) is approximately controllable if and only if rank $B_i=n_i$ for every $i=1,2,3,\ldots$

Corollary 2.1 [14] Let m=1. Then dynamical system (2.1) is approximately controllable if and only if every vector $b^i \in R^m$, $i=1,2,3,\ldots$ contains at least one nonzero element.

Now, we recall known in the literature (see [11] for details) necessary and sufficient condition of approximate controllability with nonnegative controls for dynamical system (2.1).

Theorem 2.2 [11] Dynamical system (2.1) is approximately controllable with nonnegative controls if and only if rank $B_i=n_i$ for every $i=1,2,3,\ldots$ and the columns of these matrices B_i, $i=1,2,3,\ldots$ which correspond to the real eigenvalues, form positive bases in the space R^m.

Remark 2.2 The above result implies, in particular, that the number of positive controls required for approximate controllability with nonnegative controls is at least that of the highest multiplicity of the eigenvalues plus one. Therefore, dynamical system (2.1) with one scalar nonnegative control is never approximately controllable [11]. Moreover, it should be stressed, that in general case for multiple eigenvalues, it is not so easily to verify the hypothesis that the set of given vectors forms a positive basis in the Euclidean space.

Remark 2.3 Using the concept of polar cone C^0, the results stated above can be extended for constrained controls which take their values from a given closed compact cone C with nonempty interior $\mathrm{int}C \in U_{ad}$ [11].

3. Constrained controllability

In this section we shall present results concerning constrained approximate controllability for dynamical systems (2.1). We start with the following result on approximate positive controllability.

Theorem 3.1 If there exists p and q such that eigenvalue $s_p \in R$ and coefficients b_{pqj} have the same sign for every $j=1,2,\ldots,m$, then the dynamical system (2.1) is not approximately positive controllable.

Proof. In order to prove this theorem it is sufficient to point the final state $x_f \in X^+$ which cannot be reached approximately from a given initial state $x_0 \in X^+$. We shall prove it in two steps. First, let us assume that $x_{pq} \notin X^+$. Let us take $x_0 = 0$. Therefore, by the equality (2.2) we have $v^0_{ik}(t) = 0$ for $t \geq 0$ and every $i=1,2,3,\ldots$ $k=1,2,\ldots,n_i$. Let us choose the final state $x_f \in X^+$

$$x_f = \sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i}\langle x_f, x_{ik}\rangle_X x_{ik} = \sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i} v^f_{ik} x_{ik} \in X^+$$

as follows

$x_f = \sup\{-x_{pq},0\} \in X^+$, when $b_{pqj} > 0$ for $j=1,2,\ldots,m$

$x_f = \sup(x_{pq},0\} \in X^+$, when $b_{pqj} < 0$ for $j=1,2,\ldots,m$

Therefore, $v^f_{pq} = \langle x_f,x_{pq}\rangle_X = \langle \sup\{-x_{pq},0\},x_{pq}\rangle_X < 0$, when $b_{pqj} > 0$ for $j=1,2,\ldots,m$ and $v^f_{pq} = \langle x_f,x_{pq}\rangle_X = \langle \sup\{x_{pq},0\},x_{pq}\rangle_X > 0$, when $b_{pqj} < 0$ for $j=1,2,\ldots,m$

Following (2.2) for a given p and q we have

$$v_{pq}(t) = \exp(s_p t)\langle x(0), x_{pq}\rangle_X + \int_0^t \exp(s_p(t-\tau))\left(\sum_{j=1}^{j=m} b_{pqj} u_j(\tau)\right)d\tau \tag{3.1}$$

Therefore, since the admissible controls are nonnegative i.e., $u_j(t)\geq 0$ for $j=1,2,3,\ldots,m$ and $t\geq 0$, then by (3.1) it follows that

$v_{pq}(t) > 0$, for $t\geq 0$, when $b_{pqj} > 0$ for $j=1,2,\ldots,m$

$v_{pq}(t) < 0$, for $t\geq 0$, when $b_{pqj} < 0$ for $j=1,2,\ldots,m$

Taking into account the form of the solution $x(t,0,u)$ given by equality (2.2) we have

$$\left\|x(t,0,u) - x_f\right\|_X = \left(\sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i}\left|v_{ik}(t) - v^f_{ik}\right|^2\right)^{0,5} >$$
$$\left|v_{pq}(t) - v^f_{pq}\right| > const > 0 \quad for > 0$$

Therefore, by (3.2) the final state $x_f \in X^+$ cannot be reached approximately from zero in any time using nonnegative controls.

Now, let us consider the case when eigenfunction $x_{pq} \in X^+$. Hence, similarly as in the first part of the proof, following (2.2) for a given p and q we have

$$v_{pq}(t) = \exp(s_p t)\langle x(0), x_{pq}\rangle_X + \int_0^t \exp(s_p(t-\tau))\left(\sum_{j=1}^{j=m} b_{pqj} u_j(\tau)\right)d\tau \tag{3.3}$$

Since x_{pq} is an orthonormal eigenvector, then taking $x(0) = x_{pq} \in X^+$ we have the following equality

$\langle x(0),x_{pq}\rangle_X = \langle x_{pq},x_{pq}\rangle_X = 1$

Therefore, since the admissible controls are nonnegative i.e., $u_j(t)\geq 0$ for $j=1,2,3,\ldots,m$ and $t\geq 0$, then by (3.3) it follows that

$v_{pq}(t) > 1$ for $s_p > 0$ and $b_{pqj} > 0$ for $j=1,2,3,\ldots,m$

$v_{pq}(t) < 1$ for $s_p < 0$ and $b_{pqj} < 0$ for $j=1,2,3,\ldots,m$

Since, we investigate approximate positive controllability of the dynamical system (2.1), let us choose the final state $x_f \in X^+$ and such that

$v^f_{pq} < 1$ for $s_p > 0$ and $b_{pqj} > 0$ for $j=1,2,3,\ldots,m$

$v^f_{pq} > 1$ for $s_p < 0$ and $b_{pqj} < 0$ for $j=1,2,3,\ldots,m$

Taking into account the form of the solution $x(t,0,u)$ given by equality (2.2) we have

$$\left\|x(t,0,u) - x_f\right\|_X = \left(\sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i}\left|v_{ik}(t) - v^f_{ik}\right|^2\right)^{0,5} > \tag{3.4}$$
$$\left|v_{pq}(t) - v^f_{pq}\right| > const > 0 \quad for \quad t > 0$$

Therefore, by (3.4) the final state $x_f \in X^+$ cannot be reached approximately from zero in any time using nonnegative controls.

Now, let us consider the cases when $s_p > 0$, $b_{pqj} < 0$ and $s_p<0$, $b_{pqj} > 0$. We choose the initial state $x_0 \in X^+$ and final state $x_f \in X^+$ such that $v^0_{pq} = 0$ and $v^f_{pq} > 1$, for $s_p > 0$ and $b_{pqj} < 0$ for $j=1,2,3,\ldots,m$

In that case we have $v_{pq}(t)<0$ for $t\geq 0$, and the final state $v^f_{pq}>1$ cannot be reached by nonnegative controls.

Finally, when $s_p < 0$ and $b_{pqj} > 0$ for $j=1,2,3,\ldots,m$, we choose $v^0_{pq} = 0$ and $v^f_{pq} = 0$, $v^f_{ik} > 1$ for $i,k=1,2,3,\ldots$ $i\neq p$, $k\neq q$ and uniformly stable dynamical system (2.1) and

$v^0_{pq} = 0$ and $v^f_{pq} = 0$, $v^f_{ik} < 1$ for $i,k=1,2,3,\ldots$ $i\neq p$, $k\neq q$ and not uniformly stable dynamical system (2.1).

In both cases the final state $x^f \in X^+$ cannot be reached by nonnegative controls. Hence, dynamical system (2.1) is not approximately positive controllable and our theorem follows.

From Theorem 3.1 and Remark 2.1 directly follows the next result concermimg approximate controllability of dynamical system (2.1) with nonnegative controls

Corollary 3.1. If the assumptions of Theorem 3.1 are satisfied, then the dynamical system (2.1) is not approximately controllable with nonnegative controls.

4. Positive stationary pairs

In section 3 we have obtained some negative results concerning approximate positive controllability for dynamical system (2.1). However, it is often not so important to reach the entire positive cone of the state space. It suffices to steer approximately dynamical system to particular positive states and held constant by a nonnegative control for all times. This observation directly leads to the concept of so called positive stationary pairs [9]. In this section we generally assume that the dynamical system (2.1) is positive in the sense stated in section 2.

Definition 4.1 [9] We call a pair $\{x_s,u_s\}\in(X^+\backslash\{0\})\times U^+$ positive stationary pair if $Ax_s + Bu_s = 0$. In this case $x(t,x_s,u_s) = x_s \in X^+$ is a nonzero constant solution of the equation (2.1) for $t \geq 0$, $u(t) = u_s$ and $x_s = x(0)$.

Theorem 4.1 [9] Let dynamical system (2.1) be positive and S(t) be uniformly exponentially stable positive semigroup. Then to each $u_s\in U^+\backslash\ker B$ there exists exactly one $x_s = -A^{-1}Bu_s$ such that $\{x_s,u_s\}$ is a positive stationary pair. Moreover, if $\{x_s,u_s\}$ is a positive stationary pair, and we choose $x(0)\in X^+$ and $u(t) = u_s$, $t\geq 0$, then the solution of the equation (2.1) tends to x_s as $t\to\infty$.

Corollary 4.1 Let $Re(s_1) < 0$. Then to each $u_s\in U^+\backslash\ker B$ there exists exactly one

$$x_s = \sum_{i=1}^{i=\infty} s_i^{-1}\sum_{k=1}^{k=n_i}\left\langle x_{ik}, \sum_{j=1}^{j=m} b_j u_{sj}\right\rangle_X x_{ik} \qquad (4.1)$$

such that $\{x_s,u_s\}$ is a positive stationary pair.

Proof. Since the spectrum of the linear operator $\sigma(A)$ is pure discrete point spectrum, we conclude that the inequality $Re(s_1)<0$ is a necessary and sufficient condition for so called uniform stability of linear dynamical system [1], [9]. Therefore, using general spectral formula for the operator A^{-1} and Theorem 4.1 stated above we obtain immediately equality (4.1).

Remark 4.1 Many valuable remarks and comments on the relationships between different kinds of stability (uniform exponential stability, strong stability, weak stability) of the linear abstract differential equation (2.1) and the existence of positive stationary pairs for positive dynamical systems can be found in the paper [9].

5. Parabolic typedynamical systems

In this section we shall illustrate the general theorems and corollaries stated in the previous sections 3 and 4 for the case of linear distributed parameter systems of parabolic type. We begin by describing the mathematical model of the distributed parameter system. Let Ω be a bounded, open and connected subset of R^N with a smooth boundary $\partial\Omega$ and $cl\Omega=\Omega\cup\partial\Omega$. Let Δ be the Laplacian operator on Ω and ∇ be the gradient operator on Ω. Let us consider linear distributed parameter dynamical system described by the following partial parabolic differential equation

$$w_t(z,t) = Aw(z,t) + \sum_{j=1}^{j=m} b_j(z)u_j(t) \qquad t>0 \quad z\in\Omega \qquad (5.1)$$

where $b_j\in L^2(\Omega)$, for j=1,2,3,...,m, and admissible controls are nonnegative i.e., $u_j\in L^2_{loc}([0,\infty),R^+)$, for j=1,2,3,...,m. The boundary conditions are of the following form

$$\alpha(z)w(z,t) + \beta(z)\,{}^{\partial w}/_{\partial v}(z,t) = 0 \quad t\geq 0 \quad z\in\partial\Omega \qquad (5.2)$$

It is assumed that $\alpha(z)$ and $\beta(z)$ are twice continuously differentiable on $cl\Omega$. The vector field v(z) is the outer unit normal to $\partial\Omega$ at $z\in\partial\Omega$ and ${}^{\partial}/_{\partial v} = v\nabla$ denotes differentiation in the direction of the outward norma to Ω. Specifying $\alpha(z)$ and $\beta(z)$ we obtain Dirichlet, Neumann or Robin (mixed) boundary conditions. The initial condition for equation (5.1) is given by $w(z,0) = w_0(z)$ $z\in\Omega$

The second order uniformly elliptic differential operator has the following form

$$A = \sum_{k,j=1}^{k,j=N} a_{kj}(z)D_k D_j + \sum_{k=1}^{k=N} a_k(z)D_k + a_0(z)I \qquad (5.3)$$

where $z\in R^N$, $a_{kj}(z) = a_{jk}(z)$, for j,k=1,2,3,...,N , $D_k = \partial/\partial z_k$, for k=1,2,3,...,N. The domain D(A) of the operator A is characterized explicitly by

$D(A) = \{w\in L^2(\Omega) : Aw\in L^2(\Omega)$ and $\alpha(z)w(z,t) + \beta(z)\,{}^{\partial w}/_{\partial v}(z,t) = 0,\ t\geq 0,\ z\in\partial\Omega\}$

The coefficients $a_{kj}(z)$, $a_k(z)$ and $a_0(z)$ are assumed to be twice continuously differentiable on Ω and $a_0(z)\geq 0$ for $z\in\Omega$. Moreover, since operator A is uniformly elliptic then there exists a positive constant μ such that for all vectors $\xi\in R^N$ we have

$$\sum_{k,j=1}^{k,j=N} a_{kj}(z)\xi_k\xi_j \geq \mu|\xi|^2 \quad , \quad for \ \ z\in\Omega$$

Various special cases of (5.1) could be considered i.e., the reaction diffusion dynamical system

$$w_t(z,t) = d\Delta w(z,t) + aw(z,t) + \sum_{j=1}^{j=m} b_j(z)u_j(t) \qquad (5.4)$$

where a and d are real constants.

It is well known (see e.g. [10] for details), that operator A generates an analytic positive semigroup of bounded compact operators $S(t):X\to X$ for $t\geq 0$ [10]. Moreover, since the set Ω is bounded, then the operator A has pure discrete point spectrum $\sigma(A)=\sigma_p(A)=\{s_1,s_2,s_3,...,s_i,...\}$, consisting entirely with isolated eigenvalues with finite multiplicity's $n_i<\infty$, i=1,2,3,... and the corresponding eigenfunctions $\{x_{ik}$, i=1,2,3,... , k=1,2,3,...,$n_i\}$ forms an orhonormal basis in the space $L^2(\Omega)$. An additional property of the operator A that will be important later is stated in the following lemma which is proved in [10].

Lemma 5.1. [10] There exists a real eigenvalue s_1 of the operator A and a corresponding eigenvector $x_1(z)$ is a srictly positive element in the space X i.e., satisfies $x_1(z) >> 0$ for all $z\in cl\Omega$ in the case of Neumann or Robin (mixed) boundary conditions and for all $z\in\Omega$ in case of Dirichlet boundary conditions. In the latter case, we have

$$\frac{\partial x_1}{\partial v}(z) < 0 \qquad for \ z\in\partial\Omega$$

Moreover, if s_i, i=2,3,4,... is any other eigenvalue of the operator A, then the real part of s_i , $Re(s_i)$, satisfies

$Re(s_i) < s_1$ for all i=2,3,4,...

Lemma 5.1 says that there exists a real eigenvalue of the operator A which is larger than the real part of all other eigenvalues of the operator A. We call it the principal eigenvalue of the operator A. Moreover, Lemma 5.1 says that the associated eigenvector is positive and is called the principal eigenvector of the operator A.

We may express dynamical system (5.1) with boundary conditions (5.2) as an abstract ordinary differential equation in the separable Hilbert space space $X=L^2(\Omega)$. Since operator A given by (5.3) satisfies all the assumptions stated in the previous sections it is sufficient to substitute $x(t) = w(\cdot,t) \in L^2(\Omega)=X$.

Let us denote

$$b_{1j} = \langle b_j, x_1 \rangle_{L^2(\Omega)} = \int_\Omega b_j(z) x_1(z) dz \quad for \ \ j = 1,2,3,\ldots,m \quad (5.5)$$

Now, using the general results stated in section 3 we may formulate theorem and corollaries on positive approximate controllability for distributed parameter dynamical system (5.1) with normal operator A.

Theorem 5.1. Let operator A given by (5.3) be normal. Moreover, let us assume that b_{1j} have the same sign for every j=1,2,...,m. Then the linear distributed parameter dynamical system (5.1) is not approximately positive controllable.

Proof. Let us observe that distributed parameter dynam-ical system (5.1) satisfies all the assumptions required in Theorem 3.1. Therefore, by Theorem 3.1 our dynamical system (5.1) is not approximately positive controllable.

Using results given in section 4 we have the corollary.

Corollary 5.1. If $s_1<0$, then to each $u_s \in U^+ \backslash kerB$ there exists exactly one x_s such that $\{x_s,u_s\}$ is a positive stationary pair.

6. Example

Let us consider the one dimensional heat equation on a rod of length 1 with noninsulated ends described by the following linear parabolic partial differential equation

$w_t(z,t) = w_{zz}(z,t) + b(z)u(t) \qquad 0 \le z \le \quad t \ge 0 \qquad (6.1)$

with initial condition $\qquad w(z,0) = w_0(z)$

and Dirichlet type boundary conditions $w(0,t) = w(1,t) = 0$

We wish to control distributed parameter system (6.1) by a nonnegative scalar input $u \in L^2_{loc}([0,\infty),R^+)$. We can interpret this control as an electrical heating input that for all time is proportional to a given heat distribution $b(z) \in L^2([0,1],R)$. We state this control problem as an abstract control problem on the separable Hilbert space $X = L^2([0,1],R)$. Let us denote $w(z,t) = x(t) \in X$. Let $A=d^2/dz^2$ be the linear unbounded selfadjoint differential operator on X with domain $D(A) = \{w(z) \in X : w_{zz}(z) \in X , w(0)=w(1)=0\}$. It is known [3] that the operator A has simple eigenvalues $s_i = -i^2\pi^2$ and the corresponding eigenfunctions $x_i(z) = 2^{0,5}\sin(i\pi z)$, for i=1,2,3,... forms an orthonormal basis in the space $X = L^2([0,1],R)$.

Since all the eigenvalues are real, then by Theorem 5.1 dynamical systems (6.1) is not approximately positive controllable for any $b \in X$. The same result has been proved in [9] but using quite different methods.

Moreover, let us observe that operator A generates an analytic positivesemigroup S(t), for $t \ge 0$ on X given by

$$S(t)x = \sum_{i=1}^{i=\infty} \exp(-i^2 \pi^2 t) \langle x, x_i \rangle_{L^2} x_i$$

Now, let us assume that $b \in X^+ = L^2([0,1],R^+)$. Therefore, distributed parameter system (6.1) is a positive dynamical system. Following [9] it should be stressed, that positive dynamical system (6.1) is also not approximately positive controllable. However, since $Re(s_1) = -\pi^2 < 0$, then by Corollary 5.2 for each $u_s \in R^+$ there exists exactly one $x_s = -A^{-1}bu_s \in X^+$ given by

$$x_s = \sum_{i=1}^{i=\infty} \left(-i^2\pi^2\right)^{-1} \int_0^1 \sqrt{2}\sin(i\pi z)b(z)dz \sqrt{2}\sin(i\pi z)u_s$$

such that $\{x_s,u_s\}$ is a positive stationary pair. From [9] an element $x_s \in X^+$ can be also expressed as follows

$$x_s(z) = \left(z\int_0^1\int_0^\xi b(\zeta)d\zeta d\xi - \int_0^z\int_0^\xi b(\zeta)d\zeta d\xi \right) u_s$$

Summarizing, distributed parameter dynamical system (6.1) is not approximately positive controllable and of course it is also not approximately controllable with nonnegative controls, however, for dynamical system (6.1) there exist stationary pairs.

7. Conclusions

The paper contains several results on constrained contro-llability for linear infinite-dimensional selfadjoint dyna-mical systems. Using spectral properties of normal gene-rally unbounded linear operators with pure discrete point spectrum, conditions for different kinds of constrained controllability have been formulated and proved. General results have been applied for constrained controllability considerastions for linear distributed parameter dynamical systems described by linear partial differential equations of parabolic type with various kinds of boundary conditions. Finally, simple illustrative example of one-dimensional heat equation with homogeneous Dirichlet boundary conditions has been presented. Some kinds of the presented results can be extended to cover the case of infinite-dimensional normal dynamical systems with discrete and continuous spectrum. It is also possible to extend the result for second-order infinite-dimensional dynamical systems.

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