# Ellipsiodal Bounding in Stable Predictive Control

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## Abstract

Efficient algorithms for approximate minimization of constrained infinite horizon predictive control costs are presented. The resulting control laws have guaranteed stability and asymptotic tracking, and comparable performance with existing QP-based control laws.

## 1 Introduction

Model based predictive control (MBPC) has attracted a lot of recent research attention (eg. [1, 2, 3]). This is primarily because it adopts a simple and sensible control strategy: using the model, predict and minimize a cost $J$ which penalizes future output errors, from a desired setpoint trajectory and control increments. A more important feature of MBPC however is its ability to handle hard input constraints [2, 4, 5] which apply to most practical control systems.

For a wide class of problems encountered in practice, input constraints can be expressed in terms of inequalities, say $\mathcal{C}_i$ $i = 0, 1 \ldots n_{con}$ (where $n_{con}$ is a computable constant), which are linear in the vector, say $\mathbf{f}$, of degrees of control freedom. The usual MBPC cost J can be expressed as a quadratic function of $\mathbf{f}$. The minimization of $J$ subject to $\mathcal{C}_i$ is therefore a Quadratic Programming (QP) problem. For large setpoint changes, the dimension of $\mathbf{f}$ must be sufficiently large to avoid infeasibility which can lead to instability, while the computational burden of QP can be prohibitive in high-dimensional problems. In fast sampling applications therefore, QP may not get close to the optimal solution due to limits on computation time, and the resulting control law will be suboptimal and possibly unstable.

Here we propose an alternative which also results in suboptimal control laws but has guaranteed stability and asymptotic tracking. The idea is simple: let $\mathcal{F}$ define the set of all $\mathbf{f}$ that satisfy the constraints $\mathcal{C}_i$. Then instead of minimizing $J$ subject to $\mathbf{f}$ lying in $\mathcal{F}$, minimize $J$ subject to the constraint that $\mathbf{f}$ lies in an ellipsoid contained within $\mathcal{F}$. To minimize the degree of suboptimality in the resulting predictive control algorithm, the bounding ellipsoid should clearly be as large as possible. However this requirement has to be moderated by the requirement for a simple/fast computation. In this paper we propose two ellipsoidal bounding algorithms which we then incorporate into the MBPC framework to develop a control law with guaranteed stability and asymptotic tracking. A further advantage of the proposed methodology is the derivation of an analytic expression for the solution to the minimization problem. The advantages of the results of the paper over earlier work are illustrated by means of a numerical example.

## 2 Infinite horizon MBPC

Consider the scalar, discrete-time, LTI model

$$y_{t+i} = z^{-1}\frac{b(z)}{a(z)}u_{t+i} = z^{-1}\frac{b(z)}{A(z)}\triangle u_{t+i},\ i = 1, 2, \ldots \tag{1}$$

where $A(z) = (1 - z^{-1})a(z)$, $\triangle u_t = u_t - u_{t-1}$; $y_t/u_t$ are the system output/input at time $t$, $z$ is the z-transform variable, and $z^{-1}$ is the backward shift operator. In most practical applications, inputs and/or control increments are subject to (hard) constraints:

$$\underline{u} + \epsilon \le u_{t+i} \le \overline{u},\ \underline{\triangle u} \le \triangle u_{t+i} \le \overline{\triangle u},\ i = 0, 1, \ldots \tag{2}$$

where $\underline{u}$, $\overline{u}$, $\underline{\triangle u}$, $\overline{\triangle u}$ are constants satisfying the obvious steady-state constraints:

$$\underline{u}+\epsilon \le \frac{a(1)}{b(1)}r_0 \le \overline{u}-\epsilon,\ \underline{\triangle u}+\epsilon' \le 0 \le \overline{\triangle u}-\epsilon',\ \epsilon, \epsilon' > 0. \tag{3}$$

This system description is used in many of the recent predictive control algorithms, which despite having different objectives, have in common the requirement for guaranteed stability. This in turn requires the use of control laws which yield stable input/output predictions. A description of the class of such laws is given below. We note here that, on account of the stability of the predictions to be used, it is not necessary to invoke condition (2) for all positive $i$; there always exists [6] a finite horizon $n_{con}$ with the property that if (2) holds for all $0 \le i \le n_{con}$, it will hold for all positive $i$.

### 2.1 The class of stable predictions

Let $\{\hat{u}_{t+i-1}, \hat{y}_{t+i}\}$, $i = 1, 2, \ldots$ denote a predicted input input trajectory and the corresponding predicted output trajectory at time $t$. Then from (1) we have:

$$\hat{Y}(z) = \frac{B(z)\triangle\hat{U}(z) + P(z)}{A(z)},\ B(z) = z^{-1}b(z), \tag{4}$$

where $\Delta\hat{U}(z)$ and $\hat{Y}(z)$ are the z-transforms of the predicted input and output sequences, and $P(z)$ accounts for the effect of initial conditions at $t$. Next assume for convenience (without loss of generality) that the setpoint trajectory is a constant $r_0$, so that the reference signal $r_t$ and corresponding predicted error signal $e_t = r_t - y_t$ have z-transforms:

$$R(z) = \frac{r_0}{1-z^{-1}}, \; \hat{E}(z) = \frac{Q(z) - B(Z)\Delta\hat{U}(z)}{A(z)}, \quad (5)$$

where $Q(z) = a(z)r_0 - P(z)$. Thus the predicted error behaviour will be stable if and only if:

$$Q(z) - B(z)\Delta\hat{U}(z) = A_+(z)\Phi(z), \quad (6)$$

where $\Phi(z)$ is a polynomial or stable transfer function; it is assumed that $A(z) = A_+(z)A_-(z)$ and that the roots of $A_-(z)$ [$A_+(z)$] are all inside [on or inside] the unit circle.

From equation (4) it follows that the predicted input behaviour will be stable if and only if

$$A_+(z)\Phi(z) + B_+(z)\Psi(z) = Q(z) \quad (7)$$

where $B(z) = B_+(z)B_-(z)$ is conformal to the factorization of $A(z)$, and $\Psi(z)$ is a stable transfer function.

**Theorem 2.1** ([4]) *The class of stable input and output predictions is defined by*

$$\begin{aligned} \hat{E}(z) &= \frac{\Psi_p(z)}{A_-(z)} + \frac{B_+(z)}{A_-(z)}F(z) \\ \Delta\hat{U}(z) &= \frac{\Psi_p(z)}{B_-(z)} - \frac{A_+(z)}{B_-(z)}F(z) \end{aligned} \quad (8)$$

*where $F(z)$ is an arbitrary stable transfer function and $\Phi_p(z)$, $\Psi_p(z)$ denote a pair of particular solutions to the Bezout identity of equation (7).*

*Proof:* This follows directly from equations (5), (6), and the fact that $\Phi(z) = \Phi_p(z) + B_+(z)F(z)$, $\Psi(z) = \Psi_p(z) - A_+(z)F(z)$ define the whole family of solutions to equation (7). □

## 2.2 Predictive control algorithm

In common with most predictive control strategies we consider a cost that penalizes predicted error and control activity. The value of the cost at time $t$ is given by:

$$\begin{aligned} J_t &= \sum_{i=1}^{\infty}[\hat{e}_{t+i}^2 + \lambda^2\Delta\hat{u}_{t+i-1}^2] \quad (9) \\ &= \frac{1}{j2\pi}\oint[|\hat{E}(z)|^2 + \lambda^2|\Delta\hat{U}(z)|^2]\frac{dz}{z} \\ &= \frac{1}{j2\pi}\oint\left\| \begin{bmatrix} \frac{\Phi_p(z)}{A_-(z)} \\ \frac{\lambda\Psi_p(z)}{B_-(z)} \end{bmatrix} + \begin{bmatrix} \frac{B_+(z)}{A_-(z)} \\ \frac{-\lambda A_+(z)}{B_-(z)} \end{bmatrix} F(z) \right\|_2^2 \frac{dz}{z}, \end{aligned}$$

where use has been made of Parseval's theorem and of (8); $\|.\|_2$ is the Euclidean norm and all contour integrals are taken over the unit circle ($|z| = 1$). The optimization of $J_t$ in the presence of constraints (2) is conveniently handled by replacing the stable transfer function $F(z)$, which has an infinite impulse response, by a polynomial of finite degree $n_f - 1$.

Let $\mathbf{f}$ denote the vector of coefficients of $F(z)$, then by the residue theorem, $J_t$ can be expressed as:

$$J_t = (\mathbf{f} - \mathbf{f}^*)^T S(\mathbf{f} - \mathbf{f}^*) + \alpha, \quad (10)$$

with $\alpha$ a constant and $\mathbf{f}^*$ the unconstrained optimum. Denoting by $h_i$, $g_i$, $i = 0, 1, \ldots$ the elements of the impulse responses of $H(z) = A_+(z)/B_-(z)$ and $G(z) = \Psi_p(z)/B_-(z)$, the constraints of equation (2) can be rewritten in terms of $\mathbf{f}$ using (8) as

$$\underline{\Delta u} \leq \mathbf{h}_i^T\mathbf{f} + g_i \leq \overline{\Delta u} \quad (11)$$

$$\underline{u} \leq \sum_{j=0}^{i}(\mathbf{h}_j^T\mathbf{f} + g_j) + u_{t-1} \leq \overline{u} \quad (12)$$

for $i = 0, 1, \ldots n_{con}$, where $\mathbf{h}_i^T = [h_i \; h_{i-1} \; \ldots \; h_0 \; \mathbf{0}]$, $i = 0, 1, \ldots$. We denote the set of points $\mathbf{f}$ satisfying (11,12) as $\mathcal{F}$.

**Algorithm 2.1 (IHPC)** Compute $\{(\mathbf{h_i})_{i=0}^{n_{con}}\}$. At times $t = 0, 1, \ldots$:

(i). compute $P$, solve (7) for $\Phi_p$, $\Psi_p$ and hence obtain $\mathbf{f}^*$, $g_i$ $i = 0, 1, \ldots n_{con}$

(ii). if $\mathbf{f}^* \in \mathcal{F}$, set $\mathbf{f} = \mathbf{f}^*$; otherwise use QP to compute $\arg\min_{\mathbf{f}\in\mathcal{F}} J_t$

(iii). form $F(z)$ and compute the current optimal control move $\Delta u_t$ from (8).

**Theorem 2.2** *Given feasibility at $t = 0$, ie. assuming that $\mathcal{F}$ is non-empty at $t = 0$, Algorithm 2.1 has guaranteed stability and asymptotic tracking.*

*Proof:* Feasibilty at $t = 0$ implies feasibility at all subsequent times. It is therefore easy to show that the sequence $\{(J_t)_{t=0}^{\infty}\}$, minimized subject to (11,12) converges to zero. □

# 3 Suboptimal algorithms

Infinite horizon predictive control (IHPC), in common with most other constrained predictive control algorithms, requires the solution to a quadratic constrained optimization problem at each sample instant. For large setpoint changes, it is necessary to use a large number of degrees of freedom $n_f$ in order to avoid infeasibility which in turn could result in instability. Under such conditions upper bounds on the computational burden of QP algorithms are large, and QP may not therefore be implementable in constrained control problems with high sample rates. In the following we develop an alternative approach, giving a suboptimal control law within a known and practicable number of computations. The approach is to obtain a convenient ellipsoidal bound for the constraint set $\mathcal{F}$, and use this to obtain a suboptimal but analytic solution to the minimization of $J_t$.

### 3.1 Constraint set ellipsoidal bounds

We begin with the derivation of convenient ellipsoidal bounds on the constraint set $\mathcal{F}$.

Define an ellipsoidal set $\mathcal{E}_i = \{\mathbf{f}; (\mathbf{f}-\mathbf{f}_i^0)^T\Sigma_i^{-1}(\mathbf{f}-\mathbf{f}_i^0) \leq 1\}$, and let $\mathcal{C}_i = \{\mathbf{f}; (\mathbf{h}_i^T\mathbf{f}+g_i)^2 \leq d^2\}$, where $\Sigma_i = \Sigma_i^T > 0$, and $g_i$, $d$ are constants. Then the following lemma can be used to determine an ellipsoidal set $\mathcal{E}_{i+1}$ bounding the intersection of $\mathcal{E}_i$ and $\mathcal{C}_i$.

**Lemma 3.1** *If $\mathbf{f} \in \mathcal{E}_i \cap \mathcal{C}_i$, then for all $\rho > 0$, $\mathbf{f}$ also lies in $\mathcal{E}_{i+1} = \{\mathbf{f}; (\mathbf{f}-\mathbf{f}_{i+1}^0)^T\Sigma_{i+1}^{-1}(\mathbf{f}-\mathbf{f}_{i+1}^0) \leq 1\}$, where*

$$\begin{aligned} \mathbf{f}_{i+1}^0 &= \mathbf{f}_i^0 - \frac{\rho\gamma}{\beta}\Sigma_i\mathbf{h}_i \\ \Sigma_{i+1} &= \left[1+\rho-\frac{\rho\gamma^2}{\beta}\right]\left[\Sigma_i - \frac{\rho}{\beta}\Sigma_i\mathbf{h}_i\mathbf{h}_i^T\Sigma_i\right] \end{aligned} \tag{13}$$

*and $\gamma = \mathbf{h}_i^T\mathbf{f}_i^0 + g_i$, $\beta = d^2 + \sigma\rho$, $\sigma = \mathbf{h}_i^T\Sigma\mathbf{h}_i$. Furthermore if $\rho$ is the smallest positive root of*

$$\begin{aligned} (n_f-1)\sigma^2\rho^2 + [(2n_f-1)d^2 - \sigma + \gamma^2]\sigma\rho \\ + d^2[n_f(d^2-\gamma^2)-\sigma] = 0 \end{aligned}$$

*then $\mathcal{E}_{i+1}$ has the smallest volume of all ellipsoidal sets containing $\mathcal{E}_i \cap \mathcal{C}_i$.*

*Proof:* See [7]. □

**Remark 3.1** In the above it is assumed that the intersection of $\mathcal{E}_i$ and $\mathcal{C}_i$ is non-empty. For given $\mathcal{E}_i$, $\mathcal{C}_i$, the validity of this assumption can be checked by determining whether $(|d|-|\gamma|)^2 \leq \sigma$. Furthermore it is clear that if only one of the two hyperplanes bounding $\mathcal{C}_i$ intersects $\mathcal{E}_i$, then the other can be replaced by the parallel hyperplane which is tangent to $\mathcal{E}_i$.

Constraints (11,12) define a sequence of pairs of parallel hyperplanes between which feasible $\mathbf{f}$ must lie. Given a suitable initial ellipsoidal set $\mathcal{E}_0$, an ellipsoid containing the feasible set $\mathcal{F}$ can be generated by successive application of Lemma 3.1 to the individual constraint sets $\mathcal{C}_i$, $i = 0, 1, \ldots n_{con}$. Since the resulting bounding ellipsoid $\mathcal{E}_{n_{con}+1}$ includes points which are not feasible, it must be shrunk before being used in place of $\mathcal{F}$ in the minimization of $J_t$. In the interest of minimizing the degree of suboptimality, it is important to make the shrunken ellipsoidal set as large as possible. This is undertaken below; for simplicity the treatment is given only for the case of input rate constraints, but the extension to the general case is obvious.

**Corollary 3.1** *Let $\Sigma'_{n_{con}+1} = \nu^2\Sigma_{n_{con}+1}$, $\mathcal{E}'_{n_{con}+1} = \{\mathbf{f}; (\mathbf{f}-\mathbf{f}_{n_{con}+1}^0)^T(\Sigma'_{n_{con}+1})^{-1}(\mathbf{f}-\mathbf{f}_{n_{con}+1}^0) \leq 1\}$, and*

$$\nu = \min_{i \leq n_{con}} \left\{ \frac{\min\{\gamma_i - \underline{\Delta u}, \overline{\Delta u} - \gamma_i\}}{(\mathbf{h}_i^T\Sigma\mathbf{h}_i)^{1/2}} \right\}, \tag{14}$$

*with $\gamma_i = \mathbf{h}_i^T\mathbf{f}_{n_{con}+1}^0 + g_i$. Then in the presence of constraints (11) alone, $\mathcal{E}'_{n_{con}+1}$ is the largest ellipsoidal set with the same centre and principle axes as $\mathcal{E}_{n_{con}+1}$ contained entirely within $\mathcal{F}$.*

*Proof:* The set $\mathcal{E} = \{\mathbf{f}; (\mathbf{f}-\mathbf{f}^0)^T\Sigma^{-1}(\mathbf{f}-\mathbf{f}_{n_{con}+1}^0) \leq 1\}$ can be expressed $\{\boldsymbol{\phi}; \boldsymbol{\phi}^T\boldsymbol{\phi} \leq 1\}$, $\boldsymbol{\phi} = \Lambda^{-1/2}R^T(\mathbf{f}-\mathbf{f}^0)$, where $\Lambda$, $R$ are the eigenvalue and eigenvector matrices of $\Sigma$. Correspondingly, (11) may be written:

$$\underline{\Delta u} \leq \boldsymbol{\theta}_i^T\boldsymbol{\phi} + \gamma_i \leq \overline{\Delta u}, \; i = 0, 1, \ldots n_{con},$$

with $\boldsymbol{\theta}_i = \Lambda^{1/2}R^T\mathbf{h}_i$, $\gamma_i = \mathbf{h}_i^T\mathbf{f}^0 + g_i$. It is therefore easy to show that, for all $\mathbf{f} \in \mathcal{E}$,

$$\gamma_i - (\mathbf{h}_i^T\Sigma\mathbf{h}_i)^{1/2} \leq \mathbf{h}_i^T\mathbf{f} + g_i \leq \gamma_i + (\mathbf{h}_i^T\Sigma\mathbf{h}_i)^{1/2}$$

for $i = 0, 1, \ldots n_{con}$. With $\mathcal{E}$ replaced by $\mathcal{E}'_{n_{con}+1}$, it then follows that the lagest value of $\nu$ for which (11) holds is given by (14). □

**Remark 3.2** A larger ellipsoidal bound on the interior of $\mathcal{F}$ can be found if the centre of $\mathcal{E}_{n_{con}+1}$ is redefined, in addition to the scaling of its eigenvalues. This however requires the solution to a linear program, and since the aim of this paper is to develop a simple alternative to QP-based predictive algorithms, this issue will not be pursued further here.

Combining Lemma 3.1 and Corollary 3.1, the following procedure determines an ellipsoidal set contained within $\mathcal{F}$.

**Procedure 3.1** Set $\Sigma_0 = \epsilon^{-1}\mathrm{I}$, $\epsilon \ll 1$, $\mathbf{f}_0^0 = 0$. For $i = 0, 1, \ldots n_{con}$:

(i). obtain $\mathcal{C}_i$ from (11,12), and $\mathcal{E}_{i+1}$ from (13)

(ii). if $i = n_{con}$, compute $\mathcal{E}'_{n_{con}+1}$ using (14).

The ellipsoidal bounds generated by Procedure 3.1 are likely to be similar in shape to the feasible set due to the use of minimum volume bounding ellipsoids in step (i). However the control law obtained by approximating $\mathcal{F}$ with $\mathcal{E}'_{n_{con}+1}$ in the minimization of $J_t$ will give poorer performance than the QP-based control law. To remedy this, we develop below an alternative ellipsoidal bounding algorithm, based on fitting ellipsoidal sets of maximum volume within successive constraint hyperplanes.

**Theorem 3.1** *The largest volume ellipsoidal set contained in the intersection of $\mathcal{E}_i$ and $\mathcal{C}_i$ (defined as in Lemma 3.1) is $\mathcal{E}_{i+1} = \{\mathbf{f}; (\mathbf{f}-\mathbf{f}^0_{i+1})^T\Sigma^{-1}_{i+1}(\mathbf{f}-\mathbf{f}^0_{i+1}) \leq 1\}$, where*

$$\begin{aligned} \mathbf{f}^0_{i+1} &= \mathbf{f}^0_i + \frac{(\eta+\kappa)}{2\sigma}\Sigma_i\mathbf{h}_i \\ \Sigma_{i+1} &= \mu_1\Sigma_i + \frac{(\mu_0-\mu_1)}{\sigma}\Sigma_i\mathbf{h}_i\mathbf{h}_i^T\Sigma_i \end{aligned} \tag{15}$$

$$\mu_0 = \frac{(\eta-\kappa)^2}{4\sigma}, \quad \mu_1 = \frac{(\eta-\kappa)^2/2}{\sigma - \eta\kappa - \sqrt{(\sigma-\kappa^2)(\sigma-\eta^2)}}$$

$$\kappa = \underline{\kappa}, \; \eta = \max\left\{\overline{\kappa}, \; -\left[\tfrac{n_f-1}{n_f+1}\right]\left[\underline{\kappa}^2 + \tfrac{4n_f\sigma}{(n_f-1)^2}\right]^{1/2}\right\}$$

*and $\sigma = \mathbf{h}_i^T\Sigma_i\mathbf{h}_i$, $\kappa_1 = d - (g_i + \mathbf{h}_i^T\mathbf{f}^0_i)$, $\kappa_2 = -d - (g_i + \mathbf{h}_i^T\mathbf{f}^0_i)$, $(\underline{\kappa}, \overline{\kappa}) = (\kappa_1, \kappa_2)$ if $|\kappa_1| < |\kappa_2|$, $(\underline{\kappa}, \overline{\kappa}) = (\kappa_2, \kappa_1)$ otherwise.*

Before giving the proof of Theorem 3.1, we first determine the orientation of the eigenvectors and centre of the largest volume ellipsoid contained within the intersection of $\mathcal{E}_i$ and $\mathcal{C}_i$.

Let the centre of $\mathcal{E}_i$ in a transformed space in which $\mathcal{E}_i$ is a unit spheroid be $\boldsymbol{\phi}^0_i$, and denote the hyperplane bounding $\mathcal{C}_i$ that is closest to $\boldsymbol{\phi}^0_i$ as $\delta\mathcal{C}_i$, with normal $\boldsymbol{\theta}_i$. Next consider a second transformation: $\boldsymbol{\psi} = \Lambda^{-1/2}_{i+1}R^T_{i+1}\boldsymbol{\phi}$, where $\Lambda_{i+1}$ and $R_{i+1}$ are the eigenvalue and eigenvector matrices of a symmetric positive definite matrix $S_{i+1}$. Under this transformation, the ellipsoid $\mathcal{E}_{i+1} = \{\boldsymbol{\phi}; (\boldsymbol{\phi}-\boldsymbol{\phi}^0_{i+1})^TS^{-1}_{i+1}(\boldsymbol{\phi}-\boldsymbol{\phi}^0_{i+1}) \leq 1\}$ is a unit spheroid; and since $\mathcal{E}_i \supseteq \mathcal{E}_{i+1}$ for $\det(S_{i+1}) \leq 1$ and suitably chosen $\boldsymbol{\phi}^0_{i+1}$, the problem of determining $\mathcal{E}_{i+1} \subseteq \mathcal{E}_i \cap \mathcal{C}_i$ of maximum volume is equivalent to that of determining the minimum volume $\mathcal{E}_i \supseteq \mathcal{E}_{i+1}$ for $\mathcal{E}_{i+1} \subseteq \mathcal{C}_i$ a unit spheroid.

By inspection, the minimum volume $\mathcal{E}_i$ has the vector $\boldsymbol{\psi}^0_i - \boldsymbol{\psi}^0_{i+1}$ between the centres ($\boldsymbol{\psi}^0_i$ and $\boldsymbol{\psi}^0_{i+1}$ respectively) of $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ as an eigenvector, and the eigenvalues associated with all other eigenvectors are equal. Furthermore all eigenvalues of $\mathcal{E}_i$ decrease monotonically with increasing $\|\boldsymbol{\psi}^0_i - \boldsymbol{\psi}^0_{i+1}\|_2$, and since it can be assumed without loss of generality that $\delta\mathcal{E}_{i+1}$ is tangent to $\delta\mathcal{C}_i$, it follows that the volume of $\mathcal{E}_i$ is minimized when the perpendicular distance $d(\boldsymbol{\psi}^0_i)$ from $\boldsymbol{\psi}^0_i$ to $\delta\mathcal{C}_i$ is minimized. If the perpendicular distance from $\boldsymbol{\phi}^0_i$ to $\delta\mathcal{C}_i$ is $d(\boldsymbol{\phi}^0_i)$, then

$$d(\boldsymbol{\psi}^0_i) = \frac{\|\boldsymbol{\theta}_i\|_2}{(\boldsymbol{\theta}_i^TS_{i+1}\boldsymbol{\theta}_i)^{1/2}}d(\boldsymbol{\phi}^0_i)$$

which is minimized when $\boldsymbol{\theta}_i$ is an eigenvector of $S_{i+1}$. In the space in which $\mathcal{E}_i$ is spheroidal, the normal to the hyperplanes bounding $\mathcal{C}_i$ is therefore an eigenvector of the maximum volume $\mathcal{E}_{i+1}$; and since $\boldsymbol{\theta}_i^T(\boldsymbol{\phi}^0_i - \boldsymbol{\phi}^0_{i+1}) \neq 0$, it also follows that the line between the centres of $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ is normal to the bounding hyperplanes.

*Proof of Theorem 3.1:* Under the transformation $\boldsymbol{\phi} = \Lambda^{-1/2}R_i^T(\mathbf{f}-\mathbf{f}^0_i)$, we have $\mathcal{E}_i = \{\boldsymbol{\phi};\ \boldsymbol{\phi}^T\boldsymbol{\phi} \leq 1\}$ and $\mathcal{C}_i = \{\boldsymbol{\phi};\ k_1 \leq \boldsymbol{\theta}_i{}^T\boldsymbol{\phi} \leq k_2\}$, with $\boldsymbol{\theta}_i = \Lambda^{1/2}R_i^T\mathbf{h}_i/\sigma^{1/2}$ and $k_j = \kappa_j/\sigma^{1/2}$, $j = 1,2$. From the preceding discussion, the ellipsoidal set of maximum volume in the intersection of $\mathcal{E}_i$ and $\mathcal{C}_i$, denoted $\mathcal{E}_{i+1} = \{\boldsymbol{\phi}; (\boldsymbol{\phi} - \boldsymbol{\phi}^0_{i+1})^TS^{-1}_{i+1}(\boldsymbol{\phi}-\boldsymbol{\phi}^0_{i+1}) \leq 1\}$ has the form

$$S_{i+1} = \mu_1\mathrm{I} + (\mu_0-\mu_1)\boldsymbol{\theta}_i\boldsymbol{\theta}_i{}^T, \ \boldsymbol{\phi}^0_i = \alpha\boldsymbol{\theta}_i,$$

for some constants $\alpha$ and $\mu_0, \mu_1 > 0$. Consider first the case of the boundary $\delta\mathcal{E}_{i+1}$ of $\mathcal{E}_{i+1}$ being tangential to only one the bounding hyperplanes of $\mathcal{C}_i$.

If $\delta\mathcal{E}_{i+1}$ is tangent to $\delta\mathcal{C}^j_i = \{\boldsymbol{\phi};\ \boldsymbol{\theta}_i{}^T\boldsymbol{\phi} = k_j\}$ $j = 1$ or 2, at $\boldsymbol{\phi} = \boldsymbol{\phi}_c$, then for some $\lambda_c$

$$S^{-1}_{i+1}(\boldsymbol{\phi}_c - \boldsymbol{\phi}^0_{i+1}) = \lambda_c\boldsymbol{\theta}_i.$$

From $(\boldsymbol{\phi}_c - \boldsymbol{\phi}^0_{i+1})^TS^{-1}_{i+1}(\boldsymbol{\phi}_c - \boldsymbol{\phi}^0_{i+1}) = 1$ and $\boldsymbol{\theta}_i{}^T\boldsymbol{\phi}_c = k_j$ therefore,

$$\mu_0 = (k_j - \alpha)^2. \tag{16}$$

Similarly, $\delta\mathcal{E}_{i+1}$ tangential to $\delta\mathcal{E}_i$, at $\boldsymbol{\phi} = \boldsymbol{\phi}_e$ gives for some $\lambda_e$,

$$S^{-1}_{i+1}(\boldsymbol{\phi}_e - \boldsymbol{\phi}^0_{i+1}) = \lambda_e\boldsymbol{\phi}_e.$$

Since $\boldsymbol{\theta}_\perp^T\boldsymbol{\phi}_e \neq 0$, for any $\boldsymbol{\theta}_\perp$; $\boldsymbol{\theta}_\perp^T\boldsymbol{\theta}_i = 0$, it follows that $(\mu_0^{-1} - \mu_1^{-1})(\mu_1^{-1} - 1) = \mu_0^{-1}\mu_1^{-1}\alpha^2$. Defining $\eta = 2\alpha - k_j$ and using expression (16) for $\mu_0^{-1}$, $\mu_1^{-1}$ can be expressed

$$\mu_1^{-1} = 2\frac{(1-\eta k_j) \pm [(1-k_j^2)(1-\eta^2)]^{1/2}}{(\eta - k_j)^2} \tag{17}$$

The volume of $\mathcal{E}_{i+1}$ is maximized at the value of $\eta$ that maximizes $\mu_0\mu_1^{n_f-1}$. Using (16) and (17), the maximizing value $\eta^* = \arg\sup_\eta(\mu_0\mu_1^{n_f-1})$ is given by

$$\eta^* = \pm\left[\tfrac{n_f-1}{n_f+1}\right]\left[k_j^2 + \tfrac{4n_f}{(n_f-1)^2}\right]^{1/2}.$$

The positive root gives $\alpha < k_j$, and therefore corresponds to the case of $\delta\mathcal{E}_{i+1}$ tangential to $\delta\mathcal{C}^1_i$, and the negative root similarly gives tangency to $\delta\mathcal{C}^2_i$. If the boundary of $\mathcal{E}_{i+1}$ is tangential to both hyperplanes, then by symmetry, $\alpha = (k_1 + k_2)/2$, and $\mu_0$, $\mu_1$ are given by (16) and (17), with $\eta$ replaced by the relevant $k_j$. Finally, $\mu_0\mu_1^{n_f-1}$ is monotonically increasing in $|\eta - k_j|$ for $|\eta| < |\eta^*|$ and $\mathcal{E}_{i+1}$ is therefore tangent to $\delta\mathcal{C}^j_i$ if and only if $|k_j| < |\eta^*|$. □

The following procedure uses of Theorem 3.1 to find an ellipsoidal bound on the interior of $\mathcal{F}$.

**Procedure 3.2** Set $\Sigma_0 = \epsilon^{-1}\mathrm{I}$, $\epsilon \ll 1$, $\mathbf{f}^0_0 = 0$. For $i = 0, 1, \ldots n_{con}$:

(i). obtain $\mathcal{C}_i$ from (11,12), and $\mathcal{E}_{i+1}$ from (15).

**Theorem 3.2** *Assume that $\mathcal{F}$ is non-empty at $t = 0$, ie. that the setpoint $r_0$ is feasible. Then for*

$$\epsilon \leq \min_{i \leq n_{con}} \min \left\{ \frac{\|\mathbf{h}_i\|_2}{|\overline{\Delta u} - g_i|}, \frac{\|\mathbf{h}_i\|_2}{|\overline{u} - u_{t-1} - g_i|} \right\} \quad (18)$$

*the ellipsoid $\mathcal{E}_0$ defined in Procedures 3.1 and 3.2 contains all feasible $\mathbf{f}$.*

*Proof:* From (11,12) it follows that for all $\mathbf{f} \in \mathcal{F}$, $\|\mathbf{f}\|_2$ is upper-bounded by the inverse of the RHS of (18). The constraint set $\mathcal{F}$ is therefore contained within the ellipsoid $\mathcal{E}_0$ defined in Procedures 3.1 and 3.2. □

**Remark 3.3** Simulations show that the choice of $\epsilon$ is not critical (ie. does not affect significantly the bounding ellipsoids of Lemma 3.1 and Theorem 3.1), and can therefore be taken to be some arbitrarily small value.

**Remark 3.4** It is clearly possible to construct a non-empty constraint set $\mathcal{F}$ for which Procedure 3.2 fails (although this situation has not been encountered in simulations with constraints generated on the basis of predicted system inputs). In contrast, Procedure 3.1 necessarily provides a bound on the interior of any non-empty $\mathcal{F}$ due to the inclusion ellipsoidal set employed in step (i). Thus Procedures 3.1 and 3.2 are subject to a trade-off between optimality and robustness.

### 3.2 Cost minimization

As stated earlier, we now replace the problem of minimizing $J_t$ subject to constraints (11,12) by $\min_{\mathbf{f} \in \mathcal{E}} J_t$, where $\mathcal{E} = \{\mathbf{f};\ (\mathbf{f} - \mathbf{f}^0)^T \Sigma^{-1} (\mathbf{f} - \mathbf{f}^0) \leq 1\}$ is generated at time $t$ using either of Procedures 3.1 and 3.2. Using (10), this problem is equivalent to

$$\min_{\phi \in \tilde{\mathcal{E}}} (\phi^T \phi + \alpha),\ \tilde{\mathcal{E}} = \{\phi; (\phi - \phi^0)^T \tilde{\Sigma}^{-1} (\phi - \phi^0) \leq 1\}, \quad (19)$$

where $\phi = \Lambda_S^{1/2} R_S^T (\mathbf{f} - \mathbf{f}^*)$, $\phi^0 = \Lambda_S^{1/2} R_S^T (\mathbf{f}^* - \mathbf{f}^0)$, $\tilde{\Sigma} = \Lambda_S^{1/2} R_S^T \Sigma R_S \Lambda_S^{1/2}$, and $\Lambda_S$, $R_S$ are the eigenvalue and eigenvector matrices of $S$.

**Theorem 3.3** *The solution to the minimization problem (19) is given by*

$$\mathbf{f}_{opt} = R_S \Lambda_S^{-1/2} \phi^*,\ \phi^* = \tilde{R} \mathrm{diag}\{\lambda^*/(\lambda^* - \tilde{\lambda}_i)\} \tilde{R}^T \phi^0,$$

*where $\{(\tilde{\lambda}_i)_{i=1}^{n_f}\}$ are the diagonal elements of $\tilde{\Lambda}$; $\tilde{\Lambda}$, $\tilde{R}$ are the eigenvalue and eigenvector matrices of $\tilde{\Sigma}$, and $\lambda^*$ is the negative real root of the polynomial*

$$f(\lambda) = \sum_{i=1}^{n_f} \tilde{\lambda}_i \tilde{\phi}_i^2 \prod_{j=1, j \neq i} (\lambda - \tilde{\lambda}_j)^2$$

*($\tilde{\phi}_i$ is the ith element of $\tilde{R}^T \phi$). Furthermore, $f(\lambda)$ has no negative real roots if $\mathbf{f}^* \in \mathcal{E}$ (in which case $\mathbf{f}_{opt} = \mathbf{f}^*$), and a single negative real root if $\mathbf{f}^* \notin \mathcal{E}$.*

*Proof:* If $\mathbf{f}_{opt}$ is the solution to (19), then either (i) $\mathbf{f}_{opt}$ lies on the boundary of $\tilde{\mathcal{E}}$ at the tangent point to the ellipsoid $\phi^T \phi = J_0$, where $J_0$ is a constant; or (ii) $\mathbf{f}_{opt} = \mathbf{f}^* \in \mathcal{E}$. Case (i) occurs if and only if $\phi = \lambda \tilde{\Sigma}^{-1} (\phi - \phi^0)$ and $(\phi - \phi^0)^T \tilde{\Sigma}^{-1} (\phi - \phi^0)$ for some $\lambda < 0$. Using the eigenvalue/vector decomposition of $\tilde{\Sigma}$, it can be shown that these conditions hold simultaneously only for $\lambda$ satisfying $f(\lambda) = 0$. To complete the proof we observe that there can be at most one point (corresponding to a negative root of $f(\lambda)$) at which $\mathcal{E}$ is tangent to a level set of $J_t$ with the outward normal to the boundary of $\mathcal{E}$ in the direction opposing $\nabla J_t$; that such a point exists for all $\mathbf{f}^* \notin \mathcal{F}$; and that no such points exist if $\mathbf{f}^* \in \mathcal{E}$. □

**Remark 3.5** The eigenvalue-eigenvector decomposition of $S$ can be computed off-line.

**Remark 3.6** The approximate solution to the minimization problem $\min_{\mathbf{f} \in \mathcal{F}} J_t$ obtained using Theorem 3.3 and either of Procedures 3.1 and 3.2 requires $O(n_f^3)$ operations. This is to be contrasted with the computational burden of a QP algorithm, which typically requires $O(n_f^4 (\log n_f - \log \epsilon))$ operations to attain an approximate solution within a tolerance $\epsilon$ of the solution [8].

### 3.3 Predictive control algorithm

The results of sections 3.1 and 3.2 give the following algorithm.

**Algorithm 3.1** As per Algorithm 2.1, with the optimization of step (ii) replaced by the optimization procedure of Theorem 3.3 and either of Procedures 3.1 and 3.2.

**Theorem 3.4** *Given feasibility at $t = 0$, Algorithm 3.1 has guaranteed stability and asymptotic tracking.*

*Proof:* Feasibility at $t = 0$ implies feasibility and hence the existence of a bounding ellipsoid generated using Procedure 3.1, at all subsequent times; this follows from the fact that $\mathbf{f}_{opt}$ always lies within the feasible set. By same argument it can be shown that the existence of a bounding ellipsoid generated by Procedure 3.2 at $t = 0$ ensures the existence of bounding ellipsoids generated by this algorithm at all subsequent times. It follows that sequences $\{(J_t)_{t=0}^{\infty}\}$ obtained using Algorithm 3.1 converge to zero. □

# 4 Simulation Results

The performance of the control laws of Algorithm 3.1 are contrasted with that of optimal QP-based IHPC in the following simulation example. The system model is taken as

$$a(z) = 1 - 5.5z^{-1} + 8.54z^{-2} - 3.2z^{-3}0.24z^{-4}$$
$$b(z) = 1 + 0.1z^{-1} - 3.1z^{-2} + 1.4z^{-3}$$

which has unstable poles at $z = 2, 3$ and a nonminimum phase zero at $z = -2$. The setpoint is $r_0 = 1$. Input rate constraints of $\overline{\Delta u} = -\underline{\Delta u} = 0.5$ are assumed, and $n_f = 10$ for each controller. Figure 1 shows that Algorithm 3.1 with Procedure 3.2 has marginally slower response times, more conservative control moves and higher costs than the QP-based controller, and that Procedure 3.1 leads to a higher degree of suboptimality. However, figure 1(e) shows that the computational burden of the QP-based algorithm per sample is five times that of the suboptimal controllers.

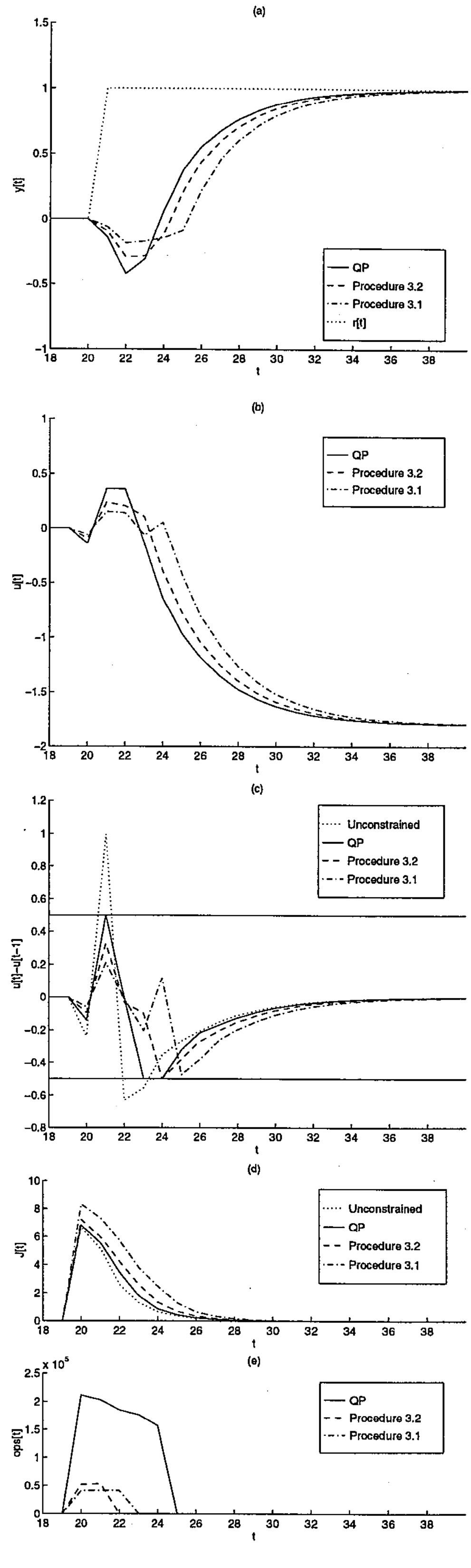


Figure 1: (a) System outputs $y_t$. (b) Control inputs $u_t$. (c) Control input rates $\Delta u_t$. (d) Prediction costs $J_t$. (e) Floating point operations per sample.

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