Normalized Coprime Factorizations for Discrete-Time Periodic Systems[1]

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Abstract

In this paper the notion of Normalized Coprime Factorization (NCF) for linear discrete-time periodic systems is studied. These systems arise in the study of linear time-varying systems with periodic coefficients. The problem is approached by the study of the linear periodic state-space representation of the system. It is shown that the NCF can be obtained through the solution to discrete-time periodic Riccati equations (DPRE). The properties of the DPRE are used to study the NCF for linear discrete-time periodic systems.

1 Introduction

Development in the theory of H_∞ design and robust stabilization problem has made the notion of normalized coprime factorizations indispensable in control theory. The description of systems using these factors has been shown to have fundamental connections to robust stabilization problem and Hankel-norm approximations [1]. In fact, the use of the NCF for the study of robust stabilization problem has resulted in a close connection between the classical robust stability problem, in the context of H_∞ perturbation, and the recent theory of robust stabilization using the gap metric. This equivalence was explored in [2].

Recently, interest has been developed in the linear time-varying (LTV) systems with periodic coefficients [5]. These systems are used to model multirate plants and digital filters that arise in the description of cyclostationary processes. In [9] an equivalence between m-input, p-output, linear, N-periodic causal, discrete-time systems and a class of discrete linear time invariant (LTI) causal systems was established. In fact, LTI theory serves as a guide to the linear periodic systems and many classical concepts first developed for time-invariant systems have been extended and applied to the periodic case. On the other hand, the results from studying periodic systems provide a guidance or proof to the theory of the LTV system.

In [3], the doubly coprime factorization for linear periodic systems is established based on its equivalent LTI representation. The optimal controller of disturbance rejection for the periodic system is then obtained with the coprime factors. From the results in [3], several questions arise worth further investigation:

1. With the lifting technique, the LTI representation (F_i, G_i, H_i, E_i) (see Section 2) has the feedforward term E_i, which is a lower triangular matrix, and it is not zero. Therefore, the doubly coprime factorization defined in [3] need to be modified.

2. The method in [3] employs the standard techniques for computing the steady-state stabilizing solution for the associated algebraic Riccati equations. However, when computing F_i, which will be defined in (2.2), successive matrix multiplication is required. In the case when the state matrices A_i are ill-conditioned, such successive matrix multiplication may produce inaccurate results, especially when the period of the system is large.

3. When the LTI optimal controller is built, it is natural to investigate how we can connect the controller to the original periodic system with inverse lifting. In particular, is the LTI controller lifting-invertible? The techniques of inverse lifting are not addressed in [3]. When lifting the periodic system, the input and output are expanded over one period. When inverse lifting the LTI controller, we want the controller to be periodic so that it can be connected with the original system. If the inverse lifting techniques exist, they might bring more computational error to the designed system.

In this paper we study the problem of the NCF for linear periodic systems with its linear periodic state-space representations. We know that there exists an NCF for discrete LTI system. However, NCF for LTI system is not useful in getting the NCF of discrete-time periodic systems because no information is given for its original periodic presentation. Therefore, it is necessary to develop a direct approach by using DPRE for the NCF of discrete-time periodic systems.

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The positive semi-definite stabilizing solution to the DPRE via a deflating subspace method is given in [4]. This method employs the cyclic QZ algorithm and is numerical stable and perturbation insensitive.

The organization of the paper is as follows. In Section 2 we review the background on the discrete-time periodic systems. In Section 3 we study the notion of coprime factorization for the discrete-time periodic systems and provide necessary and sufficient conditions under which the coprime factors exist. In Section 4 we present some results on the Riccati equations for these systems and the state-space formulae for the NCFs. Finally, Section 5 concludes the paper.

2 Preliminaries

The linear discrete-time periodic system represented by Σ_p, is described by the following equations:

$$\Sigma_p: \begin{cases} x_{i+1} = A_i x_i + B_i u_i, \\ y_i = C_i x_i + D_i u_i, \end{cases} \tag{2.1}$$

where $A_{i+N} = A_i \in \mathbb{R}^{n\times n}$, $B_{i+N} = B_i \in \mathbb{R}^{n\times m}$, $C_{i+N} = C_i \in \mathbb{R}^{p\times n}$, $D_{i+N} = D_i \in \mathbb{R}^{p\times m}$, and $N \in Z^+$. The system is said to be *reversible* if each A_i is nonsingular. The state-transition matrix of the system is given as

$$\Phi(i,l) = \begin{cases} I, & i = l \\ A_{i-1}A_i \cdots A_{l+1}A_l, & i > l. \end{cases}$$

$\Phi(i,l)$ is undefined for $i < l$. We assume throughout this paper that the periodic systems are reachable, observable and reversible. Further, the transition matrix is periodic with period N, i.e., $\Phi(i+N, l+N) = \Phi(i,l)$. The *monodromy* matrix F_i is defined to be equal to the transition matrix over one period, that is, $F_i = \Phi(i+N, i)$. The eigenvalues of each F_i are the same and are defined as the *characteristic multipliers* of the system.

It is well-known that the periodic system in (2.1) has an equivalent dynamic representation associated with N linear time-invariant systems of the form:

$$\begin{aligned} \mathcal{X}^i_{l+1} &= F_i \mathcal{X}^i_l + G_i \mathcal{U}^i_l, \\ \mathcal{Y}^i_l &= H_i \mathcal{X}^i_l + E_i \mathcal{U}^i_l, \end{aligned}$$

where

$$\mathcal{X}^i_l = x_{i+lN}, \quad \mathcal{U}^i_l = \begin{bmatrix} u_{i+lN} \\ u_{i+lN+1} \\ \vdots \\ u_{i+(l+1)N-1} \end{bmatrix},$$

$$\mathcal{Y}^i_l = \begin{bmatrix} y_{i+lN} \\ y_{i+lN+1} \\ \vdots \\ y_{i+(l+1)N-1} \end{bmatrix},$$

$$F_i = \Phi(i+N, i), \tag{2.2}$$

$$G_i = \begin{bmatrix} B_i^T \Phi^T(i+N, i+1) \\ B_{i+1}^T \Phi^T(i+N, i+2) \\ \cdots \\ B_{i+N-1}^T \end{bmatrix}^T,$$

$$H_i = \begin{bmatrix} C_i \\ C_{i+1}\Phi(i+1, i) \\ \vdots \\ C_{i+N-1}\Phi(i+N-1, i) \end{bmatrix},$$

$$E_i = \begin{bmatrix} D_i & \cdots & 0 & 0 \\ C_{i+1}B_i & \cdots & 0 & 0 \\ & \vdots & & \\ & \cdots & & D_{(i+N-1)} \end{bmatrix}.$$

Definition 2.1 *We say that the solution to the homogeneous periodic system $x_{i+1} = A_i x_i$ on $i \geq 0$ is exponentially stable if $\exists\ \rho \in [0,1)$ and $m > 0$ such that for all $k_0 \in Z^+$*

$$\|\Phi(k, k_0)\| \leq m\rho^{k-k_0}, \qquad \forall\ k \geq k_0.$$

In this paper, for simplicity we will say that the system is stable if it is exponentially stable. A Lyapunov type of characterization of the stability of the system can be obtained easily and is summarized in the following Lemma.

Lemma 2.1 *Let Σ_p be a discrete-time periodic system defined as in (2.1). Then, the following statements are equivalent*

1. *The system is stable.*
2. *There is a positive-definite time-varying matrix P_i satisfying*

   $$\alpha I \leq P_i \leq \beta I$$

   for some positive constants α and β such that the following conditions are satisfied

   $$x^T \left[A_i^T P_{i+1} A_i - P_i\right] x \quad \leq \quad -\lambda_i \|x\|^2$$

   for some positive function λ_i and $x \in \mathbb{R}^n$.

Proof: Lemma 2.1 is a direct result of [7]. □

Since the system is entirely described by the N LTI discrete-time systems, at each time i we define the reachability and observability Lyapunov equations

$$F_i^T P_i F_i + H_i^T H_i = P_i, \tag{2.3}$$

$$F_i Q_{i-1} F_i^T + G_i G_i^T = Q_{i-1}. \tag{2.4}$$

We can also define the periodic Lyapunov equations as:

$$A_i^T P_{i+1} A_i + C_i^T C_i = P_i, \tag{2.5}$$

$$A_i Q_{i-1} A_i^T + B_i B_i^T = Q_i. \tag{2.6}$$

The relationship between the above two sets of Lyapunov equations is given in the following theorem:

Theorem 2.1 *Let P_i, Q_i be a set of matrices with period N satisfying $2N$ Lyapunov equations (2.3) and (2.4) which are related with the LTI representation $(F_i,\ G_i,\ H_i,\ E_i)$ of the periodic system. Then this set of equations is equivalent to the $2N$ periodic Lyapunov equations (2.5) and (2.6) defined with respect to the original representation $(A_i,\ B_i,\ C_i,\ D_i)$ of the periodic system.*

Proof: The proof follows from the definitions of F_i, H_i, and G_i and the fact that the system is reversible, that is, A_i's are nonsingular. □

According to Theorem 2.1, we can avoid solving all $2N$ Lyapunov equations. In particular, we can solve only two Lyapunov equations and compute the rest of the solution recursively. Theoretically, we can use either the LTI Lyapunov equations or the LTV Lyapunov equations to discuss the stability of the discrete-time periodic system.

The main theme of the paper is to establish the existence of the coprime factors of the periodic systems and we will show that the conditions for the existence of the coprime factors can be expressed in terms of the notions of stabilizability and detectability of the system. These properties of the system are expressed in terms of the closed-loop stability of state and output feedback of the system. The system is detectable and stabilizable if there are bounded sequences K_i, L_i such that the closed-loop systems Σ_d and Σ_s defined below are stable. We define the periodic system Σ_d as

$$x^d_{i+1} = (A_i - L_iC_i)\, x^d_i$$

and the periodic system Σ_s is defined as

$$x^s_{i+1} = (A_i - B_iK_i)\, x^s_i.$$

The Lyapunov criteria for the stability of the systems in terms of the solution to the Lyapunov equations are given as follows:

Lemma 2.2 *If the discrete-time periodic system Σ_p is detectable and if there exists a periodic solution $P_i \geq 0$ to the N-periodic Lyapunov equation given by (2.5) then the system is stable.*

Lemma 2.3 *If the discrete-time periodic system Σ_p is stabilizable and if there exists a periodic solution $Q_i \geq 0$ to the N-periodic Lyapunov equation given by equation (2.6) then the system is stable.*

In order to establish the existence of the coprime factors of discrete-time periodic systems we need a few ancillary results. The following result establishes that a discrete-time periodic system is stabilizable by static state feedback *if and only if* it is stabilizable by dynamic state feedback.

Lemma 2.4 *Let the discrete-time periodic system Σ_p be described by equation (2.1). It is stabilizable (by static state feedback) if and only if it is stabilizable by dynamic state feedback.*

The above Lemma can now be extended easily to the case of output injection. We summarize this result below:

Corollary 2.1 *The discrete-time periodic system defined in (2.1) is stabilizable (by static output injection) if and only if it is stabilizable by dynamic output injection.*

Once the conditions for the existence of coprime factors are established, we will study the notion of "normalized" coprime factors. The conditions under which a coprime factorization is normalized will be derived. In this context we introduce the notion of an all-pass system. Suppose that the discrete-time periodic system $\Sigma_p : \mathcal{U} \mapsto \mathcal{Y}$ and $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces such as $\ell_2[0,T]$ or $\ell_2[0,\infty)$.

A system is said to be all-pass if it satisfies the following relationship:

$$\|\Sigma_p u\|^2_{\mathcal{Y}} = \|u\|^2_{\mathcal{U}},$$

for all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. Equivalently, the system satisfies $\Sigma^*_p \Sigma_p = I$ where Σ^*_p represents the adjoint of the system. We first define and compute the state-space representation of the adjoint system Σ^*_p and then characterize an all-pass system using the computed state-space representations.

Lemma 2.5 *The adjoint of the system Σ_p denoted by Σ^*_p satisfies the relation*

$$\langle \Sigma_p u, y\rangle_{\mathcal{Y}} = \langle u, \Sigma^*_p y\rangle_{\mathcal{U}}, \quad \forall\ u \in \mathcal{U},\ y \in \mathcal{Y}.$$

Furthermore, the state-space representation of the adjoint system for the system Σ_p is given by the following equations:

$$\Sigma^*_p: \left\{ \begin{array}{lcl} p_{i-1} & = & A_i^T p_i + C_i^T v_i, \\ w_i & = & B_i^T p_i + D_i^T v_i. \end{array} \right.$$

Since we will use the notion of all-pass systems to prove the results for NCFs, we first present necessary and sufficient conditions under which a periodic system is all-pass.

Theorem 2.2 *[6] Let system Σ_p be defined in (2.1) and assume the periodic solution to the Lyapunov equations (2.5) $P_i > 0$ exists. Then, Σ_p is all-pass if and only if*

$$\begin{aligned} A_i^T P_{i+1} A_i + C_i^T C_i &= P_i, && (2.7)\\ B_i^T P_{i+1} A_i + D_i^T C_i &= 0, && (2.8)\\ B_i^T P_{i+1} B_i + D_i^T D_i &= I. && (2.9) \end{aligned}$$

Theorem 2.2 established a way to construct the NCF for the periodic systems. As its result, we can find the NCF of discrete-time periodic systems based on the state-space representation (A_i, B_i, C_i, D_i). In next section we will study the notion of coprime factorization of these systems.

3 Doubly Coprime Factors

Suppose that $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W}$ are signal spaces such as $\ell_2[0,T]$ or $\ell_2[0,\infty)$. The system Σ_p is a map from the input space, $\mathcal{U}$ to the output space, $\mathcal{Y}$ and is denoted as

$$\Sigma_p : \ \mathcal{U} \rightarrow \mathcal{Y},$$

which are coprime as follows:

$$\begin{array}{ll} N_i : \mathcal{W} \rightarrow \mathcal{Y}, & M_i : \mathcal{W} \rightarrow \mathcal{U}, \\ \tilde{N}_i : \mathcal{U} \rightarrow \mathcal{W}, & \tilde{M}_i : \mathcal{Y} \rightarrow \mathcal{W}. \end{array}$$

We define the right and left factorizations of the system Σ_p respectively as:

$$\Sigma_p = N_i M_i^{-1} = \tilde{M}_i^{-1} \tilde{N}_i. \tag{3.1}$$

The factors are coprime if and only if there exist stable maps X_i, Y_i, $\tilde{X}_i$ and $\tilde{Y}_i$ defined as:

$$\begin{aligned} X_i &: \mathcal{Y} \rightarrow \mathcal{W}, \quad Y_i : \mathcal{U} \rightarrow \mathcal{W}, \quad . & (3.2) \\ \tilde{X}_i &: \mathcal{W} \rightarrow \mathcal{U}, \quad \tilde{Y}_i : \mathcal{W} \rightarrow \mathcal{Y}, & (3.3) \end{aligned}$$

such that the following relations are satisfied [12]:

$$\begin{aligned} X_i N_i + Y_i M_i &= I, \\ \tilde{N}_i \tilde{X}_i + \tilde{M}_i \tilde{Y}_i &= I. \end{aligned}$$

Furthermore, the factorization is said to be doubly coprime if the following condition is satisfied:

$$\begin{bmatrix} Y_i & X_i \\ -\tilde{N}_i & \tilde{M}_i \end{bmatrix} \begin{bmatrix} M_i & -\tilde{X}_i \\ N_i & \tilde{Y}_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.4}$$

For LTI and LTV systems it has been shown that necessary and sufficient conditions for the existence of the coprime factors is that the system is stabilizable and detectable [10, 11]. We show that the coprime factors exist for discrete-time periodic systems under the same conditions of stabilizability and detectability. We first assume that the system is stabilizable and detectable and provide a state-space representation for the factors. Since the system is assumed to be stabilizable, there exists a stabilizing state feedback K_i. Since the system is detectable, there is a stabilizing output injection F_i. We now define the stable systems which are coprime as follows:

Proposition 3.1 *Let system Σ_p be defined as in equation (2.1). Define*

$$\begin{bmatrix} N_i \\ M_i \end{bmatrix} \triangleq \left[\begin{array}{c|c} A_i - B_i K_i & B_i U_i \\ \hline C_i - D_i K_i & D_i U_i \\ -K_i & U_i \end{array}\right],$$

$$\begin{bmatrix} Y_i & X_i \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A_i - F_i C_i & B_i - F_i D_i & F_i \\ \hline U_i^{-1} K_i & U_i^{-1} & 0 \end{array}\right],$$

$$\begin{bmatrix} \tilde{N}_i & \tilde{M}_i \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A_i - F_i C_i & B_i - F_i D_i & -F_i \\ \hline V_i C_i & V_i D_i & V_i \end{array}\right],$$

$$\begin{bmatrix} \tilde{Y}_i \\ \tilde{X}_i \end{bmatrix} \triangleq \left[\begin{array}{c|c} A_i - B_i K_i & F_i V_i^{-1} \\ \hline C_i - D_i K_i & V_i^{-1} \\ K_i & 0 \end{array}\right],$$

where U_i and V_i are non-singular periodic sequences. Then, the factors N_i, M_i are right coprime, $\tilde{N}_i$, $\tilde{M}_i$ are left coprime and above equations satisfy equation (3.4).

Proof: The proof follows easily and is along the lines of Proposition 2 in [8]. □

The above proposition provides the sufficient conditions for the existence of coprime factors which are provided by construction. We now show that these conditions are also necessary for the existence of coprime factors.

Proposition 3.2 *Let the discrete-time periodic system Σ_p possess left and right coprime factors as given by (3.1), which satisfy the conditions (3.2) and (3.3). Then, the system Σ_p is stabilizable and detectable.*

Proof: The proof is similar to that of Theorem 4.6 of [10] and can be obtained by modifying the arguments presented there. □

We combine the above two propositions to give the main result of this chapter as follows:

Theorem 3.1 *A discrete-time periodic system Σ_p described in (2.1) possesses a coprime factorization if and only if is stabilizable and detectable.*

In this section we studied the notion of doubly coprime factorizations for discrete-time periodic systems and shown that the factors can be obtained from by state feedback and output injection. We have studied necessary and sufficient conditions for the construction of the coprime factors. If the feedback is chosen to be "optimal" in the sense that the feedback gain is obtained through the solution of a quadratic optimization problem, then the coprime factors can be shown to be "normalized". In next section we will study Riccati equations for discrete-time periodic systems. The connection between the coprime factors and the solution to the Riccati equation in order to obtain the NCFs will be drawn later in this paper.

4 Discrete-Time Periodic Riccati Equations and Normalized Coprime Factorizations

In this section we will study the optimization problem which leads to the DPRE. We consider the finite-horizon problem to derive the Riccati equation. We then study the conditions under which the solution to the equation is bounded and the closed-loop system is stable in the infinite-horizon case. The cost index chosen for the optimization problem is a non-negative quadratic function defined by

$$J(0,T) \;=\; \sum_{i\in[0,T]} \frac{1}{2}\left[y_i^T y_i + u_i^T u_i\right]. \qquad (4.1)$$

The initial condition of the state is assumed to be known. We now provide the modified results from [4] concerning the optimization of the above cost function with respect to the periodic system with the feedthrough term D_i.

Lemma 4.1 *[4] Assume that (A_i, B_i) is stabilizable and (A_i, C_i) is detectable for all time i. The optimal input to the discrete-time periodic system Σ_p which minimizes the cost function $J(0,T)$ defined in (4.1) is given by the optimal periodic state feedback*

$$u_i^\star \;=\; -K_i x_i$$

where

$$\begin{aligned} K_i &= T_i^{-1}(D_i^T C_i + B_i^T P_{i+1} A_i), && (4.2)\\ T_i &= I + D_i^T D_i + B_i^T P_{i+1} B_i. && (4.3) \end{aligned}$$

and $P_i \geq 0, i \in [0,T]$, with P_T given, is a periodic stabilizing solution to the DPRE given by

$$\begin{aligned} P_i \;=\; & [A_i - B_i(I + D_i^T D_i)^{-1} D_i^T C_i]^T \\ & [P_{i+1} - P_{i+1} B_i T_i^{-1} B_i^T P_{i+1}] \\ & [A_i - B_i(I + D_i^T D_i)^{-1} D_i^T C_i] \\ & + C_i^T [I - D_i (I + D_i^T D_i)^{-1} D_i^T] C_i \qquad (4.4) \end{aligned}$$

Furthermore, the optimum cost $J^\star(0,T)$ is given in terms of the initial condition and the solution P_i as

$$J^\star(0,T) = \frac{1}{2} x_0^T P_0 x_0.$$

We now turn our attention to the case when the final time T approaches infinity. Clearly, the stabilizability of the system is sufficient to ensure that the solution to the Riccati equation exists and is bounded. Furthermore, the stability of the closed-loop system is guaranteed by the detectability of the open-loop system. In this case, we derive similar conditions on the system to ensure that the solution to the periodic Riccati equation is bounded and stabilizing to the open-loop system.

Lemma 4.2 *[4] If the discrete-time periodic system Σ_p is stabilizable then the solution to the DPRE given by (4.4) is bounded for all time as T tends to infinity.*

The above result guarantees the boundedness of the solution to the Riccati equations as the finite horizon tends to infinity. The classic result on the stabilizability of the solution of the Riccati equation was given by Kalman in terms of the detectability of the system. We see that the same result can be extended to this class of systems. We now present the conditions under which the solution to the Riccati equation is stabilizing to the system.

Lemma 4.3 *[4] If the discrete-time periodic system Σ_p is stabilizable and detectable then the solution to the DPRE (4.4) is bounded for all time as T tends to infinity and is stabilizing to the system.*

It can be verified through some algebra that the Joseph stabilized DPRE has the following form:

$$\begin{aligned} P_i \;=\; & (A_i - B_i K_i)^T P_{i+1} (A_i - B_i K_i) + K_i^T K_i \\ & + (C_i - D_i K_i)^T (C_i - D_i K_i) \qquad (4.5) \end{aligned}$$

Note that the conditions provided ensure that the solution is not only bounded for all time but also guarantee the convergence of the solution to a limit. We will proceed to study the construction of NCFs using the solution to the DPRE in next section.

The positive semi-definite stabilizing solution to DPRE is computed by a deflating subspace method given in [4]. This method employs the standard QZ algorithm and retains its attractive features, such as quadratic convergence and small relative backward error. The cyclic QZ method extends the technique of simultaneous reduction to a sequence of matrices, and promises the same accuracy, numerical stability, and perturbation insensitivity as its predecessors.

We have seen that the coprime factors can be obtained by using a stabilizing feedback to the system. We will show that the choice of the stabilizing gain obtained through the solution to the Riccati equations results in a normalized coprime factorization. We first define the notion of NCF. The coprime factors of the system Σ_p, N_i, M_i and $\tilde{N}_i, \tilde{M}_i$ defined in Proposition 3.1 are said to normalized if they satisfy the following conditions:

$$\begin{aligned} N_i^* N_i + M_i^* M_i &= I, \\ \tilde{N}_i \tilde{N}_i^* + \tilde{M}_i \tilde{M}_i^* &= I. \end{aligned}$$

We are now ready to present our main result concerning the NCFs. In Section 3, it was shown that coprime factors can be obtained by the choice of a stabilizing feedback gain. We now show that the feedback gains obtained from the solution to the Riccati equations gives rise to coprime factors which are normalized. We will prove the results only for the right coprime factors. The proofs for the left coprime factors are similar and can be obtained easily.

Given a linear discrete-time periodic plant and let (A_i, B_i, C_i, D_i) be stabilizable and detectable. Theorem 4.1 shows that if K_i is chosen so that the eigenvalues $\sigma(A_i - B_i K_i)$ have negative real parts, then we can define a stable right coprime factorization for Σ_p.

Theorem 4.1 *Let the discrete-time periodic system Σ_p defined in (2.1) be stabilizable and P_i be the positive semi-definite stabilizing solution to the DPRE. Then, the NCFs are given by the following systems:*

$$\begin{bmatrix} N_i \\ M_i \end{bmatrix} \triangleq \left[\begin{array}{c|c} A_i - B_i K_i & B_i U_i \\ \hline C_i - D_i K_i & D_i U_i \\ -K_i & U_i \end{array}\right],$$

where U_i satisfy

$$U_i^T T_i U_i = I, \qquad (4.6)$$

and T_i is defined in (4.3).

Proof: Clearly, equation (2.7) follow from equation (4.5). Now, it is clear from (4.6) that D_i is nonsingular since $P_i \geq 0$. Some algebra shows that (4.2) is equivalent to (2.8). Condition (2.9) follow directly from the definitions and equation (4.6). □

We conclude that the construction of the NCFs can be obtained through the solution to the DPRE. Furthermore, the left coprime factors can be constructed from the solution to the corresponding filter Riccati equations. The coprime factors can be used to study robust stabilization under perturbations in the factors. The impetus for this approach is due to the results in the discrete-time periodic case where explicit formulae for the maximum radius for stabilization are obtained when the coprime factors are normalized. Another application for NCF is in the study of the notion of the gap metric for discrete-time periodic systems. These issues will be addressed elsewhere and are currently under study.

5 Conclusions

In this paper we have studied the notion of normalized coprime factorization for the discrete-time periodic systems. We have shown that coprime factors of discrete-time periodic systems can be obtained by stabilizing feedback and provided the state-space characterization of the coprime factors. We have shown that the solution to a quadratic optimization problem leads to the DPREs and the feedback gain obtained using the solution to these equations can be used to provide NCF for discrete-time periodic systems.

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