Parameter identification of nonlinear systems with known structure

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Abstract

This paper addresses the problem of on-line parameter estimation for nonlinear continuous–time dynamical systems of known structure. Two parameter estimation problems are formulated and explicitly solved. The resulting parameter estimation algorithms incorporate an observer; the adjustment of the observer parameters is considered as a stabilization problem for an uncertain system and is solved using sliding mode method. In the *full information parameter estimation problem* it is assumed that all the system states and their derivatives are known. In the *output measurement parameter estimation problem* we assume that only one state is known, the other states are reconstructed using an adaptive observer. The proposed algorithm is applied for parameter estimation of a Van der Pol oscillator.

1 Introduction

System identification often constitutes the first step to a successful simulation, prediction and control. Due to existence of well developed theory and excellent software packages, see [7] and [8], identification of linear dynamical systems and of systems linear in the parameters is no more difficult. Unfortunately, quite often modeled phenomena are highly nonlinear and cannot be represented using linear models or models linear in the parameters. Parametric models of nonlinear systems are common in the situation when there exists some physical insight into the system dynamics. Hence the need for parameter estimation methods for nonlinear systems. Unfortunately, identification of nonlinear systems is much more difficult and much less developed as it is for linear systems. An overview of existing methods can be found, e.g., in [12]. In this paper we concentrate on the problem of finding unknown parameters of a nonlinear dynamical system, described by a set of ordinary differential equations of known structure. A possible approach, that follows methodology used in the field of parameter identification of linear systems, is to minimize some performance index, using its gradient or Hessian, see [2]. Least Mean Squares method and Least Squares method, suitable for linear systems, [14], will fall into this category. Unfortunately, this approach is not most appropriate in practical applications for parameter estimation of systems which are nonlinear in the parameters, since, even in the exact case (no measurement noise), they can get stuck in local minima of the associated performance indices. Another, ad hoc, possibility is to use the extended Kalman filter, see [1]. Several other methods are tailored to specific system structures, assuming that the system is linear in the parameters, [5], [9], is of 1st order [15], contains nonlinearity at the output [10], or is multinomial in inputs and outputs, see [13].

The present paper addresses the problem of on-line parameter estimation for nonlinear continuous–time dynamical MISO systems of known structure, *i.e.* a special case of the systems considered in [4].

Two parameter estimation problems for the noise-free case are formulated and solved. The resulting parameter estimation algorithms incorporate an observer which is on-line adjusted and guarantees parameter and state convergence. The adjustment of the observer parameters is considered as a stabilization problem for an uncertain system and is solved using known methods. In the full information parameter estimation problem it is assumed that all the system states and their derivatives be known. The parameter update of the observer is computed using sliding mode methods, yielding exponential convergence of the state estimation error and the parameter estimates. In the output measurement parameter estimation problem we assume that only one state is known. Unlike the first problem, the state estimation error, necessary to calculate the parameter update, cannot be computed exactly. Instead, its dynamical approximation is utilized. Its convergence follows from singular perturbation theory.

The solutions to both problems possess some degrees of freedom, thus enabling time-varying weighting of the parameter updates. By introducing dummy variables it is possible to control and trade–off tracking ability versus noise rejection. The output measurement parameter estimation algorithm is tested on a simulation example, where three parameters and both states of a Van der Pol oscillator are estimated using one measurable system state. The example demonstrates parameter and state convergence, initially and after a parameter jump in a noise–free case, and in a noisy case.

2 Problem formulation

In this work we consider Lyapunov–stable systems described by equations of the form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= f(x,u,\theta)\\ y &= x_1,\end{aligned} \tag{1}$$

where $x = [x_1\ x_2\ \ldots x_n]' \in \mathbb{R}^n$ denotes the system state, $u \in \mathbb{R}^p$ the known system input, which is assumed to be a differentiable function of time, y the measured variable, $f(\cdot,\cdot,\cdot) \in \mathcal{C}^1$ a known smooth nonlinear function, and $\theta \in \Omega \subset \mathbb{R}^m$ the unknown parameter to be identified, with Ω a compact set in $\mathbb{R}^m$. Note that the system (1) is a special case of the class of systems considered in [4], where also conditions to transform a general nonlinear system into the form (1) are discussed. For this special class of nonlinear systems we consider the problem of parameter estimation. A possible way of handling such a problem is as follows.

Problem 1 *(Full Information Parameter Estimation) Given the system* (1)*, find a family of full information parameter estimation methods (FIPEM) described by equations of the form*

$$\begin{aligned}\dot{\hat{\theta}} &= h_1(x,u,\xi,\hat{\theta},\dot{x}_n),\\ \dot{\xi} &= h_2(x,u,\xi,\hat{\theta},\dot{x}_n),\end{aligned} \tag{2}$$

where $\hat{\theta} \in \mathbb{R}^m$, and $\xi \in \mathbb{R}^n$, such that, as $t \to \infty$,

$$\begin{aligned}\xi &\to x,\\ \hat{\theta} &\to \bar{\theta},\\ f(\xi,u,\hat{\theta}) &\to f(x,u,\theta).\end{aligned}$$

Note that, throughout the paper, $\hat{\theta}$ denotes the estimate of the parameter θ. The above problem is used to formulate and solve the following more natural, and more difficult, problem.

Problem 2 *(Output Measurement Parameter Estimation) Given the system* (1)*, find a family of output measurement parameter estimation methods (OMPEM) described by equations of the form*

$$\begin{aligned}\dot{\hat{\theta}} &= h_1(x_1,u,\xi,\hat{\theta},\eta),\\ \dot{\xi} &= h_2(x_1,u,\xi,\hat{\theta},\eta),\\ \dot{\eta} &= h_3(x_1,u,\xi,\hat{\theta},\eta),\end{aligned} \tag{3}$$

where $\hat{\theta} \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^n$, such that, for any pre-given $\epsilon > 0$, one has

$$\begin{aligned}\|\xi - x\|_2 &< \epsilon\\ \eta &\to 0,\\ \hat{\theta} &\to \bar{\theta},\\ \|f(\xi,u,\hat{\theta}) - f(x,u,\theta)\|_2 &< \epsilon.\end{aligned}$$

Remark 1 The variable η introduced in the OMPEM is necessary to build an estimation of the unmeasured system states, *i.e.* x_2 through x_n. Observe, moreover, that any OMPEM can be regarded as an adaptive observer for the system (1), see [11].

Remark 2 Since we do not impose any persistence of excitation conditions, solving FIPEM and OMPEM does not guarantee that $\hat{\theta} \to \theta$ as $t \to \infty$. However, it must be noted that, in the case of nonlinear parameter dependence, even the persistence of excitation conditions do not ensure convergence of the parameter estimation to the real value. In what follows we do not address the (difficult) problem of identifiability and we content ourself with an estimation which is able to *reproduce* the dynamics of the system from which the data are generated.

3 Full information parameter estimation

In the present section we give a sufficient condition for the solvability of Problem 1. In particular we give an explicit formula for a FIPEM which solves the problem.

Theorem 1
Consider the system (1). Assume the following.

(H1) For all $(\xi,u,\hat{\theta}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$,

$$\|f_{\hat{\theta}}(\xi,u,\hat{\theta})\| \neq 0. \tag{4}$$

Then the Full Information Parameter Estimation Problem (Problem 1) is solvable.

Proof. Let $h_2(\cdot)$ be defined by

$$\dot{\xi}_1 = \xi_2 \qquad \cdots \qquad \dot{\xi}_n = f(\xi,u,\hat{\theta}). \tag{5}$$

Note now that there exists a function $A_1(x,\xi,u,\hat{\theta})$ such that

$$\left| f_x(x,u,\theta)\dot{x} + f_u(x,u,\theta)\dot{u} - f_\xi(\xi,u,\hat{\theta})\dot{\xi} - f_u(\xi,u,\hat{\theta})\dot{u} \right| < A_1(x,\xi,u,\hat{\theta}), \tag{6}$$

for all $\theta \in \Omega$. Let $\dot{\hat{\theta}} = h_1(\cdot)$ be any function such that

$$f_{\hat{\theta}}(\xi, u, \hat{\theta})\dot{\hat{\theta}} = \frac{1}{p_n}\left(e_2 + \sum_{i=2}^{n} p_{i-1}e_{i+1}\right) + A_1(x, \xi, u, \hat{\theta})\,\mathrm{sgn}\left(e_1 + \textstyle\sum_{i=2}^{n+1} p_{i-1}e_i\right), \tag{7}$$

where $e \in \mathbb{R}^{n+1}$, defined by

$$e = \begin{bmatrix} x - \xi \\ f(x, u, \theta) - f(\xi, u, \hat{\theta}) \end{bmatrix} = \begin{bmatrix} x - \xi \\ \dot{x}_n - f(\xi, u, \hat{\theta}) \end{bmatrix} \tag{8}$$

is the state and dynamics estimation error, and the constants $p_i > 0$ are such that all roots of the polynomial $P(s) = 1 + \sum_{i=1}^{n} p_i s^i$ are in the left half plane. Consider the following weighted model error $\lambda \in \mathbb{R}$

$$\lambda \doteq e_1 + \sum_{i=2}^{n+1} p_{i-1}e_i. \tag{9}$$

Due to (1), (5), and (9), the time derivative of the error (9) becomes

$$\dot{\lambda} = e_2 + \sum_{i=2}^{n} p_{i-1}e_{i+1} + p_n\dot{e}_{n+1}, \tag{10}$$

where, from (8),

$$\begin{aligned} \dot{e}_{n+1} = {} & f_x(x, u, \theta)\dot{x} + f_u(x, u, \theta)\dot{u} - f_\xi(\xi, u, \hat{\theta})\dot{\xi} \\ & - f_u(\xi, u, \hat{\theta})\dot{u} - f_{\hat{\theta}}(\xi, u, \hat{\theta})\dot{\hat{\theta}}. \end{aligned} \tag{11}$$

Hence, from (11) and (7) we get

$$\begin{aligned} \frac{\dot{\lambda}}{p_n} = {} & f_x(x, u, \theta)\dot{x} + f_u(x, u, \theta)\dot{u} - f_\xi(\xi, u, \hat{\theta})\dot{\xi} - \\ & f_u(\xi, u, \hat{\theta})\dot{u} - A_1(x, \xi, u) \cdot \mathrm{sgn}(\lambda). \end{aligned} \tag{12}$$

Note that all quantities appearing in the RHS of (7) are known and that $f_{\hat{\theta}}(\xi, u, \hat{\theta})$ is known as well. According to (6), the last four terms in the RHS of (12) are majorized by the first term. Hence, λ converges to zero in finite time. As a consequence, $\dot{\hat{\theta}}$ converges to zero in finite time and $e = x - \xi$ converges exponentially to zero. Which concludes the proof. ◁

Remark 3 The function $h_1(\cdot)$ satisfying (7), always exists by Hypothesis (H2), but is not unique. The resulting freedom can be exploited to improve parameter convergence and noise rejection, what may require introduction of some optimality measures. Here we propose a simple choice of $h_1(\cdot)$. Dropping the arguments, we can rewrite equation (7) in the form $f_{\hat{\theta}}\dot{\hat{\theta}} = \varphi$. Such an equation admits the solution $\dot{\hat{\theta}} = M f_{\hat{\theta}}'\varphi/(f_{\hat{\theta}} M f_{\hat{\theta}}')$, where $M \in \mathbb{R}^{m\times m}$ is a possibly time-varying, positive definite matrix. Note that such solution minimizes $||\dot{\hat{\theta}}||_{M^{-\frac{1}{2}}}$.

Remark 4 As discussed in [14], the discontinuous function sgn$(\cdot)$ appearing in (7) can be replaced by a $\mathcal{C}^1$ approximation, leading to an improved numerical implementation of the proposed method.

Remark 5 Assumption (H2) can be relaxed by extending the parameter vector θ with a dummy parameter θ^*, *i.e.* $\theta_{ext} \doteq [\theta', \theta^*]'$, and modifying the last equation of the system (1) to

$$\dot{x}_n = f(x, u, \theta) + \theta^*.$$

Obviously, for this modified system it follows

$$\frac{\partial\left(f(x, u, \theta) + \theta^*\right)}{\partial\theta_{ext}} = [\star\ \star\ \star\ \cdots\ \star\ 1],$$

hence $||f_{\theta^*}|| > 0$ for all (x, u, θ_{ext}).

4 Output measurement parameter estimation

In practical situations, most of the system states are not measurable. Hence, the state estimation error vector (8) cannot be computed exactly. Here we show how to solve the parameter estimation problem in the output measurement form, assuming that only the state x_1 is known.

Theorem 2
Consider the system (1). Assume the following.

(H1) For all $(\xi, u, \hat{\theta}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$,

$$||f_{\hat{\theta}}(\xi, u, \hat{\theta})|| > 0. \tag{13}$$

Then the Output Measurement Parameter Estimation Problem (Problem 2) is solvable.

Proof. The proof and the construction of the OMPEM are similar to those in Theorem 1, hence they are omitted for shortness. ◁

5 A simple example: the Van der Pol oscillator

In this section we demonstrate applicability of the (OMPEM) on a simulation example. Consider the equations describing an autonomous Van der Pol oscillator, *i.e.*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= 2\omega\nu\left(1 - \mu x_1^2\right)x_2 - \omega^2 x_1. \end{aligned} \tag{14}$$

Such a system is a simple nonlinear system, which is also nonlinear in the parameters ω, ν, and μ. Hence, it often serves as an example in papers dealing with parametric identification of continuous–time nonlinear systems, see [6], [1], and [3]. The parameters, chosen in the simulations as $\theta' = [\omega\ \nu\ \mu] = [0.5\ 1\ 2]$, are to be identified by using only the output measurement, corrupted by an additive noise, *i.e.*, $x_1 + v$. Both the system and the identification algorithm (defined in Theorem 2) are simulated in ACSL. The simulation time is much bigger than the limit cycle period; the data collected over this interval constitute persistent excitation. The weighting matrix, introduced in Remark 3, is chosen such that the only estimated nonlinear parameter $\theta_1 = \omega$ is most likely to be tuned.

The first experiment is performed without any noise, *i.e.*, $v \equiv 0$. Figure 1 (top) demonstrates the trajectory convergence, while Fig. 2 shows the parameter convergence.

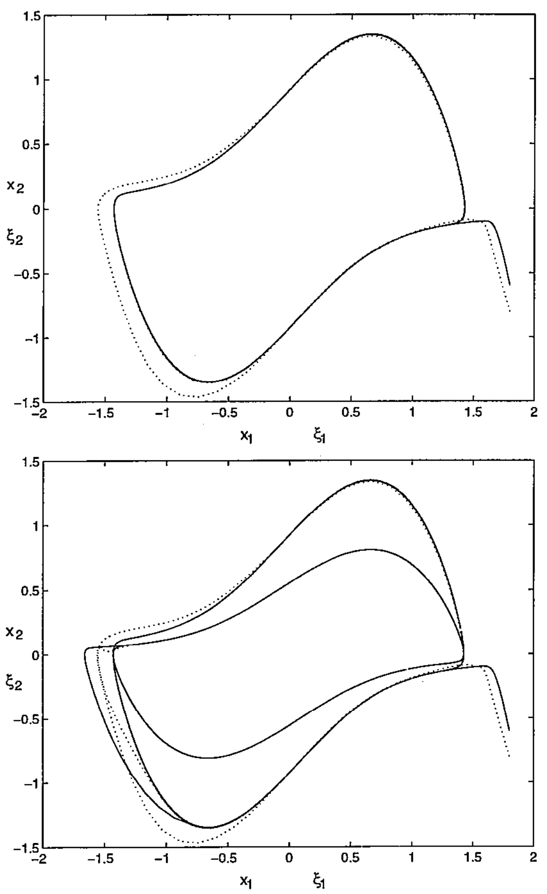

Figure 1: Phase portraits of the Van der Pol system (solid) and its identified model (dotted), in the noise-free case (top) and for parameter jump (bottom).

In the second experiment, the only measured state variable is corrupted by an additive disturbance $v(k) \in$

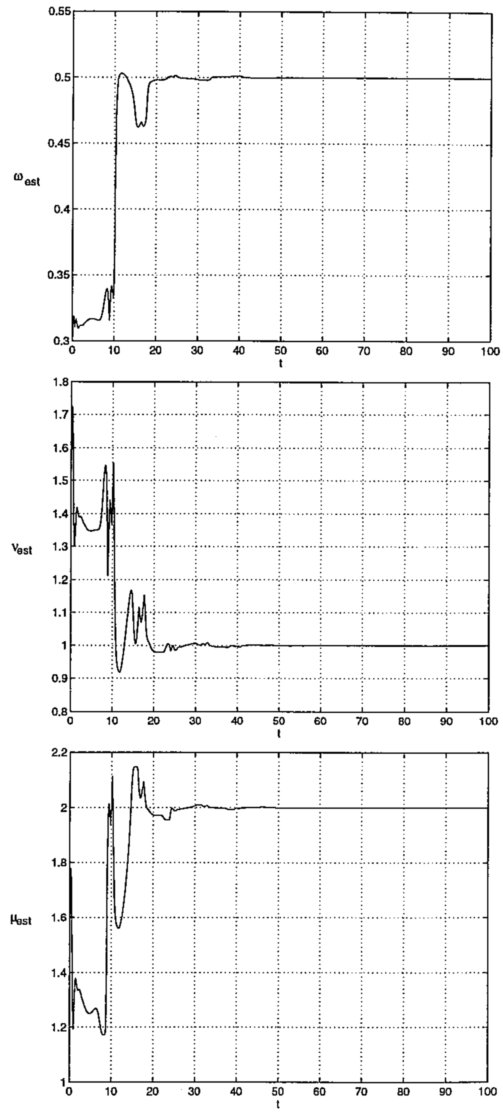

Figure 2: Estimated parameters, $\hat{\omega}$ (top), $\hat{\nu}$ (center), $\hat{\mu}$ (bottom), in the noise-free case.

$N(0, 0.025)$. Comparing Figs. 2 and 3, we note that in the "noisy" case the estimates never converge. This is a common property for parameter identification algorithms which can track time–varying dynamics, a property which is also shared by the proposed algorithm. As a matter of fact such a property is demonstrated in the third experiment. The system parameters vary according to the table:

time interval	ω	ν	μ
$0 < t \leq 70$	0.5	1	2
$70 < t \leq 150$	0.3	1	2

Figure 4 shows parameter values and estimates, while Fig. 1 (bottom) displays the corresponding phase portraits.

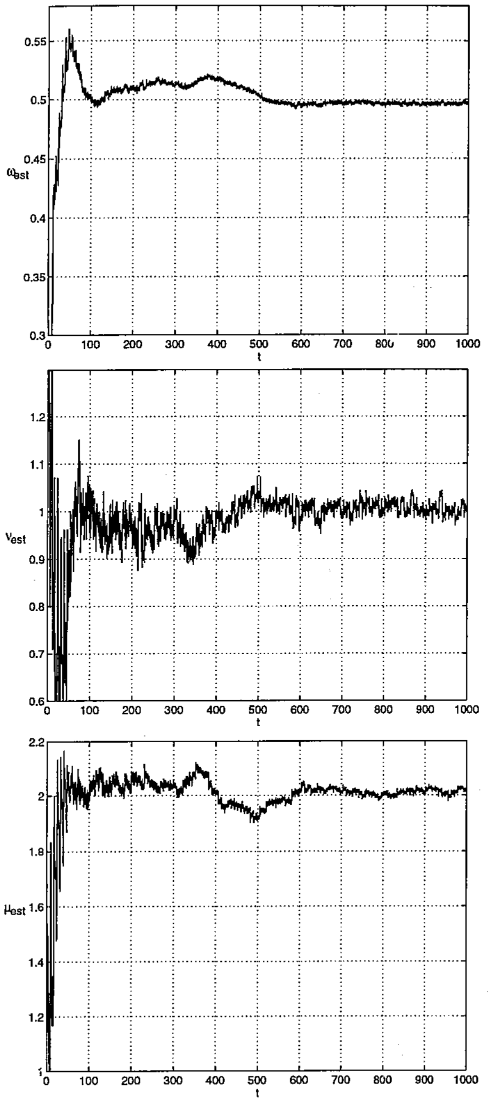

Figure 3: Estimated parameters, $\hat{\omega}$ (top), $\hat{\nu}$ (center), $\hat{\mu}$ (bottom), in the noisy case.

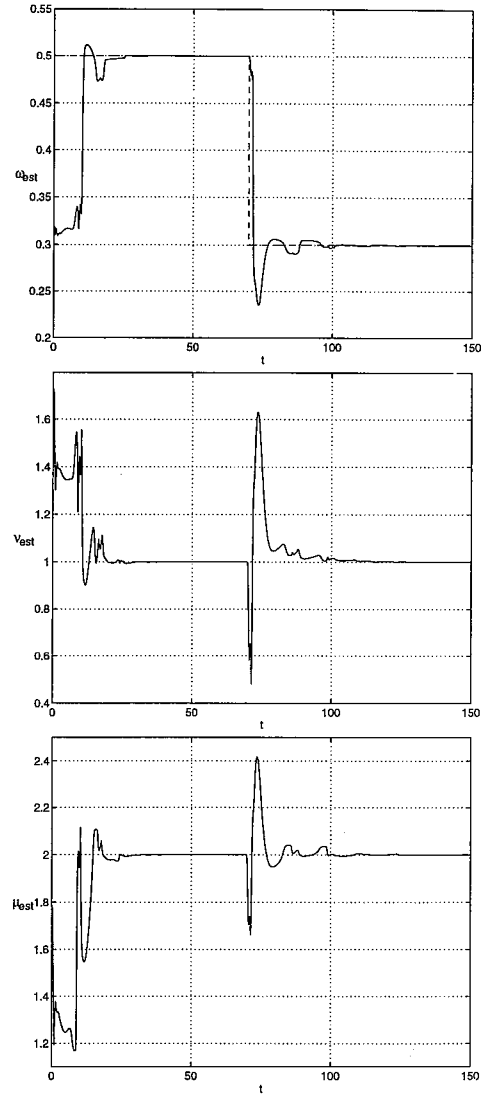

Figure 4: Estimated parameters, $\hat{\omega}$ (top), $\hat{\nu}$ (center), $\hat{\mu}$ (bottom), for parameter jump.

6 Conclusions

This paper addresses the problem of on-line parameter estimation for nonlinear continuous time dynamical systems of known structure. Two parameter estimation problems for the noise–free case are formulated and then explicitly solved. The resulting parameter estimation algorithms incorporate an observer; the adjustment of the observer parameters is considered as a stabilization problem for an uncertain system and is solved using sliding mode method. In the full information parameter estimation problem it is assumed that all the system states and their derivatives be known. In the output measurement parameter estimation problem we assume that only one state is known, the other states are reconstructed using an adaptive observer. The proposed algorithm is applied for parameter estimation of the Van der Pol oscillator, where parameter and state convergence is demonstrated, initially and after a parameter jump in a noise–free case, and in a noisy case.

The proposed method has also been used to estimate the parameters of a pendulum on a cart. The result of the experiments, carried on the pendulum built at the Automatic Control Laboratory of the ETH-Zurich, will be reported in the final version.

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