Algorithms for Phase Acquisition for QAM Constellations

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Abstract

Because of their relatively good performance, large QAM constellations are being used in many current communication applications. One of the problems associated with the use of large QAM constellations is that of carrier acquisition, which for efficiency reasons must often be done without the use of a preamble. The problem is further complicated for cross constellations, for which the high SNR corner points used by some simple carrier phase estimators are not available. In this paper we derive simple algorithms for carrier phase acquisition that can be used for both square and cross constellations, and compare their performance to that of the 4-th power estimator. The introduced algorithms convert the problem of carrier phase estimation into one of estimating the *mode* of an underlying distribution.

1 Introduction

The problem of carrier acquisition in radio-frequency (RF) communications has been well studied for many years, both for phase-shift-keying (PSK) and quadrature amplitude modulation (QAM) systems. QAM is particularly attractive for high throughput efficiency applications, because of its better performance compared to PSK as the size of the constellation increases.

With the development of high speed systems that will carry for example digital TV signals, the problem of fast carrier acquisition has become even more important, as larger and larger constellations are used for improved efficiency. Systems using QAM constellations of size 128, 256, 512 and even 1024 are being considered [1, 2]. For these large signal constellations, carrier synchronization is critical, a fact that imposes stringent constraints on the quality of the carrier acquisition algorithms to be used. This fact, coupled with the high data rates at which these systems operate, implies that not only the algorithms must perform well, but they must at the same time be simple to implement.

Carrier acquisition involves acquisition of both frequency and phase. In practical systems, frequency acquisition is performed first, leaving a signal constellation which is not rotating (or that rotates at a rate which is slow compared to the signaling rate) but has a constant phase offset that needs to be corrected by the phase synchronizer. The phase synchronization problem is invariably divided into an acquisition and a tracking part. In many practical systems, tracking is done simply and efficiently in a decision-directed (DD) mode [3, 4], and it is the acquisition problem that is more problematic, especially in applications where no preamble is allowed. The problem is even more complicated for cross constellations (i.e., with sizes 2^{2m+1}, $m = 2, 3, \cdots$) which do not have the corner points on which some simple carrier phase acquisition algorithms are based, and which contain significant phase information.

In this paper we look at the problem of carrier phase acquisition for both square and cross QAM constellations. Since QAM constellations are rotationally invariant to rotations by multiples of ninety degrees, we assume (which will be the case for any practical system) that data are differentially encoded. Thus, only phase estimates modulo $\pi/2$ need be extracted.

In Section 2 we introduce the 4-th power estimator and a modified version of it that focuses on a reduced constellation for better performance. We then use the maximum-likelihood (ML) phase estimator to motivate two suboptimal estimators, whose performance is studied in Section 3. Section 4 concludes.

2 Algorithm Development

In the following, we assume that the system is already equalized, frequency synchronized, and both timing recovery and relative gain control have been established. Our focus in this paper is on carrier phase acquisition. Under the above assumptions, the baud-rate samples of the output of a matched filter are described by

$$r_k = d_k e^{j\phi} + n_k, \quad k = 1, 2, \cdots, L, \cdots, \tag{1}$$

where d_k (a complex number) is the QAM symbol transmitted at time kT, $1/T$ is the signaling rate, and ϕ demotes the unknown phase-offset to be estimated; the n_k are complex, independent, identically distributed (i.i.d.), zero-mean, Gaussian random variables with independent real and imaginary parts having variance σ^2, modeling the effects of noise in the system and channel. Without loss of generality, we assume that $E[d_k^2] = 1$ (i.e., a unit average energy constellation), in which case the signal-to-noise ratio per symbol is $\mathrm{SNR} = 1/2\sigma^2$.

The maximum-likelihood (ML) estimator is well known for this problem, but it is hopelessly complex for any practical implementation. In the limit as the SNR goes to zero, it's been shown that the ML phase estimator reduces to the non-data aided (NDA) P-th power estimator for BPSK ($P = 2$) and QPSK ($P = 4$) constellations [7, 8], and extended to arbitrary M-PSK constellations ($P = M$) in [9]. The P-th power synchronizer was recently shown by Moeneclaey and de Jonghe [5] to be optimal (in the sense of ML) in the limit of small SNR's for arbitrary two-dimensional, rotationally symmetric constellations, such as, for example, QAM constellations (for which $P = 4$). The P-th power phase estimator produces a phase estimate according to

$$\hat{\phi} = \frac{1}{P} \arg \left[E[d^{*P}] \sum_{k=1}^{L} r_k^P \right]. \tag{2}$$

Simulation results for the performance of the 4-th power estimator in (2) are presented in Figure 1 for 128-QAM and 256-QAM. Also plotted in the figure are the approximate analytical results, and for comparison later in the paper the performance of two algorithms introduced next. The results illustrate the significantly worse performance of the power law estimator for cross-constellations, and the fact that increasing SNR above some practical level has only a minor effect on performance.

Next, we derive a suboptimal phase estimator which has good acquisition performance even for cross constellations. The log-likelihood function can be easily derived and shown to be

$$\ell(\phi) = \sum_{k=1}^{L} \ln \left\{ \sum_{d} \exp \left[-\frac{1}{2\sigma^2} \left| |r_k| - |d| e^{j(\phi+\theta_d-\omega_k)} \right|^2 \right] \right\}, \tag{3}$$

where θ_d is the argument (angle) of QAM symbol d and ω_k is the argument of r_k.

If we concentrate on the inner sum in (3), we observe that for large SNR's (such as those expected in practical systems), only terms for which the quadratic exponent is small are significant. For a given time-index k, the significant terms correspond to the symbols d for which the quadratic term is minimized. Towards minimizing the quadratic term, we have the following inequality

$$\left| |r_k| - |d| e^{j(\phi+\theta_d-\omega_k)} \right|^2 \geq [|r_k| - |d|]^2 \geq \min_d \; [|r_k| - |d|]^2, \tag{4}$$

where equality in the first inequality is achieved if and only if

$$\phi + \theta_d - \omega_k = 0, \tag{5}$$

and in the second (in addition to the condition in (5)) for symbols d whose magnitude is closest to the magnitude of r_k. Thus, at high SNR's, only symbols d whose amplitude most closely matches the amplitude of the received data r_k need be considered. It can be shown that the optimum amplitude detector makes independent amplitude decisions (no need for observing a block of data) and that in the limit as the SNR goes to infinity, the optimum thresholds for amplitude detection tend towards the midpoint between adjacent amplitudes. Even though the optimum thresholds can be computed easily for any SNR, in the sequel, for simplicity, we will use the midpoint thresholds to detect amplitudes. These were seen to be close to the optimum for practical SNR levels.

Let the symbols d that minimize the right-hand side of (4) be denoted by $\hat{d}_k$, and let $\mathcal{D}_k$ be the set containing these symbols. For QAM constellations (square or cross), the number of symbols, $|\mathcal{D}_k|$, in $\mathcal{D}_k$ is a multiple of four, with $\frac{|\mathcal{D}_k|}{4}$ of them in each of the four quadrants. Let the subset of $\frac{|\mathcal{D}_k|}{4}$ symbols $\hat{d}_k$ in the first quadrant be denoted by Δ_k. Then, the remaining symbols can be obtained by rotating symbols in Δ_k by $n \cdot \frac{\pi}{2}$, for $n = 1, 2, 3$. Let α_k represent the angles of the $\frac{|\mathcal{D}_k|}{4}$ symbols in the first quadrant. Then the angles of all the symbols $\hat{d}_k$ can be expressed as

$$\theta_{\hat{d}_k} = \alpha_k + n \cdot \frac{\pi}{2}, \quad n = 0, 1, 2, 3.$$

Figure 2 illustrates the above definitions for 64-QAM.

With the above approximation, we have so far succeeded in reducing significantly the number of terms in the inner sum in the log-likelihood function. This is a significant improvement, but still a nonlinear maximization problem must be solved to obtain a phase estimate, which is impractical for most applications. Let's assume for a moment that we are interested in a phase estimate based solely on the single data r_k. Then, the condition for equality in (5) can be used to yield the following expression for the estimates (as many as $|\mathcal{D}_k|$)

$$\hat{\phi}_k = \omega_k - \theta_{\hat{d}_k} = \omega_k - \alpha_k - n \cdot \frac{\pi}{2}, \quad n = 0, 1, 2, 3. \tag{6}$$

Since we are only interested in phase estimates modulo $\pi/2$, only the $\frac{|\mathcal{D}_k|}{4}$ angles α_k in subset Δ_k need be considered. Thus,

$$\hat{\phi}_k = (\omega_k - \alpha_k) \mod (\pi/2). \tag{7}$$

Clearly, based on a single data r_k all we have achieved is narrow down the actual phase offset to be in the vicinity of one of the $\frac{|\mathcal{D}_k|}{4}$ estimates produced by (7). This ambiguity can now be resolved in time by observing more data. Assuming that the phase offset is constant over a number of symbols, we expect that as sets of estimates are produced according to (7) there will be one phase estimate that will be common to all (not exactly the same, of course, due to noise). All that is needed then is a way to determine this common phase estimate. We do this by quantizing the first quadrant into N angles centered within equally spaced angular subintervals. These subintervals are indexed by an index $j = 1, 2, \cdots, N$, and a counter C_j is associated with each. The corresponding angle at the center of the subinterval we denote by Φ_j. Whenever a phase estimate is produced (using (7)) falling within the j-th subinterval, counter C_j is incremented by one. This is repeated for a number of symbol intervals, and when sufficient data is processed to yield a reliable estimate, one is produced according to

$$\hat{\phi} = \Phi_I, \tag{8}$$

where

$$I = \arg \max_j C_j. \tag{9}$$

Summary of the Algorithm

1. Given the received sample r_k, compute its magnitude $|r_k|$ and angle ω_k.
2. Use $|r_k|$ to find the (most likely) amplitude of the transmitted QAM symbol, i.e. the amplitude that minimizes $[|r_k| - |d|]^2$. This determines $\frac{|\mathcal{D}_k|}{4}$ symbols ($\frac{|\mathcal{D}_k|}{4} = 2$ in the example in Figure 2) in the first quadrant that match the amplitude, and their corresponding angles α_k.
3. Produce the $\frac{|\mathcal{D}_k|}{4}$ phase estimates according to $\hat{\phi}_k = (\omega_k - \alpha_k) \mod (\pi/2)$.
4. Use the $\hat{\phi}_k$ to increment the counters (initially set to zero) corresponding to the quantization intervals they fall in.
5. Repeat for the next sample until enough data is collected.
6. When enough data is collected, find the counter with the largest number of counts. The angle Φ_j corresponding to this counter is the produced estimate.

In the sequel, we will refer to the above algorithm as the *Histogram Algorithm* (HA). A moment's reflection, indicates that what the HA does is to produce in time and as data arrive an estimate of the probability density of the rotation phase ϕ, averaged over all data symbols and additive noise. The HA then finds the *mode* of this density function as the best estimate of the rotation phase. An expression for the density function of the rotation angle, $\hat{\phi}$, of the received data, as a function of the rotation angle ϕ has been derived and for 32-QAM is given by

$$g(\hat{\phi}|\phi) = E_{|d|}\left\{E_{\Delta\theta}\left[u[(\hat{\phi}-\phi)+\Delta\theta, |d|^2\mathrm{SNR}]\right]\right\}, \tag{10}$$

where

$$\begin{aligned} u(x,y) &= \frac{2e^{-y}}{\pi} + \sqrt{\frac{y}{\pi}}\left[\cos(x)e^{-y\sin^2(x)}\mathrm{erf}[\sqrt{y}\cos(x)]\right. \\ &\qquad \left. + \sin(x)e^{-y\cos^2(x)}\mathrm{erf}[\sqrt{y}\sin(x)]\right] \\ &\approx \frac{\sqrt{y}}{\mathrm{erf}(\sqrt{y})\sqrt{\pi}}\left[\cos(x)e^{-y\sin^2(x)}\right. \\ &+ \left.\sin(x)e^{-y\cos^2(x)}\right], \quad 0 \le x \le \pi/2, \end{aligned}$$

where $\Delta\theta$ is the set of all (distinct) differences between the angles of symbols having magnitude $|d|$ in the first quadrant, which can easily be determined given the constellation. Figure 3 compares this density function for 32-QAM to histogram data obtained through simulation for various sequence lengths. Clearly, as the number of data observed increases, the histogram converges to the derived density function.

The problem of estimating the mode of a distribution from data is a classical one and has been well studied in the statistics literature [11, 12, 13, 14]. In general, partitioning the data into a number of discrete bins is not desirable, since the process results in loss of information. There are in the literature a number of algorithms which perform better in estimating the mode than constructing a histogram, but they typically require more complex processing. One such algorithm which is non-parametric is pursued below, and proceeds as follows [14]:

1. Construct a vector $\mathbf{x}$ whose elements are in increasing order the angle estimates obtained through (7).
2. For some integer $J \ge 3$ compute
$$\hat{\imath} = \arg \min_i (x_{i+J} - x_i).$$
3. Declare as the estimate of the mode (and thus the estimate of the rotation angle) the angle
$$\hat{\phi} = \frac{1}{2}(x_{\hat{\imath}+J} + x_{\hat{\imath}}).$$

The optimum value of J is a trade-off between reducing estimator bias and variance, with smaller values of J reducing bias and larger reducing variance. In simulations for 128-QAM, it was found that for $L \ge 10$ the optimum value of J was approximately $(7+0.2L)$, $0.275L$, and $0.375L$ for 20dB, 25dB, and 30dB respectively. We will refer to the algorithm that estimates the mode as above as the *modified histogram algorithm* (MHA).

For the HA algorithm, the number of bins that partitions the first quadrant is a parameter to be optimized. This is a classical problem in statistics, with a larger number of bins reducing estimation bias, and a smaller number reducing estimation variance. Since the MSE is the sum of the variance and the square of the bias, an optimum value for the number of bins exists which is a function of the SNR and the length of the observed sequence. Figure ?? shows the performance of the HA for 128-QAM as a function of the number of bins for various sequence lengths and SNR's. The optimum value of the number of bins was seen empirically to be proportional to $\sqrt{L}$, and it is approximately equal to $\sqrt{L}$ for 20dB and $4\sqrt{L}$ for 25dB for 128-QAM. For 256-QAM, the optimum number of bins are approximately $0.5\sqrt{L}$ for 20dB and $5.3\sqrt{L}$ for 30dB.

The performance of the HA and MHA is illustrated through simulations in Figures 4 and 5 respectively for 128-QAM and 256-QAM. Results for other constellations were also obtained and show similar trends. For the MHA, the approximate optimum values of J were used, and for the HA the first quadrant was partitioned into 45 bins. As can be seen, the MHA performs significantly better than the HA, but for acquisition purposes the inferior performance of the HA may be adequate, providing a good tradeoff between performance and complexity. At 30dB, the MSE for the HA quickly reaches the quantization limit.

Other results for the HA and MHA algorithms are shown in Figure 2 for 128-QAM and 256-QAM.

3 Performance Comparisons

From the results presented in this paper, and other that were not included for brevity, we draw the following conclusions regarding the relative performance of the various algorithms:

- The performance of the histogram and modified histogram algorithms is significantly better than that of the power law and modified power law algorithms for moderate to high SNR's. This is especially so for cross-constellations, where the power law algorithms require an inordinate amount of time to acquire compared to either the HA or the MHA (50 symbols vs more than 10,000 symbols).

- The power law algorithms perform much better for square constellations, but are still significantly inferior to either the HA or the MHA (300 symbols for the power law algorithms vs 50 symbols for the HA or MHA) for moderate to high SNR's.

- For square constellations and low SNR the power law algorithms start performing better than the HA. For example, as seen in Figure 2, at 20dB and 256-QAM, the HA is significantly inferior to the power law algorithm.

- For cross constellations, the power law algorithm is not viable for practical implementation due to its poor performance.

4 Conclusion

Starting from the likelihood function, we have derived two new algorithms for carrier phase acquisition, applicable to both square and cross QAM constellations. The algorithms were shown to reduce the problem of carrier phase estimation to one of estimating the mode of a distribution, an analytical expression for which was also obtained. These algorithms are simple to implement, and perform well for moderate to high SNR's for both square and cross constellations. Comparisons were made to the 4-th power estimator. The introduced algorithms were seen to significantly outperform the 4-th power algorithm for practical SNR levels and cross constellations, to outperform them for moderate to high SNR's and square constellations, but to perform worse for square constellations and low SNR's.

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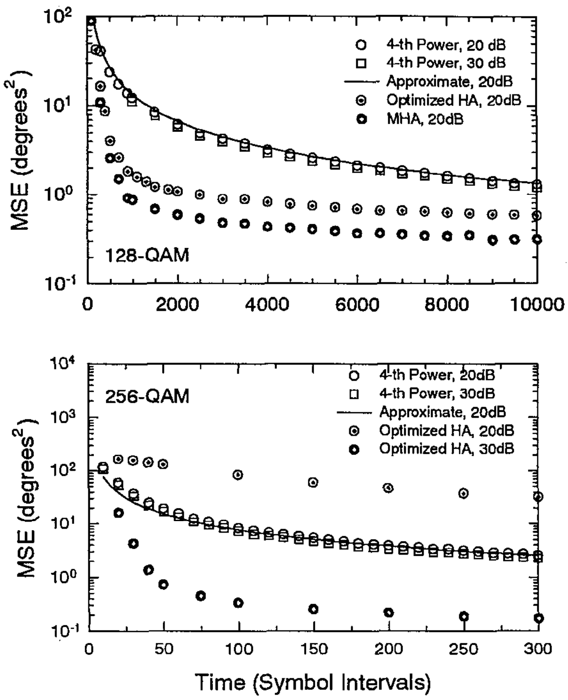

Figure 1: The performance of the 4-th power estimator for 128 and 256 QAM compared to that of the HA and MHA.

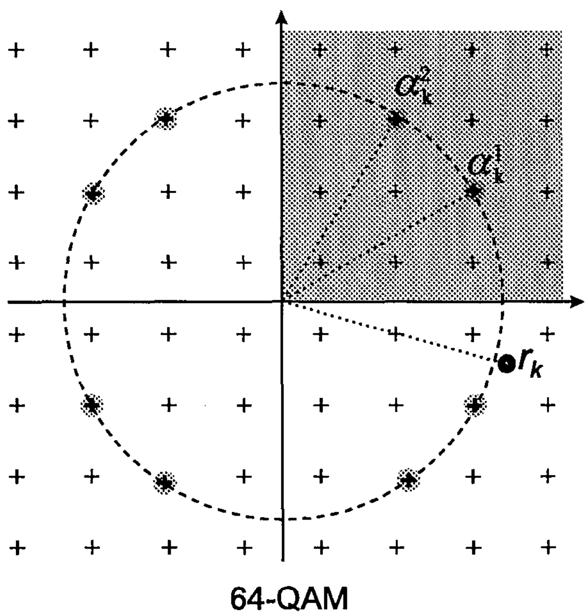

Figure 2: Illustration of sets $\mathcal{D}_k$ and Δ_k for 64-QAM.

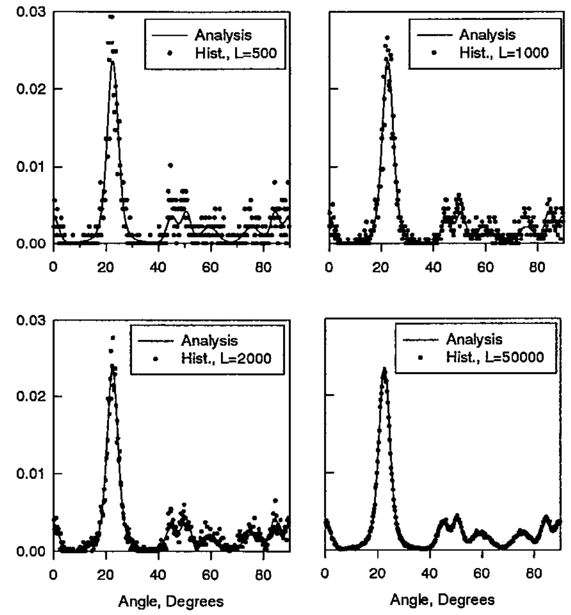

Figure 3: The histogram versus the analytical density function for various data lengths and for 32-QAM. The phase offset is 22.5 degrees.

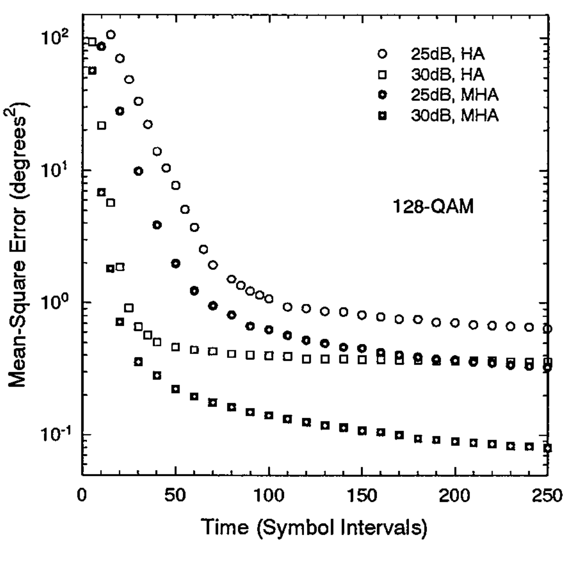

Figure 4: The performance of the HA with a 45-bin partition and the MHA for 128-QAM.

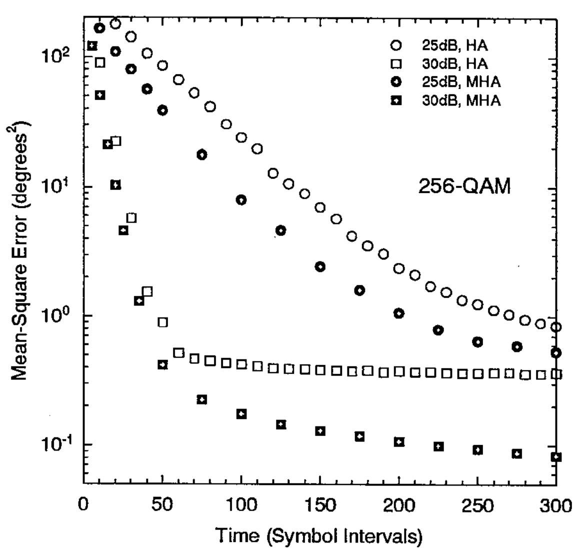

Figure 5: The performance of the HA with a 45-bin partition and the MHA for 256-QAM.