New Stability Test For 2-D Systems

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Abstract

In this paper, a new stability test for two-dimensional (2-D) discrete systems is presented. The proposed test is based on a simple 1-D test which is simpler but equivalent to that of Schur-Cohn. The test can be easily automated via a suitable computer code. Examples are given.

1. Introduction

2-D systems theory has found applications in a number of areas, for example image enhancement, digital image processing, computerised tomography, remote control, modern circuit design and so on, [4], [6]. Many mathematical problems in 2-D systems theory have already attracted increasing attention in the areas of analysis, synthesis, stability, factorization, controllability, observability, minimality, feedback control and filter design [6].

Stability of two-dimensional (2-D) systems almost arises in all the above applications. Excellent overviews of the stability problem and of the theorems and the tests associated with it can be found in [4÷6]. Some of these theorems check if some 1-D or 2-D polynomials is devoid of zeros in appropriate regions of ***C*** (the set of the complex numbers) or ***CxC*** respectively.

A shift-invariant causal single-input single output (SISO) 2-D system can be described by the transfer function:

$$G(z_1,z_2)=\frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{1}$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are coprime polynomials in the independent complex variables z_1 and z_2, It is assumed that there are no nonessential singularities of the second kind on the closed unit bidisk [4], [5], i.e. there are no points (z_1,z_2) with $|z_1|\le 1$ and $|z_2|\le 1$ such that $A(z_1,z_2)=B(z_1,z_2)=0$.

It is well known that the system (1) is (Hurwitz) stable, [4÷6], if and only if

$$B(0,z_2)\neq 0, \qquad \text{for} \quad |z_2|\le 1 \tag{2.1}$$

$$B(z_1,z_2)\neq 0, \qquad \text{for} \quad |z_1|\le 1, \quad |z_2|=1 \tag{2.2}$$

Additionally, the polynomial $B(z_1,z_2)$ is said to be *VSHP* (Very Strict Hurwitz Polynomial) if and only if (2.1) and (2.2) are fulfilled. Condition (2.1) is relatively easy to check using any 1-D stability test. Condition (2.2) is more difficult since it includes two variables. For Condition (2.2), several tests already exist which check various conditions of the coefficients of $f(z_1,z_2)$, [1], [3], [4÷7], [11]. We denote

$$B(z_1,z_2)=\sum_{i=0}^{n_1} b_i(z_2)z_1^i \tag{3}$$

By interchanging the variables z_1 and z_2, other Conditions symmetrical to (2.1) and (2.2) are found.

In this paper, a new test for checking Condition (2.2) is proposed. The analysis is based on a result concerning 1-D systems stability [2]. The theoretical results are illustrated by two examples.

2 Main Results

Let the degree in z_1 and z_2 of $B(z_1,z_2)$ be n_1 and n_2 respectively. Then $B(z_1,z_2)\neq 0$, for $|z_1|\le 1$, $|z_2|=1$ if and only if

$B_1(z_1,z_2)\neq 0$, for $|z_1|\geq 1$, $|z_2|=1$ where $B_1(z_1,z_2)=z_1^{n_1}B(z_1^{-1},z_2)$. The polynomial $B_1(z_1,z_2)$ is written as a polynomial in z_1 with coefficients which are polynomials in z_2:

$$B_1(z_1,z_2)=\sum_{i=0}^{n_1} b_{1,i}(z_2)z_1^i \tag{4}$$

with $b_{1,i}(z_2)=b_{n_1-i}(z_2)$ and $i=0,\ldots,n_1-1$

The following Theorem summarises a result obtained by Barnett [2]

Theorem 1: The polynomial $a(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ does not have any zeros outside the unit circle $|z|=1$ if and only if the matrix $C=\bar{a}_0A^n+\bar{a}_1A^{n-1}+\ldots+\bar{a}_{n-1}A+I_n$ (the bar indicates complex conjugate) is positive definite, where

$$A=\begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & . & . & 1 \\ -a_0 & -a_1 & -a_2 & . & . & -a_{n-1} \end{bmatrix} \tag{5}$$

and I_n is the $n\times n$ identity matrix.

Based on this Theorem, we can formulate a Test for checking Condition (2.2). The Test is summarised by the following Theorem:

Theorem 2: $B(z_1,z_2)\neq 0$, for $|z_1|\leq 1$, $|z_2|=1$, if and only if the matrix $C(z_2)=\bar{a}_0(z_2)A^{n_1}+\bar{a}_1(z_2)A^{n_1-1}+\ldots+\bar{a}_{n_1-1}(z_2)A+I_{n_1}$ is positive definite for all z_2 with $|z_2|=1$, where

$$A=\begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & . & . & 1 \\ -a_0(z_2) & -a_1(z_2) & -a_2(z_2) & . & . & -a_{n_1-1}(z_2) \end{bmatrix} \tag{6}$$

and $a_i(z_2)=\dfrac{b_{1,i}(z_2)}{b_{1,n_1}(z_2)}$ or equivalently

$$a_i(z_2)=\frac{b_{n_1-i}(z_2)}{b_0(z_2)} \tag{7}$$

$i=0,\ldots,n_1-1$.

Proof: It follows directly from Theorem 1.

Furthermore, we state another theorem which is an improvement of the Theorem 2.

Theorem 3: $B(z_1,z_2)\neq 0$, for $|z_1|\leq 1$, $|z_2|=1$, if and only if the matrix

$C(z_2)=\bar{a}_0(z_2)A^{n_1}+\bar{a}_1(z_2)A^{n_1-1}+\ldots+\bar{a}_{n_1-1}(z_2)A+I_{n_1}$

(i) is positive definite for a z_0, with $|z_0|=1$,

and (ii) $det\{C(z_2)\}>0$ for all z_2 with $|z_2|=1$,

where A is provided by (6) and (7).

Proof: It was pointed out by Siljak, [12], that for positivity checking of such a matrix, one requires the positivity checking of the matrix at one point, say at z_0, with $|z_0|=1$ and the positivity checking of the determinant for all z_1 with $|z_1|=1$.

Remark 1: The above Theorem provides a Test which constitutes a simplification, since it does not require the matrix $C(z_2)$ to be positive definite for all z_2, with $|z_2|=1$.

Remark 2: One should notice that the matrix C is easily evaluated using the Horner formula as follows

$$C=I_n+\ldots+A\left(\bar{a}_2I+A\left(\bar{a}_1I+\bar{a}_0A\right)\right) \tag{8}$$

Taking into account the special structure of A, for the computation of C, we find a total cost of $n+(3n-2)(n-1)$ multiplications and $n+(3n-2)(n-1)$ additions i.e. $O(3n^2)$ multiplications and $O(3n^2)$ additions. The method of Schur-Cohn includes $n^2(n+1)$ multiplications i.e. $O(n^3)$ multiplications and $n^2(n+1)/2$ additions i.e. $O(n^3/2)$ additions. Therefore, the present method is better than the classical method of Schur-Cohn with respect to the computational complexity.

Example 1: We consider the Example given in [4]

$$B(z_1,z_2) = (12+6z_2)+(10+5z_2)z_1+(2+z_2)z_1^2 \tag{9}$$

Obviously, $B(0,z_2) \neq 0$, for $|z_2| \leq 1$. Therefore Condition (2.1) holds. In order to check Condition (2.2), we form the polynomial:

$$B_1(z_1,z_2) = (2+z_2)+(10+5z_2)z_1+(12+6z_2)z_1^2 \tag{10}$$

So, we obtain

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{2+z_2}{12+6z_2} & -\dfrac{10+5z_2}{12+6z_2} \end{bmatrix} \tag{11}$$

Now, one has

$$C(z_2) = I_2 + A\left(-\frac{10+5\overline{z_2}}{12+6\overline{z_2}} - \frac{2+\overline{z_2}}{12+6\overline{z_2}}A\right) \tag{12}$$

After the necessary algebraic manipulation, one finds

$$C(z_2) = \begin{bmatrix} \dfrac{35}{36} & \dfrac{25}{36} \\ -\dfrac{25}{216} & \dfrac{85}{216} \end{bmatrix} \tag{13}$$

Clearly, this matrix is positive definite. Therefore the 2-D (discrete) system having the characteristic polynomial $B(z_1,z_2)$ is stable. The result agrees with that in [4].

Example 2: Consider the Example given in [9].

$$B(z_1,z_2) = (13-z_2)+(-12-7z_2)z_1+8z_1^2 \tag{14}$$

Since $B(0,z_2) \neq 0$, for $|z_2| \leq 1$, Condition (2.1) clearly holds. In order to check Condition (2.2), we consider the polynomial:

$$B_1(z_1,z_2) = 8+(-12-7z_2)z_1+(13-z_2)z_1^2 \tag{15}$$

So, we obtain

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{8}{13-z_2} & \dfrac{12+7z_2}{13-z_2} \end{bmatrix} \tag{16}$$

Then

$$C(z_2) = I_2 + A\left(-\frac{12+7\overline{z_2}}{13-\overline{z_2}} + \frac{8}{13-\overline{z_2}}A\right) \tag{17}$$

For convenience, if one sets $z_2 = e^{j\,0.5}$, one has only numerical manipulation and finally

$$C\left(e^{j0.5}\right)=\begin{bmatrix}0.5652 & -0.4972+j0.5179\\ 0.3411-j0.3283 & -0.3462+j0.6014\end{bmatrix} \quad (18)$$

The determinant of this matrix is -0.1960<0. Evidently, this matrix is negative definite and the corresponding 2-D (discrete) system is unstable. The result agrees with that of [9].

3 Conclusion

The test and its implementation is based on a result published by Barnett and is easy to use. Note that if one interchanges the role of the variables z_1 and z_2, another equivalent form of the method can be presented. Work is in progress by the author in the area of 2-D stability theorems formulating new methods for 2-D stability margin [10] and necessary conditions for 2-D systems stability [9].

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