Ergodic Boundary Control of Semilinear Systems[1]

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Abstract

A controlled Markov process in a Hilbert space and an ergodic cost functional are given for a control problem that is solved where the process is a solution of a parameter dependent semilinear stochastic differential equation and the control can occur only on the boundary or at discrete points in the domain. The linear term of the semilinear differential equation is the infinitesimal generator of an analytic semigroup. The noise for the stochastic differential equation can be distributed, boundary and point. Some ergodic properties of the controlled Markov process are shown to be uniform in the control and the parameter. The existence of an optimal control is verified to solve the ergodic control problem. The optimal cost is shown to depend continuously on the system parameter.

Key words: ergodic control, stochastic semilinear equations, Markov processes in Hilbert spaces, boundary control.

1 Introduction

An ergodic control problem for a stochastic process in a Hilbert space H is formulated and solved where the process is a solution of a parameter dependent semilinear stochastic differential equation in H. The problem in the general setting is motivated by ergodic control problems for processes governed by stochastic partial differential equations with control and noise occurring in the boundary conditions or at discrete points in the domain.

For example, consider the stochastic parabolic equation

$$\frac{\partial v}{\partial t}(t,\xi) = Lv(t,\xi) + F(\alpha, v(t,\xi)) + n(t,\xi) \qquad (1.1)$$

for $(t,\xi) \in \mathbf{R}_+ \times (0,1)$ with the initial and the boundary conditions

$$v(0,\xi) = v_0(\xi) \qquad (1.2)$$

$$\frac{\partial v}{\partial \xi}(t,0) = h_1(\alpha, v(t,\cdot), u(v(t,\cdot))) + \eta_1(t) \qquad (1.3)$$

$$\frac{\partial v}{\partial t}(t,1) = h_2(\alpha, v(t,\cdot), u(v(t,\cdot))) + \eta_2(t), \qquad (1.4)$$

where n denotes a space dependent Gaussian noise that is white in time, η_1 and η_2 are one dimensional standard Wiener processes and these three processes are mutually independent. Furthermore,

$$Lv = a(\xi)\frac{\partial^2}{\partial \xi^2}v + b(\xi)\frac{\partial}{\partial \xi}v + c(\xi)$$

is a second order uniformly elliptic operator where $a, b, c \in C^\infty([0,1])$, $a > 0$, $c < 0$, $F : \mathcal{A} \times \mathbf{R} \to \mathbf{R}$, $h_i : \mathcal{A} \times H \times \mathcal{K} \to \mathbf{R}$, $i = 1, 2$, where $H = L^2(0,1)$, $\mathcal{A} \subset \mathbf{R}^{d_1}$ and $\mathcal{K} \subset \mathbf{R}^k$ are compact. The control problem is to minimize the ergodic cost functional

$$J(x,u,\alpha) = \limsup_{T\to\infty} \mathbf{E}\frac{1}{T}\int_0^T c(v(t), u(v(t)))dt$$

over the set of Markov controls $\mathcal{U} = \{u : H \to \mathcal{K} \mid u \text{ is Borel measurable}\}$, where $c : H \times \mathcal{K} \to \mathbf{R}$. The $\alpha \in \mathcal{K}$ in (1.1)–(1.4) represents a parameter.

The equations (1.1), (1.3) and (1.4) are only formal because the noise terms n, η_1 and η_2 are not well defined stochastic processes (random fields). A standard approach for the rigorous treatment of the problem is to rewrite (1.1) as a controlled stochastic differential equation in the Hilbert space H, and to define the noise terms using Wiener processes with infinite dimensional state spaces and the solution to the equation as a mild solution, using the semigroup theory (cf. [10,27]).

In the present paper, this general framework is used. The controlled Markov process is defined by a Hilbert space-valued stochastic differential equation ((2.1) below). The linear term of the equation is the infinitesimal generator of an analytic semigroup. The general setting allows to cover, as special cases, stochastic boundary/point control problems as the above example (see Examples 4.1 and 4.2). The noise for the stochastic differential equation can be distributed, boundary and point. The parameter dependence occurs in the distributed and the boundary or

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the point drift terms. The control occurs only in the boundary or point drift term. The fact that the control is not distributed would seem to allow for more physically meaningful models. The noise is allowed to occur in both distributed and discrete forms to ensure more flexibility of the models. Since the H-valued Markov process depends on the control and the parameter, it is shown that some ergodic properties of the process are uniform in these quantities. For the solution of an ergodic control problem the existence of an optimal control is verified. It is shown that the optimal cost depends continuously on the system parameter.

Continuity of the optimal cost on the parameter is an important step in solving the adaptive control problem when the parameter is unknown. This verification is important to show the optimality of an adaptive control defined by means of a family of strongly consistent estimators of the unknown parameter α. In the case when the control and noise are distributed, the existence of an optimal control has been proven in [13] while the continuity of the optimal cost is new also for this case.

A brief outline of the paper is given now. In Section 2 the control problem is formulated and the basic assumptions are made and explained. The controlled process is the unique, weak, mild solution of the stochastic differential equation and induces a Markov process in H. Some estimates are made of this process and an approximation of the transition probability function for the Markov process solution of the stochastic differential equation by transition functions of the solutions of the stochastic differential equation with bounded drifts are given where the approximation is uniform in the control and the parameter. The existence and the uniqueness of the mild (backward) Kolmogorov equation for the controlled Markov process are verified in [34]. Section 3 contains the main results of the paper : the existence of an optimal control for a fixed parameter and the continuous dependence of the optimal cost on the parameter. In Section 4 two examples are given that satisfy the assumptions that are made for the control problem: In Example 4.1 the control problem (1.1)–(1.4) is treated and Example 4.2 contains a similar control problem where the control and noise occur at given discrete points in the domain rather than on the boundary.

2 Preliminaries

Consider a controlled, infinite dimensional process $(X(t),\, t \geq 0)$ that satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\mathrm{d}t &= (f(\alpha, X(t)) \\ &\quad + Bh(\alpha, X(t), u(X(t))))\mathrm{d}t \\ &\quad + B\mathrm{d}V(t) + Q^{1/2}\mathrm{d}W(t) \\ X(0) &= x \end{aligned} \tag{2.1}$$

where $X(0), X(t) \in H$, H is a separable, infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, $\alpha \in \mathcal{A} \subset \mathbf{R}^d$ is a parameter and $\mathcal{A}$ is compact, U is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$ and norm $|\cdot|_U$, $\mathcal{K}$ is a compact product of intervals in $\mathbf{R}^k$, $-A : \mathrm{Dom}(-A) \to H$ is the infinitesimal generator of an analytic semigroup $(S(t),\, t \geq 0)$ such that $A^{-1} \in \mathcal{L}(H)$, which is often denoted $A > 0$,

$$f : \mathcal{A} \times H \to H$$

$$h : \mathcal{A} \times H \times \mathcal{K} \to U$$

are Borel measurable functions, $B \in \mathcal{L}(U, D_A^{\varepsilon-1})$, the family of bounded linear operators from U to $D_A^{\varepsilon-1}$, where $\varepsilon \in (0,1]$ is given and D_A^δ for $\delta \geq 0$ is the domain of the fractional power A^δ with the topology induced by the graph norm $|x|_{D_A^\delta} = |A^\delta x|$, while for $\delta < 0$ it is a completion of H in the norm $|\cdot|_{D_A^\delta}$. It is assumed that $Q \in \mathcal{L}(H)$ is positive and self-adjoint and $(V(t),\, t \geq 0)$ and $(W(t),\, t \geq 0)$ are independent, standard cylindrical Wiener processes in the spaces U and H, respectively, that are defined on a filtered, complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. The family of controls, $\mathcal{U}$, is

$$\mathcal{U} = \{u : H \to \mathcal{K} \mid u \text{ is Borel measurable}\}.$$

The control problem is to minimize, over $u \in \mathcal{U}$, the ergodic cost functional

$$J(x, u, \alpha) = \limsup_{T \to \infty} \mathbf{E} \frac{1}{T} \int_0^T c(X(s), u(X(s)))\mathrm{d}s \tag{2.2}$$

where $c : H \times \mathcal{K} \to \mathbf{R}_+$ is bounded and Borel measurable.

The following assumptions, (A1)–(A7) are used selectively in this paper.

(A1) There exist a $\gamma \in (0, 1/2]$ and a $\Delta \in (0, 1/2]$ such that $B \in \mathcal{L}_2(U, D_A^{\gamma-1/2})$ and $Q^{1/2} \in \mathcal{L}_2(H, D_A^{\Delta-1/2})$ where $\mathcal{L}_2(\cdot,\cdot)$ is the family of Hilbert-Schmidt operators.

(A2) For each $\alpha \in \mathcal{A}$ the functions $h(\alpha, \cdot, \cdot) : H \times \mathcal{K} \to U$ is continuous and $f(\alpha, \cdot) : H \to H$ is Lipschitz continuous on the bounded subsets of H and there are constants k, k_f, k_h and $\tilde{k}(\alpha)$ such that $|f(\alpha, x)| \leq k + k_f|x|$, $|h(\alpha, x, u)|_U \leq k + k_h|x|$ and $|h(\alpha, x, u)|_U \leq \tilde{k}(\alpha)$ for all $x \in H$, $u \in \mathcal{K}$ and $\alpha \in \mathcal{A}$.

By (A1) and the analyticity of $-A$, the composition $S(r)B$ is well defined for $r > 0$ and furthermore $S(r)B \in \mathcal{L}_2(U,H)$, $S(r)Q^{1/2} \in \mathcal{L}_2(H)$ and

$$\int_0^t |S(r)B|^2_{\mathcal{L}_2(U,H)} dr + \int_0^t |S(r)Q^{1/2}|^2_{\mathcal{L}_2(H)} dr < \infty$$

for $t > 0$. Therefore, the family of operators $(Q_t, t \geq 0)$,

$$Q_t = \int_0^t S(r)BB^*S^*(r)\mathrm{d}r + \int_0^t S(r)QS^*(r)\mathrm{d}r. \tag{2.3}$$

is well defined and $Q_t \in \mathcal{L}_2(H)$ for each $t \geq 0$.

(A3) The following are satisfied

$$\mathcal{R}(\tilde{S}(t)) \subset \mathcal{R}(Q_t^{1/2}),$$

$$|Q_t^{-1/2} S(t) A^{1-\varepsilon}|_{\mathcal{L}(H)} \leq \frac{c}{t^\beta}$$

for $t \in (0,T]$ for some $T > 0$, $c > 0$ and $\beta < 1$ where $(\tilde{S}(t), t \geq 0)$ is the restriction of $(S(t), t \geq 0)$ to the space $D_A^{1-\varepsilon}$ and $\mathcal{R}(\cdot)$ is the range.

(A4) There is a continuous, increasing function $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ with $\omega(0) = 0$ such that

$$|f(\alpha,x) - f(\beta,x)| + |h(\alpha,x,u) - h(\beta,x,u)|_U$$
$$\leq \omega(|\alpha - \beta|)(1 + |x|)$$

for all $\alpha, \beta \in \mathcal{A}$, $x \in H$ and $u \in \mathcal{K}$.

(A5) For each $u \in \mathcal{U}$ and $\alpha \in \mathcal{A}$ there is an invariant measure $\mu(\alpha,u)$ for the process $(X(t), t \geq 0)$ that satisfies (2.1) and the family of measures $(\mu(\alpha,u),\ \alpha \in \mathcal{A},\ u \in \mathcal{U})$ is tight.

(A6) The function $c : H \times \mathcal{K} \to \mathbf{R}_+$ given in (2.2) is bounded and Borel measurable and $c(x,\cdot) : \mathcal{K} \to \mathbf{R}_+$ is continuous for each $x \in H$.

(A7) The set $h(\alpha, x, \mathcal{K}) \times c(x, \mathcal{K}) \subset U \times \mathbf{R}_+$ is convex for each $\alpha \in \mathcal{A}$ and $x \in H$.

Some comments on the above assumptions (A1)-(A7) are given now. Assumption (A1) is a standard condition guaranteeing that the solution of the linear version of the equation (2.1) (i.e., with $f = 0$ and $h = 0$) is an H-valued stochastic process (otherwise it is only a cylindrical process, see e.g., [12]). Note that (A1) implies that the above defined operators Q_t are trace class operators on H. They are covariance operators of the (Gaussian) probability distribution of the solution to the linear equation.

The assumption (A2) is used to verify that there exists a unique, weak, mild solution to the equation (2.1).

The assumption (A3) is used in to prove (see [34]) some suitable smoothing properties of the mild backward Kolmogorov equation corresponding to the stochastic equation (2.1), which is needed to show the ergodicity of the solutions to (2.1) and some continuity properties of the transition probability kernels. The assumption is also rather standard in the context of the perturbation methods, for instance, for $\mathbf{E}psilon = 1$ the results of Section 3 have been proven in [9, 10].

The assumption (A4) is a continuous dependence of the coefficients of the equation (2.1) on the parameter α. It is used for the verification of the results that are related to the continuous dependence of the optimal cost on the parameter.

The assumption (A5) is a kind of stability assumption that is usually needed in ergodic control problems. In [34] (A5) is verified in terms of some more explicit conditions on the coefficients of equation (2.1) (Lyapunov-type conditions).

The assumptions (A6) and (A7) are typical conditions that are used in the ergodic control theory ((A6) is sometimes called the Roxin type condition) and they are used to establish the existence of an optimal control for the given control problem.

Consider the following two stochastic differential equations

$$\mathrm{d}Z(t) + AZ(t)\mathrm{d}t = B\mathrm{d}V(t) + Q^{1/2}\mathrm{d}W(t)$$
$$Z(0) = x \tag{2.4}$$

and

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\mathrm{d}t &= f(\alpha, X(t))\mathrm{d}t + B\mathrm{d}V(t) \\ &\quad + Q^{1/2}\mathrm{d}W(t) \end{aligned}$$
$$X(0) = x. \tag{2.5}$$

Under the assumptions (A1) and (A2) it is easy to verify that each of the equations (2.4) and (2.5) has one and only one mild solution on the probability space $(\Omega, \mathcal{F}, P)$, that is, the solutions to the integral equations

$$\begin{aligned} Z(t) &= S(t)x + \int_0^t S(t-r)B\mathrm{d}V(r) \\ &\quad + \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r) \quad t \geq 0 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} X(t) &= S(t)x + \int_0^t S(t-r)f(\alpha, X(t))\mathrm{d}t \\ &\quad + \int_0^t S(t-r)B\mathrm{d}V(r) \\ &\quad + \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r) \quad t \geq 0. \end{aligned} \tag{2.7}$$

These solutions are D_A^δ-valued processes that belong to $C([0,T],L^p(\Omega,H))\cap C((0,T],L^p(\Omega,D_A^\delta))$ for any $p\geq 1$, $T>0$ and $\delta\in[0,\min(\varepsilon,\Delta,\gamma))$ (cf. [27]). Furthermore, the processes $(X(t),\,t\geq 0)$ and $(Z(t),\,t\geq 0)$ have D_A^δ-continuous versions (cf. [11, 30]) and in H they induce two Markov processes in the usual way.

Let $P_\alpha:\mathbf{R}_+\times H\times\mathcal{B}(H)\to[0,1]$ be the transition probability function for $(X(t),\,t\geq 0)$ in (2.7), that is,

$$P_\alpha(t,x,\Gamma)=\mathbf{P}_x(X(t)\in\Gamma) \tag{2.8}$$

and let $(T(t),\,t\geq 0)$ be the Markov transition semigroup for $(Z(t),\,t\geq 0)$ in (2.6), that is,

$$T_t\varphi(x)=\mathbf{E}_x\varphi(Z(t)) \tag{2.9}$$

where $x\in H$ stands for the initial value of $X(\cdot)$, $t\geq 0$ and $\varphi\in\mathcal{M}(H)$, the bounded, Borel measurable functions on H. It is clear that

$$T_t1_\Gamma(x)=N(S(t)x,Q_t)(\Gamma)$$

where $t\geq 0$, $\Gamma\in\mathcal{B}(H)$, $x\in H$ and Q_t is given by (2.3) so it is self-adjoint, nonnegative and nuclear and $N(S_tx,Q_t)$ is the Gaussian measure on H with mean S_tx and covariance Q_t.

Let $\xi_T^{\alpha,u}$ be the random variable as follows

$$\begin{aligned}\xi_T^{\alpha,u} &= \int_0^T\langle h(\alpha,X(t),u(X(t))),\mathrm{d}V(t)\rangle_U\\ &\quad-\frac{1}{2}\int_0^T|h(\alpha,X(t),u(X(t)))|_U^2\mathrm{d}t\end{aligned} \tag{2.10}$$

for $\alpha\in\mathcal{A}$, $u\in\mathcal{U}$ and $T>0$ where $(X(t),\,t\in[0,T])$ is the solution of (2.7). A weak solution of (2.1) is constructed following the standard procedure of an absolutely continuous change of probability measure (cf. [10, 15, 17, 23]). For control problems, the method was initiated in [1, 14]. Note that $\exp(\xi_T^{\alpha,u})=1$ by (A2). There is a probability measure $\mathbf{P}_x^{\alpha,u}$ on $\mathcal{F}$ such that the restriction of $\mathbf{P}_x^{\alpha,u}$ to $\mathcal{F}_T$ is given by

$$\mathbf{P}_x^{\alpha,u}(\mathrm{d}\omega)=\exp(\xi_T^{\alpha,u})\mathbf{P}(d\omega), \tag{2.11}$$

the process $(V^*(t),\,t\geq 0)$ given by

$$V^*(t)=V(t)-\int_0^t h(\alpha,X(s),u(X(s)))\mathrm{d}s$$

is a cylindrical Wiener process on U and using $\mathbf{P}_x^{\alpha,u}$ and the solution of (2.7) it follows that

$$\begin{aligned}X(t) &= S(t)x+\int_0^t S(t-r)f(\alpha,X(r))\mathrm{d}t\\ &\quad+\int_0^t S(t-r)Bh(\alpha,X(r),u(X(r)))\mathrm{d}r\\ &\quad+\int_0^t S(t-r)B\mathrm{d}V^*(r)\\ &\quad+\int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r).\end{aligned} \tag{2.12}$$

So there is a weak solution to (2.1) which is weakly unique and induces a Markov process on H whose Markov transition semigroup is denoted as

$$P_t^{\alpha,u}\varphi(x)=\mathbf{E}_x^{\alpha,u}\varphi(X(t)) \tag{2.13}$$

for $t\geq 0$ and $\varphi\in\mathcal{M}(H)$ where $\mathbf{E}_x^{\alpha,u}$ is the expectation using the probability measure $\mathbf{P}_x^{\alpha,u}$ and

$$P^{\alpha,u}(t,x,\Gamma)=P_t^{\alpha,u}1_\Gamma(x) \tag{2.14}$$

for $t\geq 0$, $\Gamma\in\mathcal{B}(H)$ and $x\in H$ is the corresponding transition probability function.

3 The Existence of an Optimal Control

Recall that the control problem is described by the system (2.1) and the cost functional

$$J(\alpha,u)=\limsup_{T\to\infty}\mathbf{E}_x^{\alpha,u}\frac{1}{T}\int_0^T c(X(s),u(X(s)))\mathrm{d}s \tag{3.1}$$

and the optimal cost is $J^*(\alpha)=\inf_{u\in\mathcal{U}}J(\alpha,u)$. If (A1)–(A3), (A5) and (A6) are satisfied then the following equality is satisfied

$$J(\alpha,u)=\int_H c(y,u(y))\mu(\alpha,u)(\mathrm{y}) \tag{3.2}$$

so the cost $J(\alpha,u)$ does not depend on the initial condition $X(0)=x\in H$. In this section the existence of an optimal control for the control problem (2.1) and (3.1) with a fixed parameter $\alpha\in\mathcal{A}$ and the continuity of the optimal cost $J^*:\mathcal{A}\to\mathbf{B}$ are presented. In Lemma 3.1 and Theorem 3.2 the parameter is fixed so it is suppressed for notational convenience.

Recall that $P(t,x,\Gamma)$ is given in (2.8) and $\eta=P(1,0,\cdot)$.

Lemma 3.1. *Let $(A_n,\,n\in\mathbf{N})$ be a sequence in $\mathcal{B}(H)$ such that $\eta(A_n)\to 0$ as $n\to\infty$. If* (A1)–(A3) *and* (A5) *are satisfied then*

$$\lim_{n\to\infty}\sup_{u\in\mathcal{U}}\mu(u)(A_n)=0 \tag{3.3}$$

Proof. See [34].

Theorem 3.2. *If* (A1)–(A3) *and* (A5)–(A7) *are satisfied for each $\alpha\in\mathcal{A}$ then there is an optimal control for the control problem given by* (2.1) *and* (3.1).

Proof. See [34].

Theorem 3.3. *If* (A1)–(A7) *are satisfied then the optimal cost $J^*:\mathcal{A}\to\mathbf{B}$ is continuous.*

Proof. See [34].

4 Some Examples

Example 4.1. Consider the scalar stochastic parabolic partial differential equation

$$\frac{\partial v}{\partial t}(t,\xi) = Lv(t,\xi) + F(\alpha, v(t,\xi)) + n(t,\xi) \quad (4.1)$$

for $(t,\xi) \in \mathbf{B}_+ \times (0,1)$ with the initial and boundary conditions

$$v(0,\xi) = v_0(\xi) \quad (4.2)$$

$$\frac{\partial v}{\partial \xi}(t,0) = h_1(\alpha, v(t,\cdot), u(v(t,\cdot))) + \dot{\beta}_1(t) \quad (4.3)$$

$$\frac{\partial v}{\partial \xi}(t,1) = h_2(\alpha, v(t,\cdot), u(v(t,\cdot))) + \dot{\beta}_2(t) \quad (4.4)$$

where n denotes a space dependent Gaussian noise that is white in time, β_1 and β_2 are one dimensional standard Wiener processes and these three processes are mutually independent, Furthermore,

$$Lv = a(\xi)\frac{\partial^2}{\partial \xi^2}v + b(\xi)\frac{\partial}{\partial \xi}v + c(\xi)$$

where $a, b, c \in C^\infty([0,1])$, $a > 0$, $c < 0$, $F : \mathcal{A} \times \mathbf{B} \to \mathbf{B}$, $h_i : \mathcal{A} \times H \times \mathcal{K} \to \mathbf{B}$ $i = 1,2$ where $H = L^2(0,1)$, $\mathcal{A} \subset \mathbf{B}^{d_1}$ is compact, $\mathcal{K} \subset \mathbf{B}^k$ is a compact product of intervals, $F(\alpha,\cdot) : \mathbf{B} \to \mathbf{B}$ is Lipschitz continuous, $h_i(\alpha,\cdot,\cdot) : H \times \mathcal{K} \to \mathbf{B}$ $i = 1,2$ is continuous and bounded for each $\alpha \in \mathcal{A}$ with at most linear growth that is uniform with respect to $\alpha \in \mathcal{A}$ and

$$|F(\alpha,\xi) - F(\beta,\xi)| + \sum_{i=1}^{2} |h_i(\alpha,x,u) - h_i(\beta,x,u)|$$
$$\leq \omega(|\alpha - \beta|)(1 + \max(|x|,|\xi|)) \quad (4.5)$$

for $\alpha, \beta \in \mathcal{A}$, $\xi \in \mathbf{B}$, $x \in H$) and $u \in \mathcal{K}$ where ω satisfies the properties in (A2). The system of equations (4.1–4.4) can be rewritten in the form of (2.1) in a natural way where $H = L^2(0,1)$, $U = \mathbf{B}^2$, $A = -L$ with

$$\mathrm{Dom}(A) = \left\{ \varphi : \varphi \in H^2(0,1), \frac{\partial}{\partial \xi}\varphi(0) = \frac{\partial}{\partial \xi}\varphi(1) = 0 \right\},$$

$f(\alpha,x)(\xi) = F(\alpha, X(\xi))$, $x \in H$, $\xi \in (0,1)$, and $h = [h_1, h_2]$. The operator B is defined as $B = \hat{A}N$ where $N \in \mathcal{L}(\mathbf{B}^2, D_A^\varepsilon)$, $\varepsilon < 3/4$ is the Neumann map corresponding to the elliptic Neumann problem

$$Lz(\xi) = 0 \qquad \xi \in (0,1) \quad (4.6)$$

$$\frac{\partial z}{\partial \xi}(0) = g_1 \qquad \frac{\partial z}{\partial \xi}(1) = g_2. \quad (4.7)$$

A simple example of a boundary input (4.3, 4.4) that satisfies all the above conditions is

$$\frac{\partial v}{\partial \xi}(t,0) = u_1(v(t,\cdot)) + \dot{\beta}_1(t) \quad (4.8)$$

$$\frac{\partial v}{\partial \xi}(t,1) = u_2(v(t,\cdot)) + \dot{\beta}_2(t) \quad (4.9)$$

where $(u_1, u_2) : H \to [-M, M]^2 = \mathcal{K}$.

Example 4.2. Consider the stochastic parabolic partial differential equation with pointwise noise and control

$$\frac{\partial v}{\partial t}(t,\xi) = Lv(t,\xi) + F(\alpha, v(t,\xi))$$

$$+ \sum_{i=1}^{N} [h_i(\alpha, v(t,\cdot), u(v(t,\cdot)) + \dot{\beta}_i(t)]\delta_{\xi_i} + n(t,\xi) \quad (4.10)$$

for $(t,\xi) \in \mathbf{B}_+ \times (0,1)$ with the initial and the boundary conditions

$$v(0,\xi) = v_0(\xi) \quad (4.11)$$

$$v(t,0) = 0 \quad (4.12)$$

$$v(t,1) = 0 \quad (4.13)$$

for $(t,\xi) \in \mathbf{B}_+ \times (0,1)$ where L, F, n, β_i and h_i are the same as in Example 4.1 and δ_{ξ_i} $i = 1,2,\ldots,N$ are the Dirac distributions at the points $\xi_i \in (0,1)$ $i = 1,2,\ldots,N$. The equation (4.10) is given a precise interpretation by using the equation (2.1) with H and f as in Example 4.1, $V(t) = (\beta_1(t), \ldots, \beta_N(t))$, $U = \mathbf{B}^N$, $h = (h_1, \ldots, h_N)$, $A = -L$ with $\mathrm{Dom}(A) = H^2(0,1) \cap H_0^1(0,1)$.

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