Parameter Estimation in a Nonlinear Structured Tree Population Model With Self Shading Effects

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Abstract

We discuss a least squares method for identifying the growth function in a nonlinear hyperbolic initial-boundary value problem that describes the dynamics of tree population with self shading effects. Furthermore, we present numerical results of estimating this parameter from computationally generated data.

1 Introduction

In this short note we consider an identification problem for the following parametrized initial boundary value problem that models forest exploitation and competition for light in trees

$$\begin{cases} u_t + (g(x, Q(t,x))u)_x + m(x, P(t))u = 0, \\ \qquad\qquad (x,t) \in (0,l] \times (0,T] \\ \\ g(0, Q(t,0))u(t,0) = C(t) \\ \qquad + \int_0^l \beta(x, P(t))u(t,x)\,dx, \quad t \in (0,T] \\ \\ u(0,x) = u^0(x), \qquad\qquad x \in [0,l] \end{cases} \tag{1.1}$$

where $u(t,x)$ is the density of the population of size x at time t, $P(t) = \int_0^l u(t,x)\,dx$ is the total population at time t and $Q(t,x) = \int_x^l u(t,x)\,dx$ is the population of individuals larger than size x. The function m denotes the mortality rate and β is the reproduction rate of an individual in the population. The function g denotes the growth rate of an individual and $C(t)$ represents the inflow of zero-size individuals from an external source. In this model, we assume that the growth rate of individuals of size x is affected by those which are larger due to shading effects, and that the death and birth rates depend only on the total population. Moreover, we assume that an individual tree is harvested when it reaches size l. The model (1.1) has been presented in [3]. The existence-uniqueness of nonnegative solutions to similar equations have been established in [4] using the classical method of characteristics.

This paper is organized as follows. In section 2 we discuss the minimization problem and the finite difference method used to approximate equation (1.1). In section 3 we present some numerical results of parameter estimates obtained using the method presented in section 2 and computationally generated data. Finally, in section 4 we close the paper with some remarks and future research issues.

2 Parameter Estimation Problem

For simplicity, we assume that all the parameters in equation (1.1) are sufficiently regular, nonnegative and are given functions except for the growth parameter g. To identify g we consider the least squares problem of minimizing the cost functional

$$J(g) = \sum_{r=1}^{n_1} \sum_{s=1}^{n_2} | \int_{x_s}^{x_{s+1}} u(t_r, x, g)dx - Z_{r,s}|^2 \tag{2.1}$$

over $g \in G$, where G is a compact subset of the space $C^1([0,l]; C[0,\infty))$ which satisfies the following assumption (necessary for the stability and convergence of the finite difference scheme presented in this section)

(A_G) Any function $g \in G$ is twice continuously differentiable in x and Lipschitzian in Q. Furthermore, the continuous function $g_x(x,Q)$ is Lipschitzian in Q, $g(x,Q) > 0, \forall x \in [0,l)$ and $g(l,Q) = 0$.

The numbers $Z_{r,s}$, $r = 1, 2, \cdots, n_1$, $s = 1, 2, \cdots, n_2$, in (2.1) are the observed total number of individuals in the size class $[x_s, x_{s+1})$ at time t_r, and $u(t,x,g)$ is the solution of the parameter dependent equation (1.1). We remark that the techniques presented here can be easily modified to allow the identification of the rest of the parameters β, m and C.

To solve the above least squares problem we start first by approximating equation (1.1). To this end, we consider the following implicit finite difference method

$$\begin{cases} \dfrac{u_j^{k+1}(g)-u_j^k(g)}{\Delta t} \\ \quad +\dfrac{g_j^k u_j^{k+1}(g)-g_{j-1}^k u_{j-1}^{k+1}(g)}{\Delta x} \\ \qquad +m_j^k u_j^{k+1}(g)=0, \quad 1\le j\le N \\ g_0^k u_0^{k+1}(g)=C^k+\sum_{i=1}^N \beta_i^k u_i^{k+1}(g)\,\Delta x \\ P^{k+1}(g)=\sum_{i=1}^N u_i^{k+1}(g)\Delta x, \\ Q_j^{k+1}(g)=\sum_{i=j+1}^N u_i^{k+1}(g)\Delta x, \end{cases} \tag{2.2}$$

where $\Delta x=\frac{l}{N}$, and $\Delta t=\frac{T}{M}$ denote the spatial and time mesh size respectively. The point $x_j=j\Delta x$, $j=0,1,2,\cdots,N$ and $t_k=k\Delta t$, $k=0,1,2,\cdots,M$. Furthermore, we denote by $u_j^k(g)$, $Q_j^k(g)$ and $P^k(g)$ the difference approximation of $u(t_k,x_j,g)$, $Q(t_k,x_j,g)$ and $P(t_k,g)$, respectively and we define

$$\begin{aligned} g_j^k &= g(x_j,Q_j^k),\ \beta_j^k=\beta(x_j,P^k), \\ m_j^k &= m(x_j,P^k) \text{ and } C^k=C(t_k). \end{aligned}$$

If we define

$$d_j^k=1+\frac{\Delta t}{\Delta x}g_j^k+\Delta t\, m_j^k, \quad 1\le j\le N$$

then (2.1) is equivalently written as the following system of linear equations

$$A^k\vec{u}^{k+1}=\vec{f}^k \tag{2.3}$$

where

$$\vec{u}^{k+1}=[u_0^{k+1},u_1^{k+1},\ldots,u_N^{k+1}]^T,$$
$$\vec{f}^k=[C^k,u_1^k,u_2^k,\ldots,u_N^k]^T$$

and the matrix A^k is given by

$$\begin{pmatrix} g_0^k & -\Delta x\beta_1^k & -\Delta x\beta_2^k & \cdots & -\Delta x\beta_N^k \\ -\frac{\Delta t}{\Delta x}g_0^k & d_1^k & 0 & \cdots & 0 \\ 0 & -\frac{\Delta t}{\Delta x}g_1^k & d_2^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -\frac{\Delta t}{\Delta x}g_{N-1}^k & d_N^k \end{pmatrix}$$

From the matrix representation and Lemma 2.1 in [1] we establish that the system of linear equations given in (2.3) has a unique non-negative solution provided that Δx, and Δt are chosen to satisfy the following condition

$$\begin{cases} \Delta x\,\dfrac{\beta_j^k}{g_0^k}+\dfrac{\Delta t}{\Delta x}g_j^k(1+\dfrac{\Delta t}{\Delta x}g_{j+1}^k+\Delta t\, m_{j+1}^k)^{-1}<1 \\ \qquad 0\le j\le N-1 \\ \Delta x\,\dfrac{\beta_N^k}{g_0^k}<1. \end{cases}$$

The above approximation is extended to a function on $[0,l]\times[0,T]$ by defining

$$\begin{aligned} U_{\Delta t,\Delta x}(t,x,g) &= u_j^k(g), \\ (t,x) &\in [t_{k-1},t_k)\times[x_{j-1},x_j), \\ k &= 1,\ldots,M,\ j=1,\ldots,N. \end{aligned}$$

Hence, for computing minimizers we define the following approximate cost functionals

$$J_{\Delta t,\Delta x}(g)=\sum_{r=1}^{n_1}\sum_{s=1}^{n_2}|\int_{x_s}^{x_{s+1}}U_{\Delta t,\Delta x}(t_r,x,g)dx-Z_{r,s}|^2 \tag{2.4}$$

over $g\in G^m$ a finite dimensional approximating sequence of the parameter space G.

3 Numerical Results

For our numerical experiments, we choose the following function forms for the parameters β, m and C

$$\begin{aligned} \beta(x,P) &= 0.5x^{2/3}P\exp(-4P) \\ m(x,P) &= (1+x)(1+P)^2 \\ C &= 0 \end{aligned}$$

In addition, we let $T=2$, $l=1$ and choose the initial condition

$$u^0(x)=\begin{cases} 2 & 0\le x\le 0.5 \\ 0 & 0.5<x\le 1. \end{cases}$$

Note that for the parameters given above the following bounds can be obtained $0\le Q\le P\le 1$, independent of any growth function $g\in G$. In fact, the lower bound follows from the nonnegativity of solutions to equations (1.1). While the upper can be obtained by integrating the first equation in (1.1), as follows

$$\frac{d}{dt}\int_0^1 udx-g(0,Q(t,0))u(t,0)+\int_0^1 m(x,P(t))u=0.$$

Hence, using the boundary condition in (1.1) we get

$$\frac{dP}{dt}=\int_0^1(\beta(x,P)-m(x,P))\,udx\le wP$$

where, $w=\sup_{x,p}(\beta(x,P)-m(x,P))$. This implies that

$$P(t)\le\|u^0\|_{L^1(0,1)}e^{wP}\le 1$$

since, $w\le 0$ and $\|u^0\|_{L^1(0,1)}=1$ for the above choice of parameters.

As for the growth function g, we assume that it has the following separable form

$$g(x,Q)=g_1(x)g_2(Q).$$

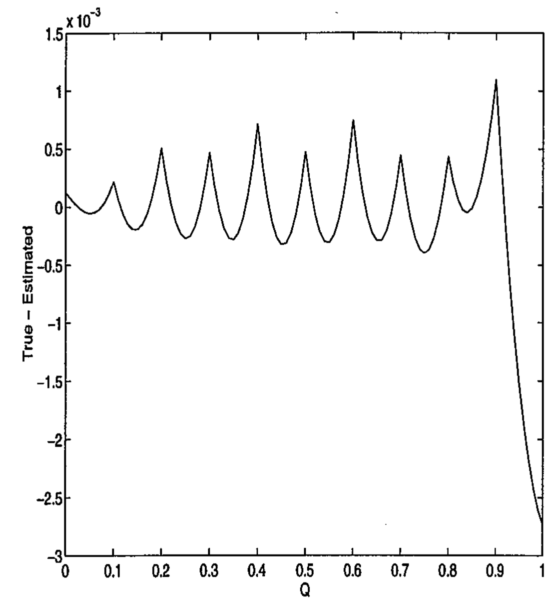

Figure 1. This graph represents the estimated versus the true parameter for the function $g_2^*(Q) = (1+Q)e^{-2Q}$.

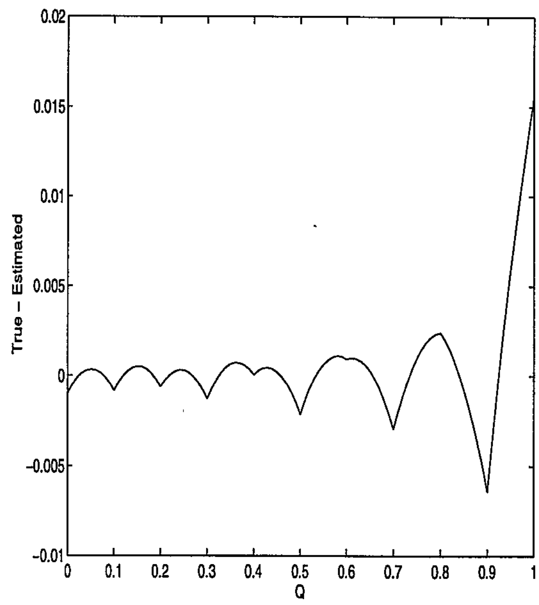

Figure 2. This graph represents the estimated versus the true parameter for the function $g_2^*(Q) = 1 - 0.5Q^2$.

Furthermore, we assume that the function $g_1(x) = (0.1+x)(1-x)$ is given and attempt to identify only $g_2(Q)$. Hence, for this special case we choose the parameter space G to be

$$\{g_2 \in C[0,1] : |g_2| \le L, \quad |g_2(Q_1) - g_2(Q_2)| \le L|Q_1 - Q_2|\}$$

where L is a fixed positive constant. This set is compact in $C[0,1]$ with the sup norm, by Arzela-Ascoli theorem. We define the finite dimensional approximating sequence of this parameter space G to be

$$G^m = \mathrm{span}\{\mathrm{B}_\mathrm{j}^\mathrm{m}\}_{\mathrm{j}=0}^{\mathrm{m}},$$

where B_j^m is the j-th linear B-spline on the interval $[0,1]$ defined with respect to the uniform mesh $\{0, \frac{1}{m}, \frac{2}{m}, \cdots, 1\}$. That is

$$B_j^m(Q) = \begin{cases} 0 & 0 \le Q \le \frac{j-1}{m} \\ mQ - j + 1 & \frac{j-1}{m} \le Q \le \frac{j}{m} \\ j + 1 - mQ & \frac{j}{m} \le Q \le \frac{j+1}{m} \\ 0 & \frac{j+1}{m} \le Q \le 1 \end{cases}$$

$j = 0, 1, \cdots, m$. Hence, if $g_2^m \in G^m$ then g_2^m is given by $g_2^m = \sum_{j=0}^m \alpha_j^m B_j^m(Q)$ and solving the identification problem (2.4) involves the choosing of the parameters $(\alpha_0^m, \alpha_1^m, \cdots, \alpha_m^m)$ from a compact subset of $\mathbb{R}^{m+1}$ so as to minimize the functional $J_{\Delta t, \Delta x}(g_2^m)$.

To test our scheme, we choose a "true" function g_2^* and generate the total population data $Z_k = P^k(g_2^*) = \sum_{i=1}^N u_i^k(g_2^*)\Delta x$, $k = 1, \cdots, 200$, using the finite difference method given in (2.2), with $\Delta x = 0.0125$ and $\Delta t = 0.01$. We then used our least squares method to attempt to identify g_2^* from the generated data Z_k, $k = 1, \cdots, 200$.

In the simulations presented here the nonlinear least squares minimization was carried out with LMDIF1, an implementation of the Levenberg-Marquardt algorithm available from NETLIB. We let $m = 10$, and as an initial guess for g_2^* we took $g_2^0 = 0.5$. In our first numerical experiment we let $g_2^*(Q) = (1+Q)\exp(-2Q)$ and presented the difference between the true and estimated parameter in Figure 1. We repeated the process for the function $g_2^*(Q) = 1 - 0.5Q^2$ and presented the results in Figure 2.

4 Concluding Remarks

In this paper, we have presented a least squares method for identifying parameters in a nonlinear tree population model with shading effects. The numerical results of parameter estimates obtained using computationally generated data sets appear to be very promising. Our future efforts will focus on using such techniques for identifying parameters from observed field data.

We point out that using similar techniques as those used in [5] together with the abstract least squares theory presented in [2] we can establish results concerning subsequential convergence of minimizers of the finite dimensional approximate cost functionals $J_{\Delta t,\Delta x}$ to a minimizer of the infinite dimensional one J. These efforts will appear in a forthcoming paper.

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