On Feedback Linearization Solution

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Abstract

Based upon a condensed Brunovský form, feedback linearization for both single-input and multi-input nonlinear systems is derived in a unified manner. The derivation appears considerably simpler than the known derivations for the multi-input case. A straightforward characterization of the coordinate transformation required in the feedback linearization is provided.

1 Introduction

Consider nonlinear systems of the form

$$\dot{x} = f(x) + g(x)\,u\,, \qquad x \in \mathbf{R}^n\,, \quad u \in \mathbf{R}^p \tag{1}$$

where f and columns of g are smooth vector fields on a neighbourhood $\mathcal{M} \in \mathbf{R}^n$ of $x(0) = x_0$. Assume that the columns of g denoted by $\{g_i\}$ are linearly independent on $\mathcal{M}$.

This paper deals with the problem of finding a coordinate transformation $\xi = T(x)$ on $\mathcal{M}$ such that in the new coordinates

$$\dot{\xi} = A\,\xi + B\,(a + b\,u) \tag{2}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$ are constant matrices and the pair $\{A, B\}$ is controllable, $a = a(T(x)) \in \mathbf{R}^p$ and $b = b(T(x)) \in \mathbf{R}^{p\times p}$ are with elements being functions of x, and b is invertible on $\mathcal{M}$. It is then trivial to use the feedback control $u = b^{-1}(v - b)$, where v is a new control input, to bring the system (2) into a pure linear control system $\dot{\xi} = A\,\xi + B\,v$.

This is the so-called *feedback linearization problem* by using static state feedback. The above problem formulation appears slightly different from but, of course, is equivalent to the those addressed in [1]-[5].

The feedback linearization problem has attracted considerable attention in 1980s. The problem was posed and solved in [1] under sufficient conditions for the single-input case. An appealing solution under necessary and sufficient conditions was further given in [4]. In [2] and [5], the results of [1] and [6] were generalized to the multi-input case and the corresponding necessary and sufficient conditions were obtained. In the book [7] a set of necessary and sufficient conditions for the feedback linearization are neatly represented. The reference [8] is an excellent survey on the feedback linearization and other related problems, see also comments provided in the bibliographical notes in [7].

Compared with the treatment of the single-input case, the known derivations of the solution for the multi-input case are considerably complicated. This prevents a ready understanding of the multi-input solution compared with that for the single-input case.

The objective of this paper is to derive the multi-input solution in a manner analogous to the single-input case. The derivation provides not only insights into the problem but also a simple procedure for constructing the coordinate transformation required in the feedback linearization.

A crutial technique in dealing with feedback linearisation problems is the use of the Brunovský form of a controllable pair $\{A,\, B\}$ as initiated in [9]. The treatment in this paper follows still this line. But, instead of the original form, a condensed Brunovský form is used, which simplifies the essential equation formulation as well as derivation and expression of the solution considerably.

The notation in this paper is quite standard. Let $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}^{1\times n}$ and $\gamma \in \mathbf{R}^{n\times 1}$ be respectively smooth function, covector field and vector field of real variables $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}'$, where $(\cdot)'$ stands for the transpose of $(\cdot)$. The Lie derivatives of α, β and γ by an n-dimensional smooth vector field f are $L_f\alpha = \frac{\partial\alpha}{\partial x} f$,

$$L_f\beta = f'\left(\frac{\partial\beta'}{\partial x}\right)' + \beta\frac{\partial f}{\partial x}, \quad ad_f\gamma = \frac{\partial\gamma}{\partial x}f - \frac{\partial f}{\partial x}\gamma\,.$$

The short notation $d\alpha = \frac{\partial\alpha}{\partial x}$ will also be used. The well-known Leibniz formula under Lie differentiation is

$$L_f\,\langle\beta,\,\gamma\rangle = \langle L_f\beta,\,\gamma\rangle + \langle\beta,\,ad_f\gamma\rangle \tag{3}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the inner product, e.g. $\langle\beta,\,\gamma\rangle = \beta\gamma$. Repeated Lie derivatives of α, β and γ by f are defined by induction: $L_f^k\alpha = L_f(L_f^{k-1}\alpha)$, $L_f^k\beta = L_f(L_f^{k-1}\beta)$, $ad_f^k\gamma = ad_f(ad_f^{k-1}\gamma)$ with $L_f^0\alpha = \alpha$, $L_f^0\beta = \beta$ and $ad_f^0\gamma = \gamma$.

For simplicity, the notations $L_f\alpha$, $L_f\beta$ and $ad_f\gamma$ are, by abuse, still used when $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_m \end{bmatrix}$,

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \tag{4}$$

where α_i, β_i and γ_i are real smooth function, covector fields and vector fields, respectively. In such circumstances, $L_f\alpha$,

$L_f\beta$ and $ad_f\gamma$ are understood as

$$L_f\alpha = \begin{bmatrix} L_f\alpha_1 \\ \vdots \\ L_f\alpha_m \end{bmatrix}, \quad L_f\beta = \begin{bmatrix} L_f\beta_1 \\ \vdots \\ L_f\beta_m \end{bmatrix}$$
$$ad_f\gamma = \begin{bmatrix} ad_f\gamma_1 & \cdots & ad_f\gamma_m \end{bmatrix}$$

and furthermore $L_\gamma\alpha$ means

$$L_\gamma\alpha = \begin{bmatrix} L_\gamma\alpha_1 \\ \vdots \\ L_\gamma\alpha_m \end{bmatrix} = \begin{bmatrix} L_{\gamma_1}\alpha_1 & \cdots & L_{\gamma_m}\alpha_1 \\ \vdots & & \vdots \\ L_{\gamma_1}\alpha_m & \cdots & L_{\gamma_m}\alpha_m \end{bmatrix}.$$

At places the notation, e.g., $\beta = \text{col.}\begin{bmatrix} \beta_1 & \cdots & \beta_m \end{bmatrix}$ is also used to indicate the same β as given in (4) to save space.

2 Condensed Brunovský form

The coordinate transformation $\xi = T(x)$ with feedback control $u = b^{-1}(v - b)$ preserves controllability of the system (1). Hence, a necessary condition for the feedback linearization is that there exists an m-tuple of integers $\{k_1, \cdots, k_m\}$ with, after reordering $\{g_i\}$, $k_1 \geq \cdots \geq k_m > 0$ and $\sum_{i=1}^m k_i = n$, such that $\operatorname{rank} \bar{G}(x) = n$ on $\mathcal{M}$, where

$$\bar{G} = \begin{bmatrix} g_1 & ad_f g_1 & \cdots & ad_f^{k_1-1} g_1 & \cdots \\ g_m & ad_f g_m & \cdots & ad_f^{k_m-1} g_m \end{bmatrix}$$

is called the controllability matrix and $\{k_1, \cdots, k_m\}$ the controllability indices. A formal proof of the requirement of nonsingularity of the controllability matrix will be given in Section 3.

A controllable pair $\{A, B\}$ with the indices $\{k_1, \cdots, k_m\}$ is said to be in the controllability *controllability form* [11] if $A = \operatorname{diag}(A_1, \cdots, A_m)$, $B = \operatorname{diag}(B_1, \cdots, B_m)$, where

$$A_i = \begin{bmatrix} a_{1,i} & \cdots & \cdots & a_{1,k_i} \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5}$$

$A_i \in \mathbf{R}^{k_i \times k_i}$, $B_i \in \mathbf{R}^{k_i \times 1}$, $a_{1,i}$s are generally non-zero constants.

If in (5) all $a_{i,j} = 0$, $\{A, B\}$ is said to be in the *Brunovský form* [10, 11]. A controllable linear system can always be transformed into the Brunovský form via a state coordinate change and static state feedback control.

By grouping together vector fields of the same order Lie derivatives in $\bar{G}$, an equivalent controllability matrix is given by

$$G = \begin{bmatrix} g_1 & \cdots & g_{p_0} & ad_f g_1 & \cdots & ad_f g_{p_1} & \cdots \\ ad_f^{k-1} g_1 & \cdots & ad_f^{k-1} g_{p_{k-1}} \end{bmatrix} \tag{6}$$

where $p = p_0 \geq \cdots \geq p_{k-1} > 0$, $k = k_p$, $\sum_{i=0}^{k-1} p_i = n$. Clearly, there exists a one-to-one correspondence between the p-tuple $\{k_1, \cdots k_p\}$ and the k-tuple of $\{p_0, \cdots, p_{k-1}\}$.

The following theorem introduces a compact version of the controllability form which is referred as to the *condensed controllability form.*

Theorem 1 *For a controllable pair $\{A, B\}$ with the indices $\{p_0, \cdots, p_{k-1}\}$ a nonsingular constant matrix P can be determined such that $\{A, B\}$ can be transformed into the condensed controllability form*

$$PAP^{-1} = \begin{bmatrix} \times & \cdots & \cdots & \times \\ E_1' & 0 & & \\ & \ddots & \ddots & \\ & & E_{k-1}' & 0 \end{bmatrix}, \quad PB = \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

where $|B_0| \neq 0$, $E_i = \begin{bmatrix} e_1, \cdots, e_{p_i} \end{bmatrix} \in \mathbf{R}^{p_{i-1} \times p_i}$, e_i is the ith column of the p_{i-1}-dimensional identity matrix, and '$\times$' denotes some matrices of no interest.

Proof. According to the matrix pencil decomposition technique [12], via orthogonal transformations, $\{A, B\}$ can be transformed into the staircase form

$$A \sim \begin{bmatrix} \times & \cdots & \cdots & \times \\ A_1' & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & A_{k-1}' & \times \end{bmatrix}, \quad B \sim \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{8}$$

where $A_i \in \mathbf{R}^{p_{i-1} \times p_i}$, $i = 1, \cdots, k-1$ have full column rank, $|B_1| \neq 0$, and '$\times$' denotes matrices whose values are of no particular interest. Here $A \sim \bar{A}$ and $B \sim \bar{B}$ mean $PAP^{-1} = \bar{A}$ and $PB = \bar{B}$, where P is a nonsingular matrix.

Denote the matrices in (7) as

$$\bar{A} = \begin{bmatrix} \times & \times & \times & \times \\ \Lambda_{k-3}' & \times & \times & \times \\ 0 & A_{k-2}' & \times & \times \\ 0 & 0 & A_{k-1}' & X \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since A_{k-1}' has full row rank, there exists a nonsingular matrix, say $\bar{P}_{k-1}^{-1}$, such that $A_{k-1}'\bar{P}_{k-1}^{-1} = E_{k-1}'$. Let

$$P_{k-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \bar{P}_{k-1} & \bar{P}_{k-1}A_{k-1}'^{+}X \\ 0 & 0 & 0 & I \end{bmatrix},$$

where $A_{k-1}'^{+} = A_{k-1}(A_{k-1}'A_{k-1})^{-1}$. Then, with P_{k-1},

$$\bar{A} \sim \begin{bmatrix} \times & \times & \times & \times \\ \Lambda_{k-3}' & \times & \times & \times \\ 0 & \bar{P}_{k-1}A_{k-2}' & \times & \times \\ 0 & 0 & A_{k-1}' & 0 \end{bmatrix}, \quad \bar{B} \sim \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, by repeating the above procedure, it is easy to verify that $\bar{A}$ and $\bar{B}$ can further transformed into the form (7) where $B_0 = \bar{P}_1^{-1} B_1$. □

Clearly, for the single-input case, as $p_0 = \cdots = p_{k-1} = E_1 = \cdots = E_{k-1} = 1$ with $k = n$ and $B_0 =$ constant $\neq 0$, PAP^{-1} and PB in (7) are reduced to A_1 and B_1 defined in (5).

Consider a linear system $\dot{x} = Ax + Bu$ with $\{A, B\}$ being in the form (7). It is readily to find a feedback control, say $u = Kx + v$, such that in the closed-loop system matrix $A + BK$ all '×' matrices in (7) can be eliminated.

For the similar reason, as no particular assumptions on matrices a and b, except nonsingularity of b, in (2) have been made, all terms '×' and B_0 in (7) can be included in a and b in (2). Hence, the following proposition is immediate.

Proposition 1 *The feedback linearization problem is solvable iff there exists a coordinate transformation $\xi = T(x)$ on $\mathcal{M}$ such that (1) can be transformed into (2) where $\{A, B\}$ is in the* condensed Brunovský form

$$A = \begin{bmatrix} 0 & & & \\ E_1' & \ddots & & \\ & \ddots & \ddots & \\ & & E_{k-1}' & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{9}$$

From this proposition, the remaining part of the paper will focus on finding the state coordinate change $\xi = T(x)$ which transforms (1) into (2) where A and B take the simple block forms shown in (9).

3 Essential Conditions

This section derives several fundamental conditions required in solving the feedback linearization problem.

Proposition 2 *If the feedback linearization problem is solvable, then there exists a k-tuple of $\{p_0, \cdots, p_{k-1}\}$ such that* rank $G(x) = n$ *on $\mathcal{M}$, where G is defined in (6).*

Proof. Let $\xi = T(x)$ be the coordinate change transforming (1) into (2). Then,

$$\dot{\xi} = f_\xi + g_\xi\, u\,, \quad f_\xi = A\,\xi + B\,a\,, \quad g_\xi = B\,b\,.$$

It can easily be verified that, for $j = 1, \cdots, n-1$,

$$ad^j_{f_\xi} g_\xi = (-1)^j A^j B\, b + \sum_{i=0}^{j-1} A^i B\, \eta_i\,,$$

where $\eta_j \in \mathbf{R}^{p \times p}$ contains some smooth vector fields of no interest, and therefore that

$$\begin{aligned} & \text{rank} \begin{bmatrix} g_\xi & ad_{f_\xi} g_\xi & \cdots & ad^{n-1}_{f_\xi} g_\xi \end{bmatrix} \\ = \; & \text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} b = n \end{aligned} \tag{10}$$

as the pair $\{A, B\}$ is assumed to be controllable and b is nonsingular. Moreover, the commutative property of coordinate changes and Lie derivatives of vector fields [13] implies that $ad^j_{f_\xi} g_\xi = dT\, ad^j_f g$ which, due to the nonsingularity of dT and (10), leads to

$$\text{rank} \begin{bmatrix} g & ad_f\, g & \cdots & ad^{n-1}_f g \end{bmatrix} = n\,.$$

This guarantees that a k-tuple of $\{p_0, \cdots, p_{k-1}\}$ exists such that rank $G = n$. □

As pointed out in Section 2, $\{A, B\}$ in (2) can be assumed, without loss of generality, being in the condensed Brunovský form (9). Hence, some simple structure restrictions on the transformation $T(x)$ are implied by (9) as shown in the following proposition.

Proposition 3 *Assume* rank $G = n$ *and G has the controllability indices $p_0, \cdots, p_{k-1}$. Let*

$$T = \text{col.} \begin{bmatrix} T_1 & T_2 & \cdots & T_k \end{bmatrix}, \qquad T_i \in \mathbf{R}^{p_{i-1}}$$

transform (1) into (2) with A and B being in the form (9). Then, the following relations hold

$$L_g\, T_i = 0, \quad L_f T_i = E_{i-1}' T_{i-1}, \; i = 2, \cdots, k \tag{11}$$
$$L_g T_1 = b, \quad L_f T_1 = a\,. \tag{12}$$

Proof. A comparison of $\dot{T} = dT(f + g\,u)$ with $\dot{T} = A\,T + B(a + b\,u)$ leads immediately to these relations. □

Use the notation, for $i = 1, \cdots, k-1$,

$$G_{i+1} = \begin{bmatrix} G_i & ad^i_f g_{(i)} \end{bmatrix}, \; g_{(i)} = \begin{bmatrix} g_1 & \cdots & g_{p_i} \end{bmatrix} \tag{13}$$

and define $G_1 = g_{(0)} = g$.

The proposition below indicates the equivalence between $L_g T_i = 0$ and $\langle dT_i, G_{i-1} \rangle = 0$ under the condition $L_f T_i = E_{i-1}' T_{i-1}$.

Proposition 4 *If $L_f T_i = E_{i-1}' T_{i-1}$, $i = 2, \cdots, k$, hold, then, for $i = 2, \cdots, k$,*

$$L_g\, T_i = 0 \qquad \Leftrightarrow \qquad \langle dT_i,\, G_{i-1} \rangle = 0\,. \tag{14}$$

Proof. Note that '⇐' is obvious, only '⇒' needs to be proved.

Clearly, (14) is true for $i = 2$. Suppose that (14) holds for $i = j$. Use the notation $\bar{G}_j = \begin{bmatrix} g & ad_f G_{j-1} \end{bmatrix}$. Then, by the construction of G given in (6), span $G_j =$ span $\bar{G}_j$. Here, span D stands for the distribution spanned by vector fields contained in D [7].

Hence, $\langle dT_{j+1}, G_j \rangle = 0$ iff $\langle dT_{j+1}, \bar{G}_j \rangle = 0$. Moreover, the left hand-side of (14) implies $\langle dT_{j+1}, g \rangle = 0$ for $j = 1, \cdots, k-1$. Hence, $\langle dT_{j+1}, \bar{G}_j \rangle = 0$ iff $\langle dT_{j+1}, ad_f G_{j-1} \rangle = 0$. By the Leibniz formula (3) and from $L_f T_{j+1} = E_j' T_j$,

$$\begin{aligned} & \langle dT_{j+1},\, ad_f G_{j-1} \rangle \\ = \; & L_f \langle dT_{j+1},\, G_{j-1} \rangle - \langle L_f(dT_{j+1}),\, G_{j-1} \rangle \\ = \; & L_f \langle dT_{j+1},\, G_{j-1} \rangle - E_j' \langle dT_j,\, G_{j-1} \rangle = 0 - 0 = 0\,. \end{aligned}$$

This inductive step verifies (14).

In the above equation, $\langle dT_j,\, G_{j-1}\rangle = 0$ is clear by the inductive assumption. $L_f \langle dT_{j+1},\, G_{j-1}\rangle = 0$ needs further verification. In fact, $\langle dT_{j+1},\, G_{j-1}\rangle = 0$ is to be proved in the following.

As $G_{j-1} = \begin{bmatrix} g_{(0)} & ad_f g_{(1)} & \cdots & ad_f^{j-2} g_{(j-2)} \end{bmatrix}$, $\langle dT_{j+1},\, G_{j-1}\rangle = 0$ is equivalent to

$$\left\langle dT_{j+1},\, ad_f^l g_{(l)} \right\rangle = 0\,, \qquad l = 0, 1, \cdots, j-2\,. \quad (15)$$

Clearly, due to the first set of equations in (11), (15) is true for $l = 0$.

Now suppose (15) is true for $l = q$. Then, for $l = q+1$ $(\leq j-2)$,

$$\begin{aligned} &\left\langle dT_{j+1},\, ad_f^{q+1} g_{(q+1)} \right\rangle \\ = \;& L_f \left\langle dT_{j+1},\, ad_f^q g_{(q+1)} \right\rangle - \left\langle L_f(dT_{j+1}),\, ad_f^q g_{(q+1)} \right\rangle \\ = \;& L_f \left\langle dT_{j+1},\, ad_f^q g_{(q+1)} \right\rangle - E_j' \left\langle dT_j,\, ad_f^q g_{(q+1)} \right\rangle \end{aligned}$$

which is identical to zero owing to the facts that $\left\langle dT_{j+1},\, ad_f^q g_{(q)} \right\rangle = 0$ implies $\left\langle dT_{j+1},\, ad_f^q g_{(q+1)} \right\rangle = 0$ as $g_{(q+1)} \subseteq g_{(q)}$ and that $\langle dT_j,\, G_{j-1}\rangle = 0$ implies $\left\langle dT_j,\, ad_f^q g_{(q+1)} \right\rangle = 0$. □

4 Coordinate Transformation

This section characterizes the coordinate transformation which solves the feedback linearization problem under necessary and sufficient conditions. The proposition below is useful.

Proposition 5 *Let T_j $(j \leq k)$ be a solution to $\langle dT_j,\, G_{j-1}\rangle = 0$ with dT_j having full row rank. Then*

$$\left\langle L_f^{j-i} dT_j,\, G_i \right\rangle = 0\,, \qquad i = 2, \cdots, j \qquad (16)$$

and, for $i = 1, \cdots, j$,

$$\operatorname{rank} \left\langle L_f^{j-i} dT_j,\, ad_f^{i-1} g_{(i-1)} \right\rangle = \dim T_j\,. \qquad (17)$$

Proof. Firstly, (16) holds for $i = j$. Suppose (16) is true for $i = l$, then, for $i = l-1$,

$$\begin{aligned} &\left\langle L_f^{j-(l-1)} dT_j,\, G_{l-2} \right\rangle \\ = \;& L_f \left\langle L_f^{j-l} dT_j,\, G_{l-2} \right\rangle - \left\langle L_f^{j-l} dT_j,\, ad_f G_{l-2} \right\rangle \\ = \;& 0 - 0 = 0\,. \end{aligned}$$

This is because G_{l-2}, $ad_f G_{l-2} \subset \operatorname{span} G_{l-1}$ by construction of the controllability matrix G. Thus, (16) is proved by induction.

To prove (17) inductively, let $i = j$, then (17) is obviously true by assumption. Now, suppose that (17) is true for $i = l$. For $i = l-1$, direct computation gives

$$\begin{aligned} &\left\langle L_f^{j-(l-1)} dT_j,\, ad_f^{l-2} g_{(l-2)} \right\rangle \\ = \;& L_f \left\langle L_f^{j-l} dT_j,\, ad_f^{l-2} g_{(l-2)} \right\rangle - \left\langle L_f^{j-l} dT_j,\, ad_f^{l-1} g_{(l-2)} \right\rangle. \end{aligned}$$

Due to (16), $\left\langle L_f^{j-l} dT_j,\, ad_f^{l-2} g_{(l-2)} \right\rangle = 0$. Moreover, note that $\left\langle L_f^{j-l} dT_j,\, ad_f^{l-1} g_{(l-1)} \right\rangle$ has full row rank and that $g_{(l-1)} \subseteq g_{(l-2)}$. Hence, $\left\langle L_f^{j-l} dT_j,\, ad_f^{l-1} g_{(l-2)} \right\rangle$ and thus $\left\langle L_f^{j-(l-1)} dT_j,\, ad_f^{l-2} g_{(l-2)} \right\rangle$ have full row rank. □

Remark 1 *For the single-input case, by setting $j = k = n$, (16) and (17) are reduced to the well-known relations $\left\langle L_f^{n-i} dT_n,\, ad_f^{j-1} g \right\rangle = 0$, $i = 2, \cdots, n$, $j = 1, \cdots, i-1$ and $\left\langle L_f^{n-i} dT_n,\, ad_f^{i-1} g \right\rangle \neq 0$, $i = 1, \cdots, n$, respectively.*

Corollary 1 *If T_j $(j \leq k)$ is a solution to $\langle dT_j,\, G_{j-1}\rangle = 0$ with dT_j having full row rank, then all covector fields in the set $\{L_f^{j-i} dT_j,\ i = 1, \cdots, j\}$ are linearly independent.*

Proof. In view of (16), direct computation gives

$$\begin{bmatrix} L_f^{j-1} dT_j \\ L_f^{j-2} dT_j \\ \vdots \\ dT_j \end{bmatrix} G = \begin{bmatrix} C_1 & \times & \cdots & \times \\ & C_2 & \ddots & \vdots \\ & & \ddots & \times \\ & & & C_j \end{bmatrix}$$

with $C_i = \left\langle L_f^{j-i} dT_j\,, ad_f^{i-1} g_{(i-1)} \right\rangle$, which leads to what is to be proved by virtue of (17). □

The following is a key, though simple, lemma.

Lemma 1 *The feedback linearization problem is solvable iff the equations*

$$\langle dT_i,\, G_{i-1}\rangle = 0, \quad L_f T_i = E_{i-1}' T_{i-1}, \quad i = 2, \cdots, k \quad (18)$$

have a solution $T = \text{col.}\begin{bmatrix} T_1 & T_2 & \cdots & T_k \end{bmatrix}$ with $\left\langle dT_i,\, ad_f^{i-1} g_{(i-1)} \right\rangle$, $i = 1, \cdots, k$, being nonsingular on $\mathcal{M}$.

Proof. If the feedback linearization problem is solvable, necessity of (18) is implied by (11) and Proposition 4. Moreover, dT and, by Proposition 2, G must be nonsingular. From the first relation in (18), it is straightforward to obtain

$$dT\, G = \begin{bmatrix} D_1 & \times & \cdots & \times \\ & D_2 & \ddots & \vdots \\ & & \ddots & \times \\ & & & D_k \end{bmatrix} \qquad (19)$$

with $D_i = \left\langle dT_i\,, ad_f^{i-1} g_{(i-1)} \right\rangle$. This verifies the requirement of nonsingularity of $\left\langle dT_i,\, ad_f^{i-1} g_{(i-1)} \right\rangle$.

Now suppose that $T = \text{col.} \begin{bmatrix} T_1 & T_2 & \cdots & T_k \end{bmatrix}$ is a solution satisfying (18) and with $\left\langle dT_i,\, ad_f^{i-1} g_{(i-1)} \right\rangle$ being nonsingular on $\mathcal{M}$. Then, direct computation leads to

$$\dot{T} = dT(f + g\,u) = \begin{bmatrix} L_f T_1 \\ E_1' T_1 \\ \vdots \\ E_{k-1}' T_{k-1} \end{bmatrix} + \begin{bmatrix} L_g T_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

where $a = L_f T_1$ and $b = L_g T_1 = \langle dT_1,\, g \rangle$ which is guaranteed to be nonsingular. That is, $\dot{T} = dT(f + g\,u)$ is indeed in the form (2) where A and B have the forms of (9). Finally, as nonsingularity of dT is implied by (19), $\xi = T(x)$ is a diffeomorphism on $\mathcal{M}$. □

In view of Lemma 1, to solve the underlying problem it needs only to find existence conditions and a solution to the equations given in (18).

Theorem 2 *The feedback linearization problem is solvable iff both the following conditions hold*

(a) $\operatorname{rank} G(x) = n$ *on* $\mathcal{M}$, *where* G *is the controllability matrix with the controllability indices of the* k*-tuple* $\{p_0, \cdots, p_{k-1}\}$ *as defined in (6);*

(b) *the distributions* $\Delta_i = \operatorname{span} G_i$, $i = 1, \cdots, k-1$, *are* involutive *on* $\mathcal{M}$, *where* G_i *is defined in (13).*

In case (a) and (b) hold, the solution of the feedback linearization problem is given by $T = \text{col.} \begin{bmatrix} T_1 & \cdots & T_k \end{bmatrix}$, $T_i \in \mathbf{R}^{p_{i-1}}$ *with*

$$T_i = \begin{bmatrix} T_{i,1} \\ T_{i,2} \end{bmatrix} = \begin{bmatrix} L_f T_{i+1} \\ T_{i,2} \end{bmatrix}, \quad i = 1, \cdots, k-1 \quad (20)$$

where both T_k *and* $T_{i,2}$ *with* dT_k *and* $dT_{i,2}$ *having full row rank are determined via*

$$\langle dT_k,\, G_{k-1} \rangle = 0\,, \quad \langle dT_{i,2},\, G_{i-1} \rangle = 0 \quad (21)$$

$$dT_{i,2} \not\subseteq \operatorname{span}\{L_f dT_{i+1},\, dT_{i+1},\, \cdots,\, dT_k\}\,, \quad (22)$$

$i = 1, \cdots, k-1$, *respectively. Here, define* $G_0 = 0$.

Proof: Necessity of (a) follows from Proposition 2. In view of Lemma 1, the existence of T_i with dT_i having full row rank and satisfying $\langle dT_i,\, G_{i-1} \rangle = 0$ is required for the underlying problem to have a solution. This verifies the necessity of (b) by virtue of the Frobenius theorem [7, 13].

Now let (a) and (b) be satisfied. Again by Frobenius' theorem, the solutions of T_k and $T_{i,2}$ with with dT_k and $dT_{i,2}$ having full row rank and satisfying (21) and (22) do exist because $\dim T_k = \dim \Delta_{k-1}^{\perp}$ and $\dim T_{i,2} = \dim \Delta_{i-1}^{\perp} - \dim \operatorname{span}\{L_f dT_{i+1},\, dT_{i+1},\, \cdots,\, dT_k\}$, $i = 1, \cdots, k-1$. Here, $\Delta^{\perp}$ represents the codistribution spanned by all covector fields orthogonal to the distribution Δ [7].

From Lemma 1, it remains only to prove that the solution given in (20) satisfies (18) and that all $\left\langle dT_i,\, ad_f^{i-1} g_{(i-1)} \right\rangle$ are nonsingular. Owing to the special structure of E_{i-1}, the second set of equations in (18) are clearly satisfied by (20).

By definition of T_k in (20), the first equation in (18) is satisfied for $i = k$. Now, assume that $\langle dT_j,\, G_{j-1} \rangle = 0$ is satisfied by T_i given in (20) for $i = j$. By the Leibniz formula (3), direct computations lead to

$$\langle dT_{j-1},\, G_{j-2} \rangle = \begin{bmatrix} \langle L_f dT_j,\, G_{j-2} \rangle \\ \langle dT_{j-1,2},\, G_{j-2} \rangle \end{bmatrix} =$$

$$\begin{bmatrix} L_f \langle dT_j,\, G_{j-2} \rangle - \langle dT_j,\, L_f G_{j-2} \rangle \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This inductive step proves that (20) satisfies (18).

Finally, by construction and due to Proposition 5, $\{dT_1, \cdots, dT_k\}$ contain linearly independent covector fields. Hence, from (19), all $\left\langle dT_i,\, ad_f^{i-1} g_{(i-1)} \right\rangle$ must be nonsingular. □

If the controllability indices $\{p_0, \cdots, p_{k-1}\}$ satisfy some simple relation, for instance in the single-input case, the solution (20) can considerably be simplified. Corollary 1 and the above theorem lead to the following corollary immediately.

Corollary 2 *If* $\operatorname{rank} G(x) = n$ *on* $\mathcal{M}$ *with the controllability indices* $\{p_0, \cdots, p_{k-1}\}$ *satisfying* $p_0 = \cdots = p_{k-1}$, *then the feedback linearization problem is solvable iff the distribution* G_{k-1} *is* involutive *on* $\mathcal{M}$ *and in which case, the solution is given by* $T = \text{col.} \begin{bmatrix} T_1 & \cdots & T_k \end{bmatrix}$ *with*

$$T_i = L_f T_{i+1}\,, \qquad i = 1, \cdots, k-1 \quad (23)$$

where T_k *with* dT_k *having full row rank is determined via* $\langle dT_k,\, G_{k-1} \rangle = 0$.

Evidently, the solution (23) is equally simple as the solution for the single-input case where $p_0 = \cdots = p_{k-1} = 1$ and $k = n$.

The following example taken from [7] illustrates the design aspect of the proposed solution.

Example 1 The system

$$\dot{x} = \begin{bmatrix} x_2 + x_2^2 \\ x_3 - x_1 x_4 + x_4 x_5 \\ x_2 x_4 + x_1 x_5 - x_5^2 \\ x_5 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & \cos(x_1 - x_5) \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u$$

as analysed in [7], satisfies all conditions of Theorem 2 around the neighbourhood $\mathcal{M}$ of $x_0 = 0$. In the following it is to show how the coordinate transformation required in the feedback linearization can easily be found by using the solution formulated via (20-22).

The controllability matrix is given by

$$G = \left[\begin{array}{c|c|c} g & ad_f g & ad_f^2 g_1 \end{array}\right] =$$

$$\left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 1+2x_2 \\ 0 & 0 & -1 & \cos(x_1-x_5) & 0 \\ 1 & \cos(x_1-x_5) & -x_1+x_5 & -x_2\sin(x_1-x_5) & x_4 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2x_2 \end{array}\right]$$

with indices $p_0 = p_1 = 2$, $p_2 = 1$.

T_3: It is easy to obtain $dT_3 = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & -1 \end{array}\right]$ as a solution satisfying $\langle dT_3, G_2\rangle = dT_3 \left[\begin{array}{cc} g & ad_f g \end{array}\right] = 0$. Hence, $T_3 = x_1 - x_5$.

T_2: As $L_f dT_3 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \end{array}\right]$, $dT_{2,2} = \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \end{array}\right] \notin \text{span}\{L_f dT_3, dT_3\}$ is obviously a solution to $\langle dT_2, G_1\rangle = dT_2 g = 0$. Thus, $T_2 = \left[\begin{array}{c} L_f T_3 \\ T_{2,2} \end{array}\right] = \left[\begin{array}{c} x_2 \\ x_4 \end{array}\right]$.

T_1: Since $p_0 = p_1$,

$$T_1 = L_f T_2 = \frac{\partial T_2}{\partial x} f = \left[\begin{array}{c} -x_1x_4 + x_4x_5 + x_3 \\ x_5 \end{array}\right].$$

Finally, it can be verified that

$$T = \left[\begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array}\right] = \left[\begin{array}{c} -x_1x_4 + x_4x_5 + x_3 \\ x_5 \\ \hline x_2 \\ x_4 \\ \hline x_1 - x_5 \end{array}\right]$$

is the desired transformation.

5 Conclusion

A condensed controllability/Brunovský form has been introduced to deal with the feedback linearization problem for multi-input nonlinear control systems. By using this condensed form in the problem formulation, the underlying problem has been treated in a more compact manner. As the derivation is in many respects similar to the known treatment in the single-input case, more direct insights have therefore been provided in the solution of the multi-input case. In fact, the derivation has been unified for both single-input and multi-input cases. Owing to the compact formulation, the proposed solution shows a clearer structure than previous solutions in [2, 5, 7].

The use of the condensed controllability/Brunovský form makes it possible to derive the multi-input solution in a manner analogous to the single-input case. It is believed that the use of the the controllability/condensed Brunovský form could benefit derivations of many related problems in multi-input multi-output control systems. Even in the linear system case, the condensed controllability/Brunovský form should be of value in designing feedback control. For instance, the dead-beat controller design for the discrete-time linear systems may be one of such examples.

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