On Singular Phenomena in Certain Time-Optimal Feedback System Operating by Discontinuous Resistance

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Abstract. The purpose of this paper is to present the solution of selected time-optimal problems of the controlled object the dynamics of which is given by: $\dot{x} = y,\ \dot{y} = f(x) + u$, where $|u| \le 1$ and motion resistance function $f(x) = 0$ if $x < 0$, and $f(x) = -A$ if $x \ge 0$ where $0 \le A < 1$. That model describes dynamics of a very important class of industrial installations. As the time-optimal problem will be understood a transfer of the initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to the target state $\mathbf{z}_1 = (x_1, y_1), x_1 \ge 0$ in a minimum time $t^* < \infty$. There has been shown that in the formula defining resistance function $f(x)$ there exists a rational value $A = A_b > 0$ that plays an essential role in time-optimal structure formation. Namely, if $A \le A_b$ then the time-optimal control process is typical, analogous as in classical case $\ddot{x} = u,\ |u| \le 1$, i.e. there exists a switching curve formed by the trajectories of time-optimal solutions reaching the target state and the time-optimal process is formed by at most one switching operation.

For the case $A > A_b$ we will examine two following singular phenomena.

a) If the target state $\mathbf{z}_1 = (0, 0)$ then there exists the switching curve dividing the state plane into two sets, however only one its branch is formed by the time-optimal solution reaching the target $\mathbf{z}_1 = (0, 0)$ and generated by the control $u \equiv -1$. None of solution forms the second branch of switching curve. It is formed by a state-locus depending on the value of A only. In dependency of the starting state $\mathbf{z}_0$ the time-optimal control process is generated by *bang-bang* control with none, one or two switching operations. This is the *first singular phenomenon*, because any small decrease of the value A over A_b requires to change the structure which would be able to generate the time-optimal process.

b) The paper shows, that if the target state $\mathbf{z}_1 = (x_1, 0)$, $x_1 > 0$ then there exists a set of the starting states from which there starts two trajectories reaching the target in the same minimum time. This is the *second singular phenomenon*.

Finally, some suggestions as to practical applications have been given too.

1 INTRODUCTION

Industrial devices, such as saddles of machine tools, tracer machines, industrial manipulators, several parts of industrial robots, or the position mechanisms of industrial automata, need to change their position in a minimum time, particularly when it is necessary to move the mechanism before another technological operation can proceed. Synthesis of a time-optimal control structure becomes therefore an important, economical problem.

Dynamics of the above devices, called *position mechanisms* depend essentially on motion resistance. From technical point of view we distinguish motion resistance depending on velocity of the mechanism or on its position. If the first type of that motion resistance is a case then the dynamics of the controlled object is given by [5],[6],[7]: $\dot{x} = y,\ \dot{y} = f(y) + u$, where x, y is position and velocity of the mechanism respectively, f is a function of motion resistance, u is a control function. In order to define as large as possible class of motion resistance, in particular all types of friction, we assume that function f is piecewise continuous. Discontinuity of the right-hand side of the above model makes the classical theory of differential equation, as well as the maximum principle, impossible to apply to the time-optimal problem. This problem has been solved with the use of differential inequality theory by assumption that both the control function and co-ordinates y and $\dot{y}$ are constrained. The solution mentioned above, has been used for feedback control system creation, based on the concept of regular closed-loop system synthesis [2],[7]. The closed-loop system created in such a way is operating analogously as that created for the classical type of the dynamic object: $\ddot{x} = u,\ |u| \le 1$.

If the second type of motion resistance is a case i.e. if they are depending on the position of the mechanism only, then the dynamics of the *position mechanisms* is defined by the following differential equation: $\dot{x} = y,\ \dot{y} = f(x) + u$. In this paper we will work with the following mapping of position mechanism dynamics:

$$\dot{x} = y,\ x(0) = x_0;\quad \dot{y} = f(x) + u,\ y(0) = y_0 \qquad (1.1a)$$

by $|u| \le 1$ and motion resistance function given as follows:

$$f(x) = 0,\ x \le 0;\quad f(x) = -A,\ x > 0 \qquad (1.1b)$$

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The model (1.1) describes dynamics of a very important class of industrial installations, namely manipulators with counterweight, outriggers of position mechanisms and a lot of the like devices.

The work deals with selected cases of the time-optimal problem of the system (1.1) that will be understood as a transfer the initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to the target state $\mathbf{z}_1 = (x_1, 0)$, $x_1 \geq 0$ in a minimum time $t^* < \infty$.

There has been shown that in the formula (1.1b) defining motion resistance function there exists a rational value $A = A_b$ that plays an essential role in time-optimal structure formation. Namely, if $A \leq A_b$ then the time-optimal control process is typical, analogous as in classical case $\ddot{x} = u$, $|u| \leq 1$. Thus, in the state plane there exists a switching curve formed by standard solutions of (1.1) reaching the target $\mathbf{z}_1$. The control process is of *bang-bang* type and to each state belonging either to several branches of this switching curve or doing to the sets resulting from partitioning the state plane by that switching curve there are admitted the time-optimal controls $u \equiv +1$ and $u \equiv -1$. The time-optimal control process is of *bang-bang* type with at most one switching operation.

For the case $A > A_b$ we will examine two following singular phenomena.

a) If the target state $\mathbf{z}_1 = (0, 0)$ then there exists also the switching curve, dividing the state plane into two sets, however only one its branch is formed by the solution of (1.1) reaching the target $\mathbf{z}_1 = (0, 0)$ and generated by the control $u \equiv -1$. None of (1.1) solution forms the second branch of switching curve. It is formed by a state-locus depending on the value of A only. In dependency of the starting state $\mathbf{z}_0$ the time-optimal control process is generated by *bang-bang* control with none, one or two switching operation. This is the *first singular phenomenon*, because any small decrease of the value A over A_b requires to change the structure which would be able to generate the time-optimal process.

b) The paper shows, that if the target state $\mathbf{z}_1 = (x_1, 0)$, $x_1 > 0$ then there exists a set of the starting states from which there starts two different trajectories reaching the target in the same minimum time. This is the *second singular phenomenon.*

Knowing the time-optimal solution of (1.1) we will tray to create the feedback control system that would have the properties like to those which have the closed-loop systems created in accordance with the principle of the following concept.

Definition 1.1. We will say that there exists the time-optimal regular synthesis of the open system [2],[7]

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z},\mathbf{u}),\ \mathbf{z} \in R^2, \quad \mathbf{u} \in \mathbf{U} \subset R^m \tag{1.2}$$

where u is a control vector-function, if there exists a vector-function $\mathbf{v}$: $R^n \to \mathbf{U}$ such that:

a) Each time-optimal solution of the open dynamic object (1.2) is a standard (Caratheodory) solution of the closed-loop system

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z})) \tag{1.3}$$

b) Each standard solution of the closed-loop system (1.3) is a time-optimal solution of the open, controlled object (1.2). ■

For the desirability of implementing the above closed-loop optimal system the following reasons are given:

i) *There is no need to compute the optimal control for every initial state separately,*

ii) *The controller acting upon* (1.2) *is sensitive to instantaneous perturbations, i.e. if at any instant of the control process the system is deviated from its optimal trajectory, the rest of the process will again lead to the desired final state (target) and will be optimal with respect to this new initial state.*

Lemma 1.1.

Given controlled object (1.1). Time-optimal control function u transferring any starting state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to any target state $\mathbf{z}_1 = (x_1, y_1) \in [0,\infty) \times 0$ in minimum time $t_{\min} < \infty$ is of the *bang-bang* type, i.e. $u = \pm 1$.

PROOF: The detailed way of proving shows paper [8]. ■

2. PRELIMINARIES

Notations 2.1.

a) The solutions of the system (1.1) generated by the control function $u \equiv +1$ and $u \equiv -1$ starting from any point $\mathbf{z}_i$ will be denoted as $\mathbf{q}_+(t; \mathbf{z}_i)$ and $\mathbf{q}_-(t; \mathbf{z}_i)$ respectively or shortly (in particular in the figures) by symbols $\mathbf{q}_+$ and $\mathbf{q}_-$.

b) Trajectories of the solutions $\mathbf{q}_+(t; \mathbf{z}_0)$ and $\mathbf{q}_-(t; \mathbf{z}_0)$ reaching the target state $\mathbf{z}_1$ will play an essential role. They will be called *Terminal Trajectories*, will be denoted $\mathbf{T}^+$ and $\mathbf{T}^-$ respectively and will be defined by:

$$\mathbf{T}^+ = \{\ \mathbf{q}_+(t; \mathbf{z}_1),\ \leq 0\} \tag{2.1}$$

$$\mathbf{T}^- = \{\ \mathbf{q}_-(t; \mathbf{z}_1),\ \leq 0\} \tag{2.2}$$

Trajectory $\mathbf{T}^-$ intersects the positive semi-y-axis in the point noted $\mathbf{z}_y = (0, y_y)$, where $y_y \geq 0$.

c) The negative and positive semi-y-axes will be noted $\mathbf{B}^+$ and $\mathbf{B}^+$ respectively

The y-axis $\mathbf{B} = \mathbf{B}^- \cup \mathbf{B}^+$ forms *the bound of motion resistance zone* and divides the state-plane into two following half-planes: $\mathbf{S}^-$ if $x < 0$ and $\mathbf{S}^+$ if $x > 0$. ■

Remark 2.2. *Properties of the solutions* $\mathbf{q}_-$ *and* $\mathbf{q}_+$.

a) The co-ordinates of the solution $\mathbf{q}_-(t;\mathbf{z}_0)$ has got the following properties. Let $y_0 > 0$. Then, there exists a time $t_1 > 0$ such that $y_-(t,\mathbf{z}_0)$ is decreasing function on $[0,\infty)$, $y_1(t_1;\mathbf{z}_0)(= 0$, but $x_-(t,\mathbf{z}_0)$ is increasing function on $[0, t_1]$ and is decreasing one on $[t_1,\infty)$.

b) The co-ordinates of the solution $\mathbf{q}_+(t;\mathbf{z}_0)$ has got the following properties. Let $y_0 < 0$. Then, there exists a time $t_1 > 0$ such that $y_+(t,\mathbf{z}_0)$ is increasing function on $[0,t_1]$, $y_+(t_1,\mathbf{z}_0) = 0$, but $x_+(t,x_0)$ is decreasing function on $[0, t_1]$ and increasing one on $[t_1,\infty)$. ■

Lemma 2.3

Given controlled object (1.1). The time-optimal control u^* bringing the controlled object from any $\mathbf{z}_0 \in R^2$ to the target state $\mathbf{z}_1 = (x_1, 0)$, $0 \leq x_1$ is of *bang-bang type*, i.e. the control function $u \equiv +1$ and $u \equiv -1$.

PROOF: The way of proving is given in the paper [8]. ■

3. SINGULAR SWITCHING CURVE

Lemma 3.1.

Given controlled object (1) and terminal trajectory $\mathbf{T}^-$ (2.2). Then, from each $\mathbf{z}_0 \in \mathbf{T}^-$ there starts the unique solution $\mathbf{q}_-(t;\mathbf{z}_0)$ that lies totally in terminal trajectory $\mathbf{T}^-$ and reaches the target $\mathbf{z}_1$ in a minimum time $t^* < \infty$.

PROOF: The above thesis has been proved in the paper [8]. ■

Now, we are going to investigate time-optimal solution of the object (1.1) for selected both starting point $\mathbf{z}_0$ and the target $\mathbf{z}_1$.

3.1. Time-optimal problem for the target $\mathbf{z}_1 = (0,0) = \mathbf{0}$

If the target state $\mathbf{z}_1 = \mathbf{0}$ is a case then the terminal trajectories are defined by (2.2). Minimum time taken for the transfer the state $\mathbf{z}_0 \in \mathbf{T}^-$ to the target $\mathbf{z}_1 = \mathbf{0}$ (obviously along the terminal trajectory $\mathbf{T}^-$) results from (3.1) after setting $\mathbf{z}_1 = (0,0)$.

At first we will examine the time-optimal trajectories starting from $\mathbf{z}_0 \in \mathbf{T}^+$. There will be distinguished two following cases of the motion resistance function:

$$\text{i) } A \in [0, A_b], \qquad \text{ii) } A \in (A_b, 1) \tag{3.1}$$

where A_b is a certain rational value, $A_b \in [0, 1)$.

Now, we are going to show that if motion resistance function satisfies (3.1, i) then there exists the time-optimal switching curve $\mathbf{T} = \mathbf{T}^+ \cup \mathbf{T}^-$ where $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined by (2.2). However, if motion resistance function satisfies (3.1, ii) then there exists the time-optimal switching curve $\mathbf{T} = \mathbf{T}_m \cup \mathbf{T}^-$ where $\mathbf{T}^-$ is given by (2.2). The branch $\mathbf{T}_m$ is a special state locus which cannot be created by whatever solution of the system (1.1). It will be defined in what follows. The switching curves shown above play the same role as that in classical time-optimal closed-loop system controlling the dynamic object described by: $\ddot{x} = u$, $|u| \leq 1$.

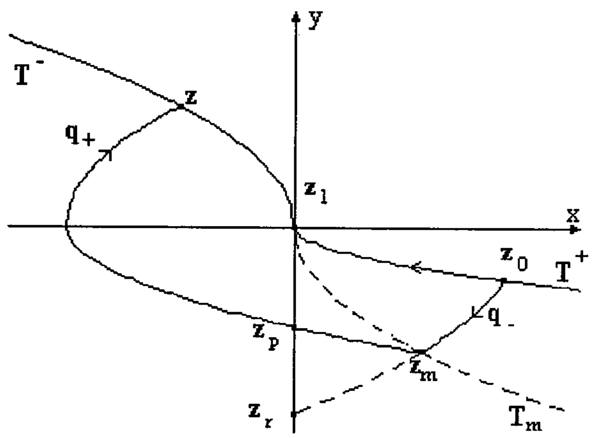

Figure 1. Control process with two switching operation

Lemma 3.2

Given the controlled object (1.1). Let starting state $\mathbf{z}_0 \in \mathbf{T}^+$ and the target state $\mathbf{z}_1 = (0,0) = \mathbf{0}$.

Thesis a) If $A \in (A_b, 1)$ then the transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ to the target $\mathbf{z}_1 = \mathbf{0}$ in minimum time $t^* < \infty$ is performed along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to a state $\mathbf{z}_m \in \mathbf{S}^+$, afterwards along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_m)$ solution to $\mathbf{z}_n \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$ (see Fig. 1).

Thesis b) If $A \in [0, A_b]$ then the transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ is performed along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution, i.e. along the curve $\mathbf{T}^+$.

PROOF: A way of argument grounded on Lemma 3.1, Remark 2.2 and is given in the paper [8] ■

Let us perceive that if $A \in (A_b, 1)$ is a case then the locus of the states $\mathbf{z}_m$ forms a switching curve noted (in accordance with the Fig.1) $\mathbf{T}_m$. This switching curve $\mathbf{T}_m$ is defined by the following formula:

$$\mathbf{T}_m = \{(x,y): x = g(y), \quad y \in [y_r, 0]\} \tag{3.2}$$

where $g: R^1 \to R^1$ and $\mathbf{z}_r = (0, y_r)$ is a point in which the trajectory $\mathbf{T}_m$ intersects negative semi-y-axis. $\mathbf{B}^-$.

It should be emphasised that the switching curve $\mathbf{T}_m$ is none of the trajectories which may be formed by any one solution of (1.1) (see Fig. 1).

Remark 3.3

From Lemma 3.2 it follows that if motion resistance function satisfies inequality (3.1, i), i.e. $A \in [0, A_b]$ then the time-optimal transfer of each state $\mathbf{z}_0 \in \mathbf{T}^+$ holds along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution, i.e. along the curve $\mathbf{T}^+$ without any switching of the control function u.

However, if $A \in (A_b, 1)$ then the time-optimal transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ to the target $\mathbf{z}_1 = \mathbf{0}$ must be performed with two switchings of control function u, i.e. from $\mathbf{z}_0 \in \mathbf{T}^+$ there starts the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution which reaches the state $\mathbf{z}_m \in \mathbf{T}_m \subset \mathbf{S}^+$ where the switching operation is being executed. From $\mathbf{z}_n$ there starts the trajectory of the $\mathbf{q}_+(t; \mathbf{z}_m)$ solution the trajectory of which intersects semi-y-axis $\mathbf{B}^-$, penetrates into $\mathbf{S}^-$ set and tends to reach the curve $\mathbf{T}^-$ in the point $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^-$ where there should be executed the second switching operation. From $\mathbf{z}_n$ there starts the trajectory of the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$ solution which brings the object along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1 = \mathbf{0}$. ■

Let us define in the state plane some special sets for the cases of motion resistance function quoted above, that will be of use in the next part of the text.

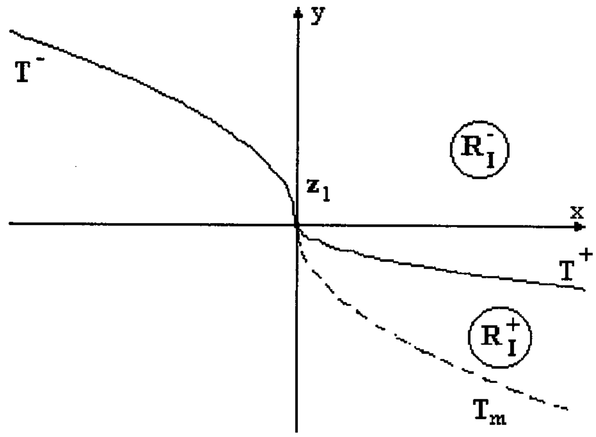

Figure 2. State plane partitioning

(1) If $A \in [0, A_b]$ then we will note (see Fig. 2):

$$\mathbf{T}_\mathrm{I} = \mathbf{T}^- \cup \mathbf{T}^+ \tag{3.2}$$

$$\mathbf{R}_\mathrm{I}^- = \{(x,y): (x',y) \in \mathbf{T}_\mathrm{I} \Rightarrow x > x'\} \tag{3.3}$$

$$\mathbf{R}_\mathrm{I}^+ = \{(x,y): (x',y) \in \mathbf{T}_\mathrm{I} \Rightarrow x < x'\} \tag{3.4}$$

(2) If $A \in (A_b, 1)$ then we will note (see Fig. 3):

$$\mathbf{T}_\mathrm{II} = \mathbf{T}^- \cup \mathbf{T}_m \tag{3.5}$$

$$\mathbf{R}_\mathrm{II}^- = \{(x,y): (x',y) \in \mathbf{T}_\mathrm{II} \Rightarrow x > x'\} \tag{3.6}$$

$$\mathbf{R}_\mathrm{II}^+ = \{(x,y): (x',y) \in \mathbf{T}_\mathrm{II} \Rightarrow x < x'\} \tag{3.7}$$

Theorem 3.4

Given controlled object (1.1) and target state $\mathbf{z}_1 = (0, 0) = \mathbf{0}$.

Thesis a). If the motion resistance function (1.2) satisfies inequality (3.1, i), i.e. $A \in [0, A_b]$ then the time-optimal control function

$$u^*(x,y) = \begin{vmatrix} +1, & (x,y) \in \mathbf{T}^+ \cup \mathbf{R}_\mathrm{I}^+ \\ -1, & (x,y) \in \mathbf{T}^- \cup \mathbf{R}_\mathrm{I}^- \end{vmatrix} \tag{3.8}$$

where $\mathbf{T}^+$, $\mathbf{T}^-$, $\mathbf{R}_\mathrm{I}^+$, $\mathbf{R}_\mathrm{I}^-$ are defined by (2.1),(2.2),(3.3) and (3.4) respectively.

Thesis b). If the motion resistance function (1.2) satisfies inequality (3.1, ii), i.e $A \in (A_b, 1)$ then the time-optimal control function

$$u^*(x,y) = \begin{vmatrix} +1, & (x,y) \in \mathbf{T}_m \cup \mathbf{R}_{\mathrm{II}}^+ \\ -1, & (x,y) \in \mathbf{T}^- \cup \mathbf{R}_{\mathrm{II}}^- \end{vmatrix} \tag{3.9}$$

where $\mathbf{T}^-$, $\mathbf{T}_m$, $\mathbf{R}_{\mathrm{II}}^+$ and $\mathbf{R}_{\mathrm{II}}^-$ are defined by (2.1)(2.2), (2.16), (3.6) and (3.7) respectively

PROOF: The detailed way of proving has been given in the paper [8]. ■

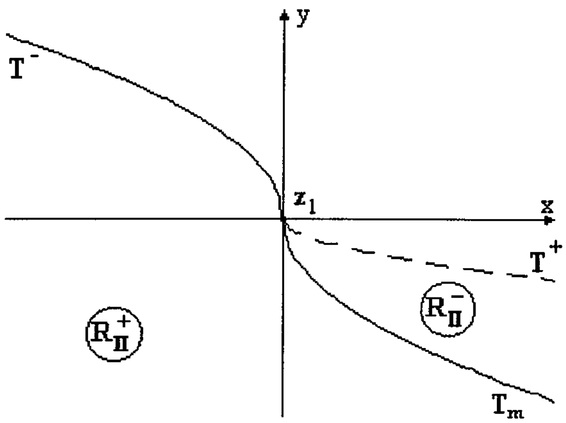

Figure 3. State plane partitioning

3.2. Time-optimal problem for the target state $\mathbf{z}_1 = (x_1, 0),\ x_1 > 0$.

For this case of the target state $\mathbf{z}_1$ the terminal trajectories are defined by (2.1),(2.2). In this chapter we will examine the time-optimal trajectories starting from $\mathbf{z}_0 = (0, 0) = \mathbf{0}$. As previously, there will be distinguished two cases of motion resistance function defined by (3.1).

Lemma 3.6

Given the controlled object (1.1,(1.2). Let starting state $\mathbf{z}_0 = \mathbf{0}$ and the target state $\mathbf{z}_1 = (x_1, 0),\ x_1 > 0$.

Thesis a) If $A \in (A_b, 1)$ then the transfer the object from starting state $\mathbf{z}_0 = \mathbf{0}$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ holds along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to a certain state $\mathbf{z}_s \in \mathbf{S}^-$ next along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_s)$ solution over the point $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$ to $\mathbf{z}_n = (x_n, y_n) \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$ (see Fig. 3).

Thesis b) If $A \le A_b$ then the transfer the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ is performed along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution to a point $\mathbf{z}_n \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along $\mathbf{T}^-$ curve to the target $\mathbf{z}_1$ (see Fig. 4).

PROOF: The proof bases on Lemma 2.3 and the properties of $\mathbf{q}_-$ and $\mathbf{q}_+$ solutions shown in Remark 2.2. The detailed way of proving may be found in the paper [8]. ■

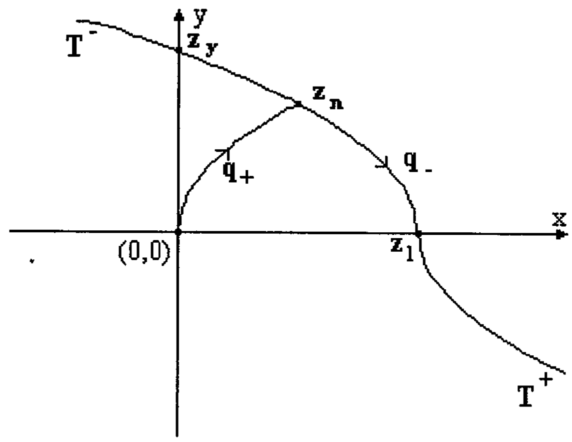

Figure 4. Time-optimal process with one switching operation

Remark 3.7

If the motion resistance function satisfies (3.1,i), i.e. $A \in [0, A_b]$ the optimal transfer of the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ should be executed along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution to the point $\mathbf{z}_n \in \mathbf{T}^-$ and from $\mathbf{z}_n$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$, i.e. along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$. This control process is realised with one switching operation in the state $\mathbf{z}_n \in \mathbf{T}^-$.

If the motion resistance function satisfies (3.1,ii), i.e. $A \in (A_b, 1)$ then the optimal transfer of the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ should be executed along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_0)$ throw the set $\mathbf{S}^-$ to the point $\mathbf{z}_s \in \mathbf{S}^-$, from $\mathbf{z}_s$ along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_n)$ solution over the point $\mathbf{z}_n \in \mathbf{T}^-$ and from $\mathbf{z}_n$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$, i.e. along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$. This control process is realised with two switching operations executed in the points $\mathbf{z}_s$ and $\mathbf{z}_n$ one. ■

4. NON-UNIQUENESS PHENOMENON

Let us denote $\mathbf{T}_0^-$ the trajectory of such $\mathbf{q}_-(t;\mathbf{z}_0)$ solution that reaches the target $\mathbf{z}_1 = \mathbf{0}$ (see Fig. 5). This trajectory has been already described by (2.2) after setting $0 \rightarrow \mathbf{z}_1$. Obviously, $\mathbf{T}_0^- \subset \mathbf{S}^- \cup \{\mathbf{0}\}$.

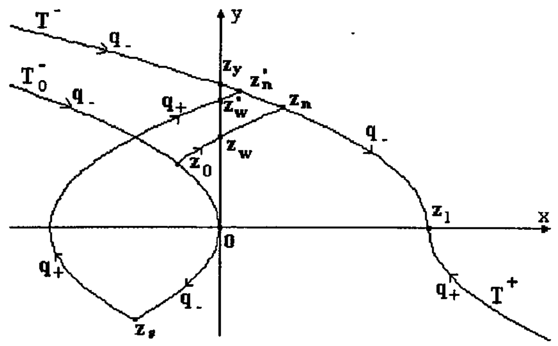

Figure 5. Non-unique trajectories

Theorem 4.1

Given controlled object (1.1). Let us assume that $A \in (A_b, 1)$. There exists such a point $\mathbf{z}_0 \in \mathbf{T}_0^- \setminus \{\mathbf{0}\}$ from which there start two different *bang-bang* solutions the trajectories of which reach the target $\mathbf{z}_1 = (x_1, 0)$, $x_1 > 0$. in the same minimum time $t^* < \infty$.

PROOF: The detailed way of proving may be found in the paper [8]. ■

This phenomenon has been shown in the Fig. 5. The first time-optimal trajectory starting from $\mathbf{z}_0 \in \mathbf{T}_0^-$ after intersecting semi y-axis $\mathbf{B}^+$ in the point $\mathbf{z}_w \in \mathbf{B}^+$ penetrates into the set $\mathbf{S}^-$ and tends to intersect the curve $\mathbf{T}^-$ in the point $\mathbf{z}_n \in \mathbf{T}^-$ and from that point reaches the target $\mathbf{z}_1$ along $\mathbf{T}^-$. The second time optimal trajectory starting from the same state $\mathbf{z}_0 \in \mathbf{T}_0^-$ runs over the set $\mathbf{S}^- \cup \{0\}$ and after executing the switching operation in the point $\mathbf{z}_s \in \mathbf{S}^-$ reaches the point $\mathbf{z}_n' \in \mathbf{T}^- \cap \mathbf{S}^-$ along the trajectory of $\mathbf{q}_+$ solution. From this point the system is brought to the target $\mathbf{z}_1$ along the curve $\mathbf{T}^-$.

Repeating the same way of computing as that done in the proof of Theorem 4.1 we state that there exists a subset of the states $\mathbf{z}_0 = (x_0, y_0) \in \mathbf{S}^-$ from which there start the trajectories of non-unique time-optimal solutions. These co-ordinates x_0, y_0 may be found from solution of 4-the degree algebraic equation. Unfortunately, those co-ordinates cannot be defined in an open form such as that in the proof of Theorem 4.1. They may be calculated in numerical way only.

From the point of view of time-optimal closed-loop system synthesis knowing the values of these co-ordinates does not play an essential role. More important is knowledge, that in the state plane there does exist the state from which there start the non-unique time-optimal trajectories. The *singular phenomenon* of existence of non-unique time-optimal trajectories will be a basic point in the next paragraph where there will be given some proposals as to practical applications.

5. CONCLUDING REMARKS

In a real dynamic system unexpected variations of resistance may appear. This fact and singular phenomena shown above imply to create sub-optimal closed-loop system, with the same properties as those of the system created in accordance with *regular synthesis.* Such a system should reach the target in a time close to minimum one. Thus, we estimate a possible interval of resistance function variations $A \in [A_{\min}, A_{\max}]$. Next, we create the curves $\mathbf{T}^-$, $\mathbf{T}^+$ and $\mathbf{T}_m$ substituting into their descriptions $A_{\min} \rightarrow A$. Denote these curves by $\mathbf{T}_{\min}^-$, $\mathbf{T}_{\min}^-$ and $\mathbf{T}_{m0}$ respectively. The model (4),(5) with the switching curves $\mathbf{T}_{\min}^- \rightarrow \mathbf{T}^-$, $\mathbf{T}_{\min}^+ \rightarrow \mathbf{T}^+$ and $\mathbf{T}_{m0} \rightarrow \mathbf{T}_m$ becomes a differential equation with discontinuous right-hand side. Thus, the classical method of solution of this differential equation becomes inappropriate. None of standard solution may map cited new curves. Analysis of control process generated by closed-loop system formed in such a way requires to use an other concept of solution. Here, following idea of generalised solutions of non-linear and discontinuous differential equations is applied.

Definition. Let $\mathbf{x}: \mathbf{I} \rightarrow R^n$ (**I** is an interval in R^1) be an absolutely continuous function on each compact subinterval of **I**. Then **x** is called a solution of differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t,\mathbf{x}(t)),\ \mathbf{g}: R^1 \times R^n \rightarrow R^n$$

iff $\dot{\mathbf{x}}(t) \in \mathrm{F}(\mathbf{g}(t,\mathbf{x}(t)))$ almost everywhere on **I** where: operator

$$\mathrm{F}(\mathbf{g}(t,\mathbf{x})) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(\mathbf{Z})=0} \overline{cvx}\, \mathbf{g}(t,(\mathbf{x}+\varepsilon\mathbf{B}) \setminus \mathbf{Z}),$$

B is the open unit ball in R^n, $\mu(\mathbf{Z})$ is Lebesgue measure of the set **Z**, $\overline{cvx}\,\mathbf{M}$ denotes closure of the convex hull of $\mathbf{M} \subset R^n$. The solution formed in such a way is called F-solution. ■

For each $A \in (A_{\min}, A_{\max}]$ the trajectories $\mathbf{T}_{\min}^-$, $\mathbf{T}_{\min}^+$ are formed by F-solution only which are unique ones. Technical interpretation of the F-solution will be done for

$x_0 \in \mathbf{T}^-_{\min}$. The structure of the control function (3.8) [or (3.9)] by $\mathbf{T}^-_{\min} \to \mathbf{T}^-$, $\mathbf{T}^+_{\min} \to \mathbf{T}^+$ implies that from each $x_0 \in \mathbf{T}^-_{\min}$ there starts the standard solution generated by $u = -1$ the trajectory of which should penetrate into $\mathbf{R}^+$ and implies also that none of the above solution exists in $\mathbf{R}^+$. Practically, the trajectory of this solution on leaving $x_0 \in \mathbf{T}^-_{\min}$ penetrates into $x_0 \in \mathbf{T}^-_{\min}$ $\mathbf{R}^+$ where it is immediately forced to re-penetrate $\mathbf{T}^-_{\min}$. The real closed -loop system generates therefore a trajectory which starts to oscillate around $\mathbf{T}^-_{\min}$ with a certain frequency and amplitude depending on the delay time inherent in the switching operation, which evidently exists in every real structure. This trajectory of F-solution is therefore *a limit of the real oscillatory process (sliding, chattering) when the delay time tends to zero, i.e. when the frequency tends to infinity*. The same interpretation may be offered for F-solution forming the curve $\mathbf{T}^+_{\min}$. The closed-loop-system formed in accordance with presented concept is independent from whatever variation of resistance. Numerical simulations shown that the controlled processes are close to the minimum time ones.

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