# State Observers for Nonlinear Systems with Smooth/Bounded input

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## Abstract

It is known that for nonlinear systems the drift-observability property (i.e. observability for zero input) is not sufficient to guarantee the existence of an asymptotic observer for any input. Many authors studied conditions on systems structure that ensure uniform observability (i.e. observability for any input). Conditions are available that define restrict classes of uniformly observable systems. This work considers the problem of state observation with exponential error rate for smooth nonlinear systems that do not meet conditions of uniform observability: conditions are given on the input, instead of on the system structure. It is shown that drift-observability, together with a smoothness/boundedness condition on the input, is sufficient to ensure the existence of an exponential observer. Three types of observers are presented, that can be constructed under drift-observability assumption.

## 1 Introduction

Many authors pointed out the peculiarities of the state observation problem for nonlinear systems [4–11]. A main property is that state reconstructability in general depends on the input function, in that drift-observability (i.e. observability for zero input) is not a sufficient condition for existence of an asymptotic observer for any input, as it is for linear systems. This fact induced some authors [1,9,10] to find conditions that ensure state reconstructability for any input. Classes of nonlinear systems are then defined for which observers can be constructed that work independently of the input applied (uniformly observable systems). However, such classes are characterized by limitative conditions, that can be met with difficulty in applications.

Following the researches presented in [1–3], this paper studies conditions on the input, and not on the system structure, that guarantee state observation with exponential error decay. It is shown that, given systems that are not uniformly observable, an exponential observer can still be constructed if the input satisfies some boundedness and/or smoothness condition. The properties required for the system are basically two: 1) smoothness of the vector fields and functions that define the system, 2) drift-observability. Due to pages limitation, the results presented in this paper concern only single-input single-output nonlinear systems of the type

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1}$$
$$y(t) = h(x(t)), \tag{2}$$

where $x(t) \in X \subseteq I\!R^n$, $u(t) \in U \subseteq I\!R$ and $y(t) \in I\!R$, $g(x)$ and $f(x)$ are $C^\infty(X)$ vector fields and $h(x)$ is a $C^\infty(X)$ function.

Throughout the paper the symbol $O_{a\times b}$ denotes a matrix of zeroes of dimension $a \times b$, the symbol $I_k$ denotes the $k \times k$ identity matrix, and $\mathrm{cn}(P)$ denotes the condition number of a matrix $P$.

## 2 Preliminaries

In this paper it is assumed that the reader is familiar with the concept of Lie derivative of a function along a vector field. Consider the vector function (square mapping)

$$\Phi(x) \triangleq [h(x) \quad L_f h(x) \quad \cdots \quad L_f^{n-1} h(x)]^{\mathrm{T}}. \tag{3}$$

Denoting with $Y_n$ the vector of the first $n$ output derivatives (from 0 to $n-1$)

$$Y_n = \left[ y \; \dot{y} \; \ldots \; y^{(n-1)} \right]^{\mathrm{T}}, \tag{4}$$

it is easy to verify that, for $u(t) \equiv 0$,

$$Y_n = \Phi(x). \tag{5}$$

Thus, from a theoretical point of view, if the vector $Y_n$ where known at a given time $t$, the invertibility of the mapping $\Phi(\cdot)$ would allow exact state reconstruction.

This property justifies the following definition.

**Definition 1** *If the map $\Phi(x)$ is a diffeomorfism from an open set $\Omega \subseteq I\!R^n$ in $\Phi(\Omega)$, the system (1) (2) is said $\Omega$-drift-observable. If $\Omega \equiv I\!R^n$ than the system (1) (2) is said globally drift-observable.*

An important consequence of this definition is the nonsingularity in $\Omega$ of the Jacobian of the map $\Phi(\cdot)$

$$Q(x) \triangleq \frac{d\Phi(x)}{dx}. \quad (6)$$

In the state observation problem it is important the following concept, that is a weaker version of the well-known concept of relative degree (see e.g. [12]).

**Definition 2** *The system (1) (2) is said to have observation relative degree $r$ in a set $\Omega \in I\!R^n$ if*

$$\begin{array}{ll} \forall x \in \Omega & L_g L_f^s h(x) = 0, \quad s = 0, 1, \ldots, r-2, \\ \exists x \in \Omega \ : & L_g L_f^{r-1} h(x) \neq 0. \end{array} \quad (7)$$

Consider now the expression of the output derivatives in the general case in which $u \not\equiv 0$. From the definition of observation relative degree it follows that the output derivatives from 0 to $r-1$ are functions of the state $x$ only, while the $r$-th derivative is also function of the input

$$\begin{array}{rcll} y^{(k)} & = & L_f^k h(x), & k = 0, 1, \ldots, r-1, \\ y^{(r)} & = & L_f^r h(x) + L_g L_f^{r-1} h(x) u. & \end{array} \quad (8)$$

It is readily proved that higher order derivatives are functions of the state $x$, of the input $u$ and of its time derivatives until a suitable order. More precisely, if $U_s$ denotes the vector composed of the first $s$ time-derivatives of the input (from 0 to $s-1$)

$$U_s \triangleq \begin{bmatrix} u \ \dot{u} \ \ldots \ u^{(s-1)} \end{bmatrix}^{\mathrm{T}}, \quad (9)$$

it can be readily proved that the $k$-th output derivative can be written, for $k \geq r$, as

$$y^{(k)} = L_f^k h(x) + \psi_k(x, U_{k-r+1}) \quad (10)$$

where the function $\psi_k(x, U_{k-r+1})$ is recursively defined as

$$\begin{array}{l} \psi_k \triangleq 0, \quad \text{for } k = 0, 1, \ldots, r-1, \\ \psi_r(x, U_1) \triangleq L_g L_f^{r-1} h(x) u, \\ \psi_k(x, U_{k-r+1}) \triangleq L_g L_f^{k-1} h(x) u + L_f \psi_{k-1}(x, U_{k-r}) + \\ \quad + L_g \psi_{k-1}(x, U_{k-r}) u + \begin{bmatrix} 0 & \frac{\partial \psi_{k-1}}{\partial U_{k-r}} \end{bmatrix} U_{k-r+1}, \quad k > r. \end{array} \quad (11)$$

Using the scalar functions $\psi_k(x, U_{k-r+1})$ for $k = 0, 1, \ldots, n-1$ a $n$-components vector function $\Psi(x, U_{n-r})$ can be defined such that

$$Y_n = \bar{\Phi}(x, U_{n-r}) \triangleq \Phi(x) + \Psi(x, U_{n-r}). \quad (12)$$

(The $j$-th component of $\Psi(x, U_{n-r})$, with $r+1 \leq j \leq n$, is $\psi_{j-1}(x, U_{j-r})$.) If $r = n$ the function $\Psi$ vanishes, and the (12) can be simply written as (5). It is also easy to check from definitions (11) that the function $\Psi(x, U_{n-r})$ satisfies the property

$$\Psi(x, 0) = 0, \qquad \forall x \in I\!R^n. \quad (13)$$

In general, drift-observability of system (1) (2)does not imply invertibility of (12) for $x$. In general, invertibility of (12) for $x$ strongly depends on the input, through the vector of derivatives $U_{n-r}$, that can be considered as parameters in the mapping $\bar{\Phi}(x, U_{n-r})$. Thus, the following definition can be given.

**Definition 3** *If for any $U_{n-r}$ in an open set $\bar{\mathcal{U}} \subseteq I\!R^{n-r}$ the map $Y_n = \bar{\Phi}(x, U_{n-r})$ in (12) is a diffeomorfism from an open set $\Omega \subseteq I\!R^n$ in $\bar{\Phi}(\Omega, \bar{\mathcal{U}})$, the system (1) (2) is said $\bar{\mathcal{U}}$-uniformly $\Omega$-observable. If $\Omega \equiv I\!R^n$ and $\mathcal{U}^{n-r} \equiv I\!R^{n-r}$ than the system (1) (2) is said globally uniformly observable.*

An important consequence is that in $\Omega \times \bar{\mathcal{U}}$ is nonsingular the Jacobian

$$\bar{Q}(x, U_{n-r}) \triangleq \frac{\partial \bar{\Phi}(x, U_{n-r})}{\partial x}. \quad (14)$$

If a system is $\bar{\mathcal{U}}$-uniformly $\Omega$-observable, the knowledge of vectors $Y_n$ and $U_{n-r} \in \bar{\mathcal{U}}$ would allow state reconstruction. Note that in the case $r = n$ the maps $\Phi$ and $\bar{\Phi}$ coincide, so that drift-observability guarantees state reconstructability for any input. Moreover the following theorem holds.

**Theorem 4** *If system (1) (2) is $\Omega$-drift observable, then there exists a sufficiently small spherical neighborhood $\bar{\mathcal{U}}$ of the origin such that the system is $\bar{\mathcal{U}}$-uniformly $\Omega$-observable.*

**Proof.** From (12) the map $\bar{\Phi}(x, U_{n-r})$ satisfies the property

$$\bar{\Phi}(x, 0) = \Phi(x). \quad (15)$$

As a consequence $\Omega$-invertibility of $\Phi(x)$ ensures that $\bar{\Phi}(x, U_{n-r})$ can be solved for $x \in \Omega$ if $U_{n-r} = 0$. Since, by smoothness assumption for system (1) (2), the map $\bar{\Phi}(x, U_{n-r})$ is continuous w.r.t. $U_{n-r}$, then it can be solved for $x \in \Omega$ if $U_{n-r}$ is sufficiently close to the origin. This means that there exists a spherical neighborhood $\bar{\mathcal{U}}$ of the origin with sufficiently small radius that ensures $\bar{\mathcal{U}}$-uniform $\Omega$-observability. •

In the set $\Omega$, as long as $U_{n-r}$ belongs to $\bar{\mathcal{U}}$, if the system (1) (2) is $\bar{\mathcal{U}}$-uniformly $\Omega$-observable, the map $\eta = \bar{\Phi}(x, U_{n-r})$ can be considered as a time-varying change of coordinates ($U_{n-r}$ is a function of time). The inverse map is denoted as $x = \bar{\Phi}^{-1}(\eta, U_{n-r})$. Since $\eta_j = y^{(j-1)}$ for $j = 1, \ldots, n$, and then $\dot{\eta}_j = \eta_{j+1}$ for $j = 1, \ldots, n-1$, in $\eta$-coordinates the system is written

$$\begin{array}{rcl} \dot{\eta} & = & A_n \eta + B_n m(\eta, U_{n-r+1}), \\ y & = & C_n \eta, \end{array} \quad (16)$$

where the term $m(\eta, U_{n-r+1})$ is

$$\left(L_f^n h(x) + \psi_n(x, U_{n-r+1})\right)\Big|_{x=\bar{\Phi}^{-1}(\eta, U_{n-r})}, \tag{17}$$

and matrices $A_n \in \mathbb{R}^{n\times n}$, $B_n \in \mathbb{R}^n$ and $C_n \in \mathbb{R}^n$ are Brunowsky matrices

$$\begin{aligned} A_n &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ C_n &= [1 \quad 0 \quad \cdots \quad 0]. \end{aligned} \tag{18}$$

If the coordinate change $z = \Phi(x)$ is considered instead, the system (1) (2) can be written in the new coordinates as

$$\begin{aligned} \dot{z} &= A_n z + B_n L_f^n h(\Phi^{-1}(z)) + Q(x)g(x)\big|_{x=\Phi^{-1}(z)} u, \\ y &= C_n z. \end{aligned} \tag{19}$$

The product of the Jacobian $Q(x)$ by the matrix $g(x)$ is

$$Q(x)g(x) = \begin{bmatrix} L_g h(x) \\ \vdots \\ L_g L_f^{n-1} h_j(x) \end{bmatrix}. \tag{20}$$

From the definition of observation relative degree in $\Omega$, the first $r-1$ rows of vector (20) are identically zero in $\Omega$, so that last equation can be rewritten as

$$Q(x)g(x) = F\,H(x), \tag{21}$$

$$F \triangleq \begin{bmatrix} O_{(r-1)\times(n-r+1)} \\ I_{n-r+1} \end{bmatrix}, \quad H(x) \triangleq \begin{bmatrix} L_g L_f^{r-1} h(x) \\ \vdots \\ L_g L_f^{n-1} h(x) \end{bmatrix}. \tag{22}$$

It is also useful to define the function

$$L(x) \triangleq L_f^n h(x), \tag{23}$$

so that system (19) can be rewritten

$$\begin{aligned} \dot{z} &= A_n z + B_n L(\Phi^{-1}(z)) + FH(\Phi^{-1}(z))u, \\ y &= C_n z. \end{aligned} \tag{24}$$

The pair $A_n, C_n$ defined in (18) is observable, and it is an easy matter to assign eigenvalues to the matrix $A_n - KC_n$, that has the companion structure

$$A_n - KC_n = \begin{bmatrix} -k_1 & 1 & \cdots & 0 \\ -k_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -k_{n-1} & 0 & \cdots & 1 \\ -k_n & 0 & \cdots & 0 \end{bmatrix}. \tag{25}$$

Let $K(\lambda)$ denote the vector that assigns eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$. Matrix $A_n - K(\lambda)C_n$ is diagonalized by the Vandermonde matrix

$$V_n \triangleq V_n(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_n^{n-1} & \cdots & \lambda_n & 1 \end{bmatrix}, \tag{26}$$

so that

$$V_n(\lambda)(A_n - K(\lambda)C_n)V_n(\lambda)^{-1} = \mathrm{diag}\{\lambda\} = \Lambda. \tag{27}$$

It is well known that a Vandermonde matrix $V_n(\lambda)$ is singular if and only if two or more eigenvalues in the set $\lambda$ coincide. It is also well known that the smaller is the difference between eigenvalues in $\lambda$ the larger is the norm of $V_n^{-1}(\lambda)$. For reasons that will be made clear in the following section it is important to choose eigenvalues for matrix $A_n - K(\lambda)C_n$ while keeping bounded the norm of the inverse of the Vandermonde matrix $V_n(\lambda)$. In [1] it is shown that if the $n$ eigenvalues are chosen as $\lambda_j = \lambda_j(\sigma) = -\sigma^j$, for $j = 1, \ldots, n$, with $\sigma > 0$, then

$$\lim_{\sigma\to\infty} \|V_n^{-1}(\lambda(\sigma))\| = 1. \tag{28}$$

**Remark 5** *In [9,10] it is shown that a system (1) (2) is observable for any $u(t)$ if and only if in $z$-coordinates the vector function $Q(\Phi^{-1}(z))g(\Phi^{-1}(z))$ has a triangular structure. This condition restricts the class of nonlinear systems under investigation. For this reason this property is not required in the following development. Pathological inputs are excluded by suitable conditions.*

# 3 The Observer for Systems with Bounded Input

In this section it is shown that a dynamic system of the type

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + Q^{-1}(\hat{x})K\left(y - h(\hat{x})\right), \tag{29}$$

with the constant gain matrix $K$ properly chosen, is an exponential observer for the system (1) (2), provided that the input is suitably *small* and some technical conditions are satisfied. The results reported in this section are a modified version of those presented in [2] for the SISO case, and in [3] for the MIMO case. Both local and global results are available. For shortness only global results are here reported.

A first result is given by the following theorem.

**Theorem 6** *For system (1) (2) assume that the following hypotheses hold:*

*1) The system is drift-observable in $\mathbb{R}^n$, and the map $z = \Phi(x)$ is uniformly Lipschitz together with its inverse $x = \Phi^{-1}(z)$ in $\mathbb{R}^n$, with constants $\gamma_\Phi$ and $\gamma_{\Phi^{-1}}$ respectively;*
*2) the functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$, defined in (22–23), are uniformly Lipschitz in $\mathbb{R}^n$, with Lipschitz constants $\gamma_{\bar{H}}$ and $\gamma_{\bar{L}}$ respectively;*
*3) a constant $u_M > 0$ exists such that $|u(t)| \leq u_M$ $\forall t \geq 0$;*
*4) for a given $\alpha > 0$ a vector $K \in \mathbb{R}^n$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ exist that satisfy the following $H_\infty$ Riccati-like inequality*

$$\begin{aligned}(A_n - KC_n)P + P(A_n - KC_n)^{\mathrm{T}} + B_nB_n^{\mathrm{T}}+ \\ +u_M^2 FF^{\mathrm{T}} + 2\alpha P + \gamma^2P^2 \leq 0,\end{aligned} \tag{30}$$

*where $\gamma^2 = \gamma_{\bar{L}}^2 + \gamma_{\bar{H}}^2$.*
*Then the dynamic system (29) is such that*

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t}\|x(0) - \hat{x}(0)\| \tag{31}$$

*with $\mu = \sqrt{\mathrm{cn}(P)}\gamma_\Phi\gamma_{\Phi^{-1}}$.*

**Proof.** For system (1) (2) and for observer (29) consider the following coordinate transformations and the following definitions of observation errors

$$\begin{aligned} z = \Phi(x), \quad e_x \triangleq x - \hat{x}, \\ \hat{z} = \Phi(\hat{x}), \quad e_z \triangleq z - \hat{z}. \end{aligned} \tag{32}$$

From assumption (1) they are such that

$$\|e_z\| \leq \gamma_\Phi\|e_x\|, \qquad \|e_x\| \leq \gamma_{\Phi^{-1}}\|e_z\|. \tag{33}$$

System (1) (2) can be written in $z$-coordinates as (19), while the observer is written

$$\dot{\hat{z}} = A_n\hat{z} + B_nL(\Phi^{-1}(\hat{z})) + FH(\Phi^{-1}(\hat{z}))u + K(y - C_n\hat{z}). \tag{34}$$

The dynamics of the observation error in $z$-coordinates is governed by the linear perturbed equation

$$\dot{e}_z = (A_n - KC_n)e_z + B_nv_1(z,\hat{z}) + Fv_2(z,\hat{z})u, \tag{35}$$

where

$$\begin{aligned} v_1(z,\hat{z}) &\triangleq L(\Phi^{-1}(z)) - L(\Phi^{-1}(\hat{z})), \\ v_2(z,\hat{z}) &\triangleq H(\Phi^{-1}(z)) - H(\Phi^{-1}(\hat{z})). \end{aligned} \tag{36}$$

From assumption (2) the perturbations satisfy the inequalities

$$\|v_1\| \leq \gamma_{\bar{L}}\|e_z\|, \qquad \|v_2\| \leq \gamma_{\bar{H}}\|e_z\|, \tag{37}$$

In order to prove that $e_z(t)$ exponentially goes to zero, consider the positive definite function of $e_z$

$$\nu(e_z) = e_z^{\mathrm{T}}P^{-1}e_z, \tag{38}$$

where the positive definite symmetric matrix $P$ satisfies inequality (30). The derivative of $\nu$ along the error trajectory is

$$\begin{aligned}\dot{\nu} = e_z^{\mathrm{T}}(P^{-1}(A - KC) + (A - KC)^{\mathrm{T}}P^{-1})e_z+ \\ +2v_1B^{\mathrm{T}}P^{-1}e_z + 2uv_2^{\mathrm{T}}F^{\mathrm{T}}P^{-1}e_z.\end{aligned} \tag{39}$$

The following inequalities can be easily checked

$$\begin{aligned} 2v_1B_n^{\mathrm{T}}P^{-1}e_z &\leq e_z^{\mathrm{T}}P^{-1}B_nB_n^{\mathrm{T}}P^{-1}e_z + v_1^2, \\ 2uv_2^{\mathrm{T}}F^{\mathrm{T}}P^{-1}e_z &\leq u_M^2e_z^{\mathrm{T}}P^{-1}FF^{\mathrm{T}}P^{-1}e_z + v_2^{\mathrm{T}}v_2, \end{aligned} \tag{40}$$

and substituted in (39). Using (33) and inequality (30) in assumption (4), after simple transformations one has

$$\dot{\nu} \leq -2\alpha\nu, \quad \Rightarrow \quad \nu(t) \leq e^{-2\alpha t}\nu(0), \tag{41}$$

(last implication is due to Gronwall's inequality). From this, recalling the definition (38) of $\nu$, we have

$$\|e_z(t)\| \leq \sqrt{\mathrm{cn}(P)}e^{-\alpha t}\|e_z(0)\|. \tag{42}$$

Given the properties (33), inequality (42) becomes

$$\|e_x(t)\| \leq \mu e^{-\alpha t}\|e_x(0)\|, \tag{43}$$

with $\mu = \sqrt{\mathrm{cn}(P)}\gamma_\Phi\gamma_{\Phi^{-1}}$. This proves the thesis. •

**Corollary 7** *If all assumptions made in theorem 6 hold, with $\alpha = 0$ in assumption (4), then the dynamic system (29) is such that*

$$\lim_{t\to\infty}\|x(t) - \hat{x}(t)\| = 0. \tag{44}$$

**Remark 8** *An automatic choice of $K$ can be adopted by taking $K = \beta^2PC^{\mathrm{T}}$, for a given $\beta$. With this choice inequality (30) becomes a true $H_\infty$ Riccati inequality*

$$\begin{aligned} A_nP + PA_n^{\mathrm{T}} + B_nB_n^{\mathrm{T}} + u_M^2FF^{\mathrm{T}}+ \\ +2\alpha P + \gamma^2P^2 - 2\beta^2PC^{\mathrm{T}}CP \leq 0, \end{aligned} \tag{45}$$

*in which the matrix $P$ is the only unknown.*

The assumption regarding the uniform Lipschitz property in $\mathbb{R}^n$, rather strong but essential to proof the global convergence of the observation error to zero, can be relaxed to prove local convergence. This topic is not treated here.

Looking at the assumptions of theorems 6 it can be recognized that a central point in the construction of an observer of the form (29) is the existence of a pair $K, P$ that solves inequality (30).

An interesting point is that the $H_\infty$ Riccati-like inequality admits solution $(K, P)$ for any $\alpha > 0$ and $\gamma > 0$ if the term $FF^{\mathrm{T}}$ is not present in the expression.

**Lemma 9** *For any triple $\alpha, \beta, \gamma$ of positive real the $H_\infty$ Riccati-like inequality*

$$\begin{aligned}(A_n - KC_n)P + P(A_n - KC_n)^{\mathrm{T}} + \\ +\beta^2 B_n B_n^{\mathrm{T}} + 2\alpha P + \gamma^2 P^2 \leq 0,\end{aligned} \quad (46)$$

*admits solution $(K, P)$ with $P$ symmetric positive definite.*

**Proof.** Choose matrix $K$ so to assign a set of real eigenvalues $\lambda$, and set $P = (V_n(\lambda)^{\mathrm{T}} V_n(\lambda))^{-1}$. Left-multiplying (46) by $V_n(\lambda)$ and right-multiplying it by $V_n^{\mathrm{T}}(\lambda)$ the $H_\infty$ Riccati-like inequality becomes

$$2\Lambda + \beta^2 V_n B_n B_n^{\mathrm{T}} V_n^{\mathrm{T}} + 2\alpha I_n + \gamma^2 V_n^{-1} V_n^{-\mathrm{T}} \leq 0. \quad (47)$$

To satisfy the matrix inequality (47) it is sufficient to verify the scalar inequality

$$2\max\{\lambda\} \leq -\beta^2 \|V_n B_n\|^2 - 2\alpha - \gamma^2 \|V_n^{-1}\|^2. \quad (48)$$

The product $V_n B_n$ and the norm $\|V_n B_n\|$ are easy to compute

$$V_n B_n = [1 \ \cdots \ 1]^{\mathrm{T}} \in I\!R^n, \quad \Rightarrow \quad \|V_n B_n B_n^{\mathrm{T}} V_n^{\mathrm{T}}\| = n. \quad (49)$$

The choice of eigenvalues that satisfies (28) can be adopted so to keep the norm of matrix $V_n^{-1}$ next to 1 as desired. Assuming $\sigma > 1$ then $\max\{\lambda\} = -\sigma$, and inequality (48) can be rewritten

$$-\sigma \leq -\alpha - \frac{1}{2} n\beta^2 - \frac{1}{2}\gamma^2 \|V_n^{-1}(\sigma)\|, \quad (50)$$

where $V_n(\sigma)$ has been indicated as function of the scalar parameter $\sigma$ that defines all the eigenvalues. Thanks to (28), inequality (50) can be satisfied for $\sigma$ sufficiently large. This proves the lemma. •

Theorem 6 and the properties of the $H_\infty$ Riccati-like inequality (30) originate two important results: 1) existence of exponential observers for systems driven by sufficiently small input; 2) existence of an observer with assigned exponential rate for systems that have observation relative degree equal to $n$ and a bounded input.

The theorems that state these results are reported below.

**Theorem 10** *For system (1) (2) assume that the following hypotheses hold:*
*1) the system is drift-observable and the map $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$ are uniformly Lipschitz in $I\!R^n$ with constants $\gamma_\Phi$ and $\gamma_{\Phi^{-1}}$, respectively;*
*2) the functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$ are uniformly Lipschitz in $I\!R^n$, with Lipschitz constants $\gamma_{\overline{H}}$ and $\gamma_{\overline{L}}$ respectively;*
*Then for any $\alpha > 0$ there exists a vector $K \in I\!R^n$ such that the dynamic system (29), for a suitable $\mu > 0$, gives*

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\| \quad (51)$$

*provided that $|u(t)| \leq u_M \ \forall t$, with $u_M$ sufficiently small.*

**Proof.** From theorem 6, it is sufficient to show that for any positive $\alpha$ a sufficiently small $u_M$ exists such that the $H_\infty$ Riccati-like inequality (30) can be satisfied. This can be done by considering, for a given $\beta$, the inequality

$$\begin{aligned}(A_n - KC_n)P + P(A_n - KC_n)^{\mathrm{T}} + \\ +B_n B_n^{\mathrm{T}} + 2\alpha P + (\gamma^2 + \beta^2)P^2 \leq 0,\end{aligned} \quad (52)$$

which admits solution $K, P$, as proved in lemma 46.

Since $FF^{\mathrm{T}} \leq I_n \leq \frac{1}{\lambda_{\min}^2(P)} P^2$, as it can be easily verified, one has

$$\beta^2 \lambda_{\min}^2(P) FF^{\mathrm{T}} \leq \beta^2 P^2, \quad (53)$$

and thus the solution for inequality (30) exists with $u_M \leq \beta^2 \lambda_{\min}^2(P)$. This completes the proof. •

**Remark 11** *The sufficient conditions for the existence of an exponential observer given in theorem 10 do not include the condition of observability for any input. However, a bound on the input has to be satisfied. Evidently this smallness condition automatically excludes the presence of inputs that make indistinguishable some system states.*

**Theorem 12** *For the system (1) (2) assume that the following hypotheses hold:*
*1) the system is drift-observable and the map $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$ are uniformly Lipschitz in $I\!R^n$ with constants $\gamma_\Phi$ and $\gamma_{\Phi^{-1}}$, respectively;*
*2) the observability relative degree in $I\!R^n$ is $r = n$;*
*3) the matrix functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$ are uniformly Lipschitz in $I\!R^n$, with Lipschitz constants $\gamma_{\overline{H}}$ and $\gamma_{\overline{L}}$ respectively;*
*4) a constant $u_M > 0$ exists such that $\|u(t)\| \leq u_M$ $\forall t \geq 0$;*

*Then for any $\alpha > 0$ a gain vector $K \in I\!R^n$ exists such that the dynamic system (29) is such that*

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\| \quad (54)$$

*for a suitable $\mu > 0$.*

**Proof.** From theorem 6 it is sufficient to prove that with the given assumptions the $H_\infty$ Riccati-like inequality (30) can always be satisfied. This happens because when $r = n$, then $F = B_n$ (see definitions (22–23), and thus inequality (30) can be rewritten

$$\begin{aligned}(A_n - KC_n)P + P(A_n - KC_n)^{\mathrm{T}} + \\ +(1 + u_M^2)B_n B_n^{\mathrm{T}} + 2\alpha P + \gamma^2 P^2 \leq 0.\end{aligned} \quad (55)$$

Lemma 9 ensures existence of solution $(K, P)$. •

# 4 The Observer for Systems with Bounded/Smooth Input

Theorem 10 states that for systems with any relative degree an exponential observer can be designed if the input is sufficiently small. Moreover it could be shown that the smaller is the input, the faster can be chosen the exponential rate.

On the other hand, if the system has observation relative degree $r = n$, and a known bound on the input (not necessarily *small*) then an observer with arbitrary exponential rate can be designed.

In this section it is shown that in the case of relative degree $r < n$, an exponential observer with arbitrary exponential rate can be obtained if the derivatives of the input up to order $n - r$ are known and bounded. Obviously, in many application the derivatives of the input are not known, and this observer can not be constructed. For this reason this observed is called *theoretical*.

An observer that uses estimates of input derivatives is presented after. In this case the observation error is not driven to zero, but its norm can be reduced, with exponential rate, below a prescribed bound. This kind of observer is called *practical*.

## 4.1 Theoretical Observer

In this section it is assumed that the input function is differentiable $n - r$ times, with derivatives uniformly bounded in $[0, +\infty)$. The observer considered has the form

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + \bar{Q}^{-1}(\hat{x}, U_{n-r})K\big(y - h(\hat{x})\big), \quad (56)$$

and can be constructed as long as $U_{n-r}$ allows invertibility of the Jacobian of the map $\bar{\Phi}(x, U_{n-r})$. Only local uniform observability is strictly required (remember that, from theorem (4) drift-observable systems are locally uniformly observable).

For brevity, only global properties are considered in this section, although local results can be derived as well.

**Theorem 13** *For system (1) (2) assume that the following hypotheses hold:*
*1) the vector of input derivatives is bounded by a positive constant $\bar{u}_M$, i.e. $\|U_{n-r+1}(t)\| \leq \bar{u}_M \;\; \forall t \geq 0$;*
*2) the system is globally uniformly observable and the map $\eta = \bar{\Phi}(x, U_{n-r})$ and its inverse $x = \bar{\Phi}^{-1}(\eta, U_{n-r})$ are uniformly Lipschitz w.r.t. $x$ and $\eta$, respectively, in all $\mathbb{R}^n$, under condition $\|U_{n-r}\| \leq \bar{u}_M$, implied by hypothesis 1. Let $\gamma_{\bar{\Phi}}$ and $\gamma_{\bar{\Phi}^{-1}}$, respectively, be the Lipschitz constants for $\|U_{n-r}\| \leq \bar{u}_M$;*
*3) the function $m(\eta, U_{n-r+1})$ defined in (17) is uniformly Lipschitz w.r.t. $\eta$ in $\mathbb{R}^n$, under assumption 1. Let $\gamma_m$ be its Lipschitz constant for $\|U_{n-r+1}\| \leq \bar{u}_M$.*
*Then for any $\alpha > 0$ a gain vector $K \in \mathbb{R}^n$ exists such that the dynamic system (56) is such that*

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t}\|x(0) - \hat{x}(0)\| \quad (57)$$

*for a suitable $\mu > 0$.*

**Proof.** From assumption 2 the system (1) (2) is globally uniformly observable and the map $\eta = \bar{\Phi}(x, U_{n-r})$ can be considered a time-varying change of coordinates. In $\eta$-coordinates system (1) (2) and observer (56) can be rewritten

$$\begin{aligned} \dot{\eta} &= A_n\eta + B_n m(\eta, U_{n-r+1}), \\ y &= C_n\eta, \\ \dot{\hat{\eta}} &= A_n\hat{\eta} + B_n m(\hat{\eta}, U_{n-r+1}) + K(y - C_n\hat{\eta}). \end{aligned} \quad (58)$$

Defining the function

$$v(\eta, \hat{\eta}, U_{n-r+1}) \triangleq m(\eta, U_{n-r+1}) - m(\hat{\eta}, U_{n-r+1}), \quad (59)$$

the observation error $e_\eta = \eta - \hat{\eta}$ in $\eta$ coordinates is described by a linear perturbed system

$$\dot{e}_\eta = (A_n - KC_n)e_\eta + B_n v, \quad (60)$$

in which the perturbation, by assumptions 1 and 3, satisfies the inequality

$$|v| \leq \gamma_m\|e_\eta\|. \quad (61)$$

In order to prove that a properly chosen gain matrix $K$ drives $e_\eta(t)$ to zero with an assigned exponential rate $\alpha$, consider a pair $(K, P)$ that solves the $H_\infty$ Riccati-like inequality

$$\begin{aligned} (A_n - KC_n)P + P(A_n - KC_n)^{\mathrm{T}} + \\ + B_nB_n^{\mathrm{T}} + 2\alpha P + \gamma_m^2 P^2 \leq 0. \end{aligned} \quad (62)$$

Existence of solution for (62) for any $\alpha$ and $\gamma_m$ is guaranteed by lemma 9. Consider now the following positive definite function of the observation error

$$\nu(e_\eta) = e_\eta^{\mathrm{T}} P^{-1} e_\eta. \quad (63)$$

Taking the derivative of $\nu$ along the error trajectory, after few passages that use also (62), one has $\dot{\nu} \leq -2\alpha\nu$, and therefore

$$\nu(t) \leq e^{-2\alpha t}\nu(0). \quad (64)$$

Recalling definition (63) of $\nu$

$$\|e_\eta(t)\| \leq \sqrt{\mathrm{cn}(P)}e^{-\alpha t}\|e_\eta(0)\|, \quad (65)$$

and using Lipschitz conditions in assumption (2) in the original coordinates inequality (57) is obtained with $\mu = \sqrt{\mathrm{cn}(P)}\gamma_{\Phi}\gamma_{\Phi^{-1}}$, and the thesis is proved. •

As mentioned before, the observer (56) can be implemented only if input derivatives up to order $n-r-1$ are known. It follows, obviously, that the observer can be always implemented if $r \leq n-1$, since in this case no input derivative is needed (if $r = n$ the observer (56) coincides with observer (29)). Moreover, the observer can be implemented in all cases in which the generation model of input $u$ is known (e.g. the input $u$ is generated by a smooth controller or simply by a preprocessing filter).

## 4.2 Practical Observer

As in the previous section also here existence and boundedness of the first $n-r$ derivatives is assumed for the input function in $[0, +\infty)$. With this assumption the input can be thought as generated by the system

$$\begin{aligned} \dot{U}_{n-r} &= A_{n-r} U_{n-r} + B_{n-r} u^{(n-r)}, \\ u &= C_{n-r} U_{n-r}, \end{aligned} \tag{66}$$

where $A_{n-r}$, $B_{n-r}$, $C_{n-r}$ is a Brunowsky triple of order $n-r$. The asymptotic reconstruction of the input derivatives can be made using an observer for system (66). Let $x_a \in I\!R^{n-r}$ be an auxiliary state and $x_e = [x^{\mathrm{T}} \; x_a^{\mathrm{T}}]^{\mathrm{T}}$ be an extended state $x_e \in I\!R^{2n-r}$. Considered now the observation problem applied to the augmented system

$$\begin{aligned} \dot{x}_e &= \bar{f}(x_e) + \bar{g}(x_e) w, \\ y &= \bar{h}(x_e), \\ u &= [0 \; C_{n-r}] x_e. \end{aligned} \tag{67}$$

where $\bar{h}(x_e) \triangleq h(x)$ and

$$\bar{f}(x_e) \triangleq \begin{bmatrix} f(x) + g(x) C_{n-r} x_a \\ A_{n-r} x_a \end{bmatrix}, \quad \bar{g}(x_e) \triangleq \begin{bmatrix} 0 \\ B_{n-r} \end{bmatrix}. \tag{68}$$

The auxiliary variable $x_a$ coincides with the vector of input derivatives $U_{n-r}$, while the new input $w$ is the $(n-r)$-th input derivative, i.e. $w = u^{(n-r)}$, and is unknown. Thus, the problem into consideration is transformed into a state observation problem with an unknown input $w$ and two known outputs $y$ and $u$.

If system (1) (2) has observation relative degree $r$ in a set $\Omega \subseteq I\!R^n$, from definitions (68) it follows that

$$\begin{aligned} L_{\bar{g}} L_{\bar{f}}^k \bar{h}(x_e) &= 0, \qquad k = 0, 1, \ldots, n-2 \\ L_{\bar{g}} L_{\bar{f}}^{n-1} \bar{h}(x_e) &= L_g L_f^{r-1} h(x), \end{aligned} \tag{69}$$

and therefore $\exists x_e \in \Omega \times I\!R^{n-r} \; : \quad L_{\bar{g}} L_{\bar{f}}^{n-1} \bar{h}(x_e) \neq 0$. This means that system (67) has observation relative degree $n$. The mapping $\bar{\Phi}(\cdot, \cdot)$ defined in (12) can be written as $\bar{\Phi}(x_e) = \triangleq \bar{\Phi}(x, U_{n-r})|_{U_{n-r} = x_a}$, or

$$\bar{\Phi}(x_e) = \bar{\Phi}(x, x_a) \triangleq \begin{bmatrix} \bar{h}(x_e) \\ L_{\bar{f}} \bar{h}(x_e) \\ \vdots \\ L_{\bar{f}}^{n-1} \bar{h}(x_e) \end{bmatrix}. \tag{70}$$

Defining the square map

$$\begin{bmatrix} z \\ x_a \end{bmatrix} = \Phi_e(x_e) \triangleq \begin{bmatrix} \bar{\Phi}(x, x_a) \\ x_a \end{bmatrix}, \tag{71}$$

it can be easily recognized that $z_j = y^{(j-1)}$, $j = 1, \ldots, n$ and, as a consequence, system (67) in the $(z, x_a)$-coordinates is written as

$$\begin{aligned} \dot{z} &= A_n z + B_n \big(\bar{m}(z, x_a) + \bar{n}(z, x_a) w\big), \\ \dot{x}_a &= A_{n-r} x_a + B_{n-r} w, \\ y &= C_n z, \\ u &= C_{n-r} x_a. \end{aligned} \tag{72}$$

Defining the matrices

$$\bar{Q}(x, x_a) \triangleq \frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}x}, \qquad \bar{Q}_a(x, x_a) \triangleq \frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}x_a}, \tag{73}$$

the Jacobian of the map $\Phi_e$ and its inverse can be written

$$\frac{\mathrm{d}\Phi_e}{\mathrm{d}x_e} = \begin{bmatrix} \bar{Q} & \bar{Q}_a \\ \mathrm{O} & I_{n-r} \end{bmatrix}, \tag{74}$$

$$\left( \frac{\mathrm{d}\Phi_e}{\mathrm{d}x_e} \right)^{-1} = \begin{bmatrix} \bar{Q}^{-1} & -\bar{Q}^{-1} \bar{Q}_a \\ \mathrm{O} & I_{n-r} \end{bmatrix}. \tag{75}$$

The observer proposed for system (67), and therefore for (1) (2), is

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x}) C_{n-r} \hat{x}_a + \bar{Q}^{-1}(\hat{x}, \hat{x}_a) \cdot \\ &\quad \cdot \Big( K_1 \big(y - h(\hat{x})\big) - \bar{Q}_a(\hat{x}, \hat{x}_a) K_2 \big(u - C_{n-r} \hat{x}_a\big) \Big), \\ \dot{\hat{x}}_a &= A_{n-r} \hat{x}_a + K_2 \big(u - C_{n-r} \hat{x}_a\big). \end{aligned} \tag{76}$$

This is a block triangular system. The second block estimates the vector $U_{n-r}$ of input derivatives. In $(z, x_a)$-coordinates the observer becomes

$$\begin{aligned} \dot{\hat{z}} &= A_n \hat{z} + B_n \big(\bar{m}(\hat{z}, \hat{x}_a) + \bar{n}(\hat{z}, \hat{x}_a) w\big) + K_1 \big(y - h(\hat{x})\big), \\ \dot{\hat{x}}_a &= A_{n-r} \hat{x}_a + K_2 \big(u - C_{n-r} \hat{x}_a\big), \end{aligned} \tag{77}$$

Defining the functions

$$\begin{aligned} v_m(z, \hat{z}, x_a, \hat{x}_a) &= \bar{m}(z, x_a) - \bar{m}(\hat{z}, \hat{x}_a), \\ v_n(z, \hat{z}, x_a, \hat{x}_a) &= \bar{n}(z, x_a) - \bar{n}(\hat{z}, \hat{x}_a), \end{aligned} \tag{78}$$

the dynamics of the observation errors is described by equations

$$\dot{e}_z = (A_n - K_1 C_n) e_z + B_n (v_m + v_n w) \tag{79}$$

$$\dot{e}_a = (A_{n-r} - K_2 C_{n-r}) e_a + B_{n-r} w. \tag{80}$$

For the linear system (80), the following lemma can be given (the proof is not reported for brevity).

**Lemma 14** *Assume that in (80) a bound $w_M > 0$ exists such that $|w(t)| \le w_M$, $\forall t \ge 0$. For a given positive $\alpha_2$ let $(K_2, P_a)$ be a solution of the Lyapunov inequality*

$$(A_{n-r} - K_2 C_{n-r})P_a + P_a(A_{n-r} - K_2 C_{n-r})^{\mathrm{T}} + \\ + B_{n-r}B_{n-r}^{\mathrm{T}} + 2\alpha_2 P_a \le 0, \tag{81}$$

*($P_a$ symmetric and positive definite). Then*

$$||e_a(t)||^2 \le \mathrm{cn}(P_a)e^{-2\alpha_2 t}||e_a(0)||^2 + \frac{||P_a||}{2\alpha_2}w_M^2. \tag{82}$$

**Remark 15** *Note that with the choice of eigenvalues $\lambda_i = -\sigma^i$, $i = 1, \dots, n-r$ for matrix $A_{n-r} - K_2(\lambda)C_{n-r}$, $||P_a||$ can be made arbitrarily close to 1 (see (28) and proof of theorem 9). Therefore, lemma 14 asserts that a gain $K_2$ can be chosen so that the error $e_a$ decays below a prescribed bound with a prescribed exponential rate $\alpha_2$. The bigger the constant $\alpha_2$ the faster is the convergence and the smaller is the final error bound.*

In the next theorem the following function definition is needed

$$\varepsilon(t; \alpha_1, \alpha_2) \triangleq \begin{cases} \dfrac{e^{-2\alpha_1 t} - e^{-2\alpha_2 t}}{-2(\alpha_1 - \alpha_2)}, & \text{if } \alpha_1 \ne \alpha_2, \\ te^{-2\alpha_1 t}, & \text{if } \alpha_1 = \alpha_2 . \end{cases} \tag{83}$$

Let $\alpha = \min(\alpha_1, \alpha_2)$. It can be easily proved that $\varepsilon(t; \alpha_1, \alpha_2) \le 1/(2\alpha e)$ $\forall t \ge 0$.

Let $\bar{\mathcal{U}}_{n-r}(\bar{u}_M, w_M)$ be the set of input functions $u$ such that $||U_{n-r}(t)|| \le \bar{u}_M$ and $u^{(n-r)}(t) \le w_M$ for $t \ge 0$.

**Theorem 16** *For system (1) (2) assume that the following hypotheses hold:*

*1) the map $z = \bar{\Phi}(x, x_a)$ admits inverse $x = \bar{\Phi}^{-1}(z, x_a)$ for all $x_a \in \mathbb{R}^{n-r}$ (global uniform observability). $\bar{\Phi}$ and $\bar{\Phi}^{-1}$ are Lipschitz w.r.t. both arguments, with Lipschitz constants $\gamma_{\bar{\Phi}}$ and $\gamma_{\bar{\Phi}^{-1}}$, respectively;*

*2) the functions $\bar{m}(z, x_a)$ and $\bar{n}(z, x_a)$ are Lipschitz w.r.t. both arguments; let $\gamma_{\bar{m}}$ and $\gamma_{\bar{n}}$ be the Lipschitz constants;*

*3) $u \in \bar{\mathcal{U}}_{n-r}(\bar{u}_M, w_M)$;*

*Then there exist gain matrices $K_1$ and $K_2$ for the observer (76) is such that for $t \ge 0$*

$$||e_x(t)|| \le c_1 e^{-\alpha_1 t}||e_x(0)|| + \\ + \left(c_1 e^{-\alpha_1 t} + c_2\sqrt{\varepsilon(t; \alpha_1, \alpha_2)}\ \right)||e_a(0)|| + c_3. \tag{84}$$

*Moreover, $K_1$ and $K_2$ can be chosen so to make constants $c_2$ and $c_3$ arbitrarily small.*

**Proof.** From lemma 14 for any $\alpha_2 > 0$ a gain $K_2$ exists that ensures observation error decay for system (80) according to the law (82).

The state observation error dynamics in $z$-coordinates (79) consists of a linear system with nonlinear perturbations $v_{\bar{m}}$ and $v_{\bar{n}}$.

Assumption (2) states that

$$|v_{\bar{m}}| \le \gamma_{\bar{m}} \cdot \left\| \begin{bmatrix} e_z \\ e_a \end{bmatrix} \right\|, \qquad |v_{\bar{n}}| \le \gamma_{\bar{n}} \left\| \begin{bmatrix} e_z \\ e_a \end{bmatrix} \right\|, \tag{85}$$

and therefore

$$\begin{aligned} v_{\bar{m}}^2 &\le \gamma_{\bar{m}}^2(e_z^{\mathrm{T}}e_z + e_a^{\mathrm{T}}e_a), \\ v_{\bar{n}}^2 &\le \gamma_{\bar{n}}^2(e_z^{\mathrm{T}}e_z + e_a^{\mathrm{T}}e_a). \end{aligned} \tag{86}$$

Now, given positive constants $\alpha_1$ and $\beta$, consider a solution pair $(K_1, P)$ ($P$ symmetric and positive definite) of the $H_\infty$ Riccati-Like inequality

$$(A_n - K_1 C_n)P + P(A_n - K_1 C_n)^{\mathrm{T}} + \\ + 2\beta^2 B_n B_n^{\mathrm{T}} + 2\alpha_1 P + \tfrac{\gamma^2}{\beta^2}P^2 \le 0, \tag{87}$$

where $\gamma^2 = \gamma_{\bar{m}}^2 + \gamma_{\bar{n}}^2 w_M^2$ (solution is ensured by lemma 9).

Consider also the positive definite function of the error $e_z$

$$\nu = e_z^{\mathrm{T}}P^{-1}e_z. \tag{88}$$

It is not difficult to derive the following inequality

$$\dot{\nu}(t) \le -2\alpha_1\nu(t) + \frac{\gamma^2}{\beta^2}||e_a(t)||^2. \tag{89}$$

Substitution of (82) in (89), after few computations based on Gronwall inequality, gives

$$\nu(t) \le e^{-2\alpha_1 t}\nu(0) + \mu_1||e_a(0)||^2\varepsilon(t; \alpha_1, \alpha_2) + \\ + \tfrac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2, \tag{90}$$

$$\text{where} \quad \mu_1 = \mathrm{cn}(P_a)\tfrac{\gamma^2}{\beta^2}, \qquad \mu_2 = \tfrac{||P_a||}{2\alpha_2}\tfrac{\gamma^2}{\beta^2}w_M^2,$$

and from definition (88)

$$||e_z(t)||^2 \le \mathrm{cn}(P)e^{-2\alpha_1 t}||e_z(0)||^2 + \\ + ||P||\left(||e_a(0)||^2\mu_1\varepsilon(t; \alpha_1, \alpha_2) + \tfrac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2\right). \tag{91}$$

Using the following inequalities, implied by assumption (1),

$$\begin{aligned} ||e_z||^2 &\le \gamma_{\bar{\Phi}}^2(||e_x||^2 + ||e_a||^2), \\ ||e_x||^2 &\le \gamma_{\bar{\Phi}^{-1}}^2(||e_z||^2 + ||e_a||^2), \end{aligned} \tag{92}$$

easy computations show that the observation error in original coordinates satisfies inequality

$$||e_x(t)||^2 \le \gamma_{\bar{\Phi}}^2\gamma_{\bar{\Phi}^{-1}}^2\mathrm{cn}(P)e^{-2\alpha_1 t}(||e_x(0)||^2 + ||e_a(0)||^2) + \\ + \gamma_{\bar{\Phi}^{-1}}^2||P||\left(\mu_1||e_a(0)||^2\varepsilon(t; \alpha_1, \alpha_2) + \tfrac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2\right). \tag{93}$$

This inequality easily implies (84), with

$$
\begin{aligned}
c_1 &= \gamma_{\bar{\Phi}}\gamma_{\bar{\Phi}^{-1}}\sqrt{\mathrm{cn}(P)}, \\
c_2 &= \gamma_{\bar{\Phi}^{-1}}\sqrt{\|P\|\mathrm{cn}(P_a)}\,\tfrac{\gamma}{\beta}, \\
c_3 &= \gamma_{\bar{\Phi}^{-1}}\tfrac{\sqrt{\|P\|\|P_a\|}\,w_M}{2\beta\sqrt{\alpha_1\alpha_2}}.
\end{aligned}
\tag{94}
$$

The proof is completed observing that $\alpha_2$ can be chosen arbitrarily large while keeping $\|P_a\|$ arbitrarily close to 1, while $\alpha_1$ and $\beta$ can be made arbitrarily large while keeping $\|P\|$ arbitrarily close to 1. As a consequence constants $c_2$ and $c_3$ can be made arbitrarily small. •

**Remark 17** *Inequality (84) can be expressed by stating that the observation error exponentially tends to be bounded by* $c_3$.

# 5 Conclusions

This work considers the problem of state observation with exponential error rate for smooth nonlinear systems that do not meet conditions of uniform observability. It is shown that drift-observability, together with a smoothness/boundedness condition on the input, is sufficient to ensure the existence of an exponential observer. Three types of observers are presented, that can be constructed under drift-observability assumption only. The first observer presented is suitable for systems with maximal relative degree or for general nonlinear systems driven by sufficiently small input. The second type of observer requires the input derivatives up to a certain order, and gives exponential error decay in the case of input sufficiently smooth. The third observer presented, applicable in the case of smooth input, does not require input derivatives, and ensures exponential decay of the observation error below a prescribed level. Computer simulations, not reported in this paper due to lack of space, show good behavior of the last two observers in situations in which the first observer does not work.

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