Reachability and minimum energy control of nonnegative 2D Roesser type models

Tadeusz Kaczorek

Warsaw University of Technology, Faculty of Electrical Engineering
Institute of Control and Industrial Electronics
00–662 Warszawa, Koszykowa 75, Poland
kaczorek@nov.isep.pw.edu.pl

Abstract

A new class of nonnegative 2D Roesser type models is introduced. Necessary and sufficient conditions are established for the reachability of the nonnegative 2D Roesser type model for zero boundary conditions. It is shown that the nonnegative 2D Roesser type model having not nilpotent system matrix is unreachable for nonzero boundary conditions. The minimum energy control problem is formulated and solved for the nonnegative 2D Roesser type model with zero boundary conditions. The considerations are illustrated by means of a numerical example.

1 Introduction

The most popular models of two–dimensional (2D) systems are the models introduced by Roesser [19], Fornasini and Marchesini [4,5]. The reachability and controllability of positive of discrete–time linear systems have been considered in [1–3]. The reachability and controllability and the minimum energy control of 2D linear system have been considered in many papers and books [6–18]. The minimum energy control problem for the classical 2D Roesser model was formulated and solved by Klamka [17] and next the method was extended for 2D linear systems with variable coefficients [10] and other type of 2D models [11–18]. Recently Valcher and Fornasini in [20] have investigated some interesting properties of homogeneous 2D positive system described by the second Fornasini–Marchesini type models. In this paper a nonnegative 2D Roesser type model is introduced. Necessary and sufficient conditions are established for the reachability of the nonnegative 2D Roesser type model for zero boundary conditions. It is shown that the nonnegative 2D Roesser type model having not nilpotent system matrix is unreachable for nonzero boundary conditions. The minimum energy control problem is solved for the nonnegative 2D Roesser type model with zero boundary conditions.

2 Preliminaries

Let $R_+ := [0, +\infty)$ be the set of nonnegative numbers and let $Z_+ := \{0, 1, 2, ...\}$ be the set of nonnegative integers. Denote by R_+^n the set of n–tuples of nonnegative numbers. The set of nonnegative matrices of size n by m will be denoted by $R_+^{n\times m}$.

Consider the 2D Roesser model

$$x_{ij}^{(1)} = Ax_{ij} + Bu_{ij}, \quad y_{ij} = Cx_{ij} + Du_{ij} \tag{1}$$

$$x_{ij}^{(1)} = \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix}, x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \quad i, j \in Z_+$$

where $x_{ij}^h \in R^{n_1}$ is the horizontal state vector, $x_{ij}^v \in R^{n_2}$ is the vertical state vector, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$A_1 \in R^{n_1\times n_1}$, $B_1 \in R^{n_1\times m}$, $C_1 \in R^{p\times n_1}$, $A_4 \in R^{n_2\times n_2}$, $B_2 \in R^{n_2\times m}$, $C_2 \in R^{p\times n_2}$.

Definition 1 *The 2D Roesser model (1) is called nonnegative 2D Roesser type model if for all boundary conditions*

$$x_{0j}^h \in R_+^{n_1},\ j \in Z_+,\ x_{i0}^v \in R_+^{n_2},\ i \in Z_+ \tag{2}$$

and for all $u_{ij} \in R_+^m$, $i, j \in Z_+$ we have $x_{ij} \in R_+^n$, $n = n_1 + n_2$ and $y_{ij} \in R_+^p$ for $i, j \in Z_+$, where R_+^n denotes the set of n–tuples of the nonnegatives real numbers.

Proposition 1 *The 2D Roesser model (1) is nonnegative if and only if*

$$A \in R_+^{n\times n}, B \in R_+^{n\times m}, C \in R_+^{p\times n}, D \in R_+^{p\times m} \tag{3}$$

Proof From (1) it follows that $x_{ij} \in R_+^n$ and $y_{ij} \in R_+^p$, $i, j \in Z_+$ for all boundary conditions (2) and $u_{ij} \in R_+^m$, $i, j \in Z_+$ if and only if (3) holds. □

The transition matrix T_{ij} for (1) is defined as follows [19, 6, 17]

$$T_{ij} = \begin{cases} I_n \ (\textit{the identity matrix}) \\ \quad \textit{for } i = j = 0 \\ T_{10}T_{i-1,j} + T_{01}T_{i,j-1} \\ \quad \textit{for } i, j \geq 0 \ \ (i + j \neq 0) \\ T_{ij} = 0 \ (\textit{the zero matrix}) \\ \quad \textit{for } i < 0 \textit{ or/and } j < 0 \end{cases} \tag{4}$$

where

$$T_{10} := \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, T_{01} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

Proposition 2 *The transition matrix T_{ij} for the nonnegative 2D Roesser type model is a nonnegative matrix, i.e.*

$$T_{ij} \in R_+^{n\times n} \quad \textit{for} \quad \textit{all} \quad i,j \in Z_+ \tag{5}$$

Proof From (4) it follows that $T_{10} \in R_+^{n\times n}$, $T_{01} \in R_+^{n\times n}$, $T_{11} = T_{10}T_{01} + T_{01}T_{10} \in R_+^{n\times n}$ and recurrently $T_{ij} \in R_+^{n\times n}$ for all $i,j \in Z_+$. □

The solution to (1) with boundary conditions (2) is given by

$$x_{ij} = x_{bc}(i,j) + \sum_{(p,g)\in D_{ij}} M_{i-p,j-q} u_{pq} \tag{6}$$

where

$$x_{bc}(i,j) \quad : \quad = \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \\ + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix}, \tag{7}$$

$$M_{i-p,j-q} := T_{i-p-1,j-q} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i-p,j-q-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

Definition 2 *The nonnegative 2D Roesser type model (1) is called reachable for zero boundary conditions (ZBC) at the point (h,k), $h,k \in Z_+$, if for zero boundary conditions (2) and every $x_f \in R_+^n$ there exists a sequence of inputs $u_{ij} \in R_+^m$ for*

$$(i,j) \in D_{hk} := \{(i,j) \in Z_+ \times Z_+,\ 0 \le i \le h, \\ 0 \le j \le k \quad \textit{and} \quad i+j \ne h+k\} \tag{8}$$

such that $x_{hk} = x_f$.

Definition 3 *The nonnegative 2D Roesser type model (1) is called reachable for any nonzero boundary conditions (NBC) at the point (h,k), $h,k \in Z_+$, if for any nonzero boundary conditions (2) and every $x_f \in R_+^n$ there exists a sequence of inputs $u_{ij} \in R_+^n$ for $(i,j) \in D_{hk}$ such that $x_{hk} = x_f$.*

Conditions will be established under which the nonnegative 2-D Roesser type model is reachable for zero boundary conditions and the minimum energy control problem will be solved.

3 Reachability of the nonnegative 2–D Roesser type model

Theorem 1 *The nonnegative 2D Roesser type model (1) is reachable for ZBC at the point (h,k) if and only if*

$$\begin{matrix} i) & \textit{rank}\, R(h,k) = n \\ ii) & R_r(h,k) \in R_+^{hkm\times n} \end{matrix} \tag{9}$$

where

$$R(h,k) := [M_{hk}, M_{h-1,k}, M_{h,k-1}, \ldots, M_{10}, M_{01}] \tag{10}$$

and $R_r(h,k)$ is a right inverse of $R(h,k)$, $R(h,k) \times \times R_r(h,k) = I_n$.

Proof From (6) for $i = h$, $j = k$, $x_{hk} = x_f$ and $x_{bc}(h,k) = 0$ we have

$$x_f = R(h,k)\, u(h,k) \tag{11}$$

where

$$u(h,k) := \left[u_{00}^T,\ u_{10}^T,\ u_{01}^T,\ \ldots,\ u_{h-1,k}^T,\ u_{h,k-1}^T\right]^T$$

T denotes the transposition.

Note that for a nonnegative 2D Roesser type model $M_{pq} \in R_+^{n\times m}$ and $R(h,k) \in R_+^{n\times hkm}$. From (9) it follows that there exists $R_r(h,k)$ of $R(h,k)$ and from (11) we have $u(h,k) = R_r(h,k)x_f \in R_+^{hkm}$ for any $x_f \in R_+^n$ if and only if $R_r(h,k) \in R_+^{hkm\times n}$. □

Remark 1 *Let the condition (9) be satisfied and let R_n be a nonsingular matrix consisting of n columns of $R(h,k)$ such that $R_n^{-1} \in R_+^{n\times n}$. Moreover let $u_n \in R_+^n$ be a vector consisting of those entries of $u(h,k)$ which correspond to the columns selected in R_n. Assuming the remaining columns of $u(h,k)$ zero from (11) we obtain $x_f = R_n u_n$. In this case the condition ii) of theorem 1can be substituted by $R_n^{-1} \in R_+^{n\times n}$. It is well-known [1, 2] that $R_n^{-1} \in R_+^{n\times n}$ if and only if R_n is has one nonzero entry in each row and column, i.e. is an $n \times n$ monomial matrix.*

Remark 2 *It is well–known [20] that a finite memory Roesser model has a nilpotent matrix A, i.e.*

$$\det \begin{bmatrix} I_{n_1}z_1 - A_1 & -A_2 \\ -A_3 & I_{n_2}z_2 - A_4 \end{bmatrix} = z_1^{n_1} z_2^{n_2} \tag{12}$$

Using (7) it is easy to show that if (12) holds then $x_{bc}(i,j) = 0$ for $i > n$, $j > n$.

Theorem 2 *The nonegative 2D Roesser model (1) having not nilpotent matrix A is unreachable at the point (h,k) for NBC.*

Proof From (6) for $i = h$, $j = k$ and $x_{hk} = x_f$ and NBC we have

$$x_f - x_{bc}(h,k) = R(h,k)\,u(h,k) \tag{13}$$

By assumption $x_f \in R_+^n$ and $x_{bc}(h,k) \in R_+^n$ are any vectors and $x_f - x_{bc}(h,k) \notin R_+^n$.

If A is not nilpotent matrix then there does not exist a sequence $u_{ij} \in R_+^m$ satisfying (13) for $x_f - x_{bc}(h,k)$ with at least one negative component. Therefore, the nonnegative 2D Roesser model is unreachable for NBC. □

Using a different approach a similar result has been obtained in [9] (see also [13]).

4 Minimum energy control

Consider the nonnegative 2D Roesser type model (1) and the performance index

$$I(u) := \sum_{(p,q)\in D_{hk}} u_{pq}^T Q u_{pq} \tag{14}$$

where Q is the $m \times n$ symmetric positive definite weighting matrix such that $Q^{-1} \in R_+^{m\times m}$.

The minimum energy problem for the nonnegative 2D Roesser type model with zero boundary conditions (2) can be stated as follows. Given the matrices A, B of (1), the weighting matrix Q and the point (h,k), find a sequence $u_{ij} \in R_+^m$ for $(i,j) \in D_{hk}$ which transfer the model from zero boundary conditions to the desired local state $x_f = x_{hk}$ and minimizes the performance index (14).

To solve the problem we define the matrix

$$\begin{aligned} W_Q(h,k) \quad : &= \sum_{(p,q)\in D_{hk}} M_{h-p,k-q} Q^{-1} M_{h-p,k-q}^T = \\ &= R(h,k)\,Q_d R^T(h,k) \end{aligned} \tag{15}$$

where $M_{h-p,k-q}$ and $R(h,k)$ are defined by (7) and (10), respectively.

$$Q_d := diag\left[Q^{-1}, ..., Q^{-1}\right] \in R_+^{hkm\times hkm}$$

Using (15) it is easy to show that for the nonnegative 2D Roesser type model the matrix $W_Q(h,k) \in R_+^{n\times n}$ is nonsingular if and only if the matrix $R(h,k)$ has full row rank.

Define the sequence of inputs

$$\widehat{u}_{ij} := Q^{-1} M_{h-i,k-j}^T W_Q^{-1}(h,k)\,x_f \quad for \quad (i,j) \in D_{hk} \tag{16}$$

Note that $\widehat{u}_{ij} \in R_+^m$ for any $x_f \in R_+^n$ if

$$W_Q^{-1}(h,k) \in R_+^{n\times n} \tag{17}$$

Theorem 3 *Let us assume that*

i) the nonnegative 2D Roesser type model is reachable for ZBC at the point (h,k),

ii) $Q^{-1} \in R_+^{m\times m}$ and (17) holds,

iii) $\overline{u}_{ij}$ $(i,j) \in D_{hk}$ is any sequence of inputs which transfer the model from zero boundary conditions to the desired local state $x_f = x_{hk}$.

Then the sequence of inputs (16) accomplishes the same task and

$$I(\widehat{u}) \le I(\overline{u}) \tag{18}$$

Moreover, the minimum value of (14) is given by

$$I(\widehat{u}) = x_f^T W_Q^{-1}(h,k)\,x_f \tag{19}$$

Proof First we shall show that the sequence of inputs (16) provides $x_{hk} = x_f$.

From (16) it follows that

$$\begin{aligned} \widehat{u}(h,k) \quad : &= \left[\widehat{u}_{00}^T, \widehat{u}_{10}^T, \widehat{u}_{01}^T, ..., \widehat{u}_{h-1,k}^T, \widehat{u}_{h,k-1}^T\right]^T = \\ &= Q_d R^T(h,k)\,W_Q^{-1}(h,k)\,x_f \end{aligned} \tag{20}$$

Substituting (16) into (11) and using (15) we obtain

$$\begin{aligned} x_{hk} &= R(h,k)\,\widehat{u}(h,k) = \\ &= R(h,k)\,Q_d R^T(h,k)\,W_Q^{-1}(h,k)\,x_f = x_f \end{aligned}$$

Since both $\overline{u}_{ij}$ and $\widehat{u}_{ij}$ transfer the model from zero boundary conditions to x_f then

$$R(h,k)\,\overline{u}(h,k) = R(h,k)\,\widehat{u}(h,k)$$

and

$$R(h,k)\left[\overline{u}(h,k) - \widehat{u}(h,k)\right] = 0 \tag{21}$$

From (21) and (20) we have

$$\begin{aligned} &\left[\overline{u}(h,k) - \widehat{u}(h,k)\right]^T R^T(h,k)\,W_Q^{-1}(h,k)\,x_f = \\ = &\left[\overline{u}(h,k) - \widehat{u}(h,k)\right]^T Q_d^{-1}\widehat{u}(h,k) = 0 \end{aligned} \tag{22}$$

Using (22) it is easy to show that

$$\begin{aligned} &\overline{u}^T(h,k)\,Q_d^{-1}\overline{u}(h,k) = \\ = &\ \widehat{u}^T(h,k)\,Q_d^{-1}\widehat{u}(h,k) + \\ + &\left[\overline{u}(h,k) - \widehat{u}(h,k)\right]^T Q_d^{-1}\left[\overline{u}(h,k) - \widehat{u}(h,k)\right] \end{aligned}$$

or

$$\begin{aligned} &\sum_{(p,q)\in D_{hk}} \overline{u}_{pq}^T Q \overline{u}_{pq} = \\ = &\sum_{(p,q)\in D_{hk}} \widehat{u}_{pq}^T Q \widehat{u}_{pq} + \\ + &\sum_{(p,q)\in D_{hk}} \left[\overline{u}_{pq} - \widehat{u}_{pq}\right]^T \times \\ \times &\ Q\left[\overline{u}_{pq} - \widehat{u}_{pq}\right] \end{aligned} \tag{23}$$

The inequality (18) holds since the last term in (23) is always nonnegative.

To obtain the minimum value of (14) we substitute (20) into (14)

$$
\begin{aligned}
I(\widehat{u}) &= \sum_{(p,q)\in D_{hk}} \widehat{u}_{pq}^T Q \widehat{u}_{pq} = \\
&= \widehat{u}^T(h,k)\, Q_d^{-1} \widehat{u}(h,k) = \\
&= \left[Q_d R^T(h,k)\, W_Q^{-1}(h,k)\, x_f\right]^T \times \\
&\times\; Q_d^{-1}\left[Q_d R^T(h,k)\, W_Q^{-1}(h,k)\, x_f\right] = \\
&= x_f^T W_Q^{-1}(h,k)\, R(h,k)\, Q_d R^T(h,k) \times \\
&\times\; W_Q^{-1}(h,k)\, x_f = x_f^T W_Q^{-1}(h,k)\, x_f
\end{aligned}
$$

since by (15) $R(h,k)\, Q_d R^T(h,k)\, W_Q^{-1}(h,k) = I$. □

5 Example

Consider the nonnegative 2D Roesser type model (1) with

$$
A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},
$$

$$
B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

for $h = k = 1$.

Using (7) and (10) we obtain

$$
\begin{aligned}
R(1,1) &= \begin{bmatrix} M_{11} & M_{10} & M_{01} \end{bmatrix} = \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; rank\, R(1,1) = 3
\end{aligned}
$$

By theorem 1 the model is reachable for ZBC at the point $(1,1)$.

Let the performance index has the form (14) with $h = k = 1$ and $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $x_f = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{3} \end{bmatrix}^T$.

In this case using (15) and (20) we obtain

$$
\begin{aligned}
W_Q(1,1) &= R(1,1)\, Q_d R^T(1,1) = \\
&= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{u}(1,1) &= \begin{bmatrix} u_{00} & u_{10} & u_{01} \end{bmatrix}^T = \\
&= Q_d R^T(1,1)\, W_Q^{-1}(1,1)\, x_f = \\
&= \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{3} \end{bmatrix}^T
\end{aligned}
$$

and the minimum value of the performance index is equal

$$
I(\widehat{u}) = x_f^T W_Q^{-1}(1,1)\, x_f = \frac{11}{6}
$$

6 Concluding remarks

Necessary and sufficient conditions have been established for the reachability of the nonnegative 2D Roesser type model for zero boundary conditions. It has been shown that the nonnegative 2D Roesser type model having not nilpotent matrix A is unreachable for nonzero boundary conditions (2). The minimum energy control problem has been formulated and solved for the model with zero boundary conditions. The results presented can be extended for the n–D ($n > 2$) Roesser type model with constant and variable coefficients. An extension of the above considerations for singular 2D linear systems is also possible.

It is well–known [6] that the first Fornasini–Marchesini model is a special case of the Roesser model. Therefore, the results obtained in this paper for the nonnegative 2D Roesser type model can be immediately extended for the nonnegative first 2D Fornasini–Marchesini type model.

7 References

[1] P.G. Caxson, *Positive input reachability and controllability a positive systems*, Linear Algebra and its Applications, vol. 94, 1987, pp.35–35.

[2] M.P. Fanti, B. Maione and B. Turchiano, *Controllability of linear single–input positive discrete–time systems*, Int. J. Control, vol. 50, No 6, 1989, pp.2523–2542.

[3] M.P. Fanti, B. Maione and B.Turchiano, *Controllability of multi–input positive discrete–time systems*, Int. J. Control, vol. 51, No 6, 1990, pp.1295–1308.

[4] E. Fornasini, G. Marchesini, *State space realization of two–dimensional filters*, IEEE Trans. Autom. Control, AC–21, 1976, pp.484–491.

[5] E. Fornasini, G. Marchesini, *Doubly indexed dynamical systems: State space models and structural properties*, Math. Syst. Theory 12, 1987.

[6] T. Kaczorek, *Linear Control Systems*, vol. 2, Research Studies Press and J. Wiley, New York 1993.

[7] T. Kaczorek, *When the local controllability of the general model of 2–D linear systems implies its local reachability*, Systems and Control Letters, vol. 23, 1994.

[8] T. Kaczorek, *When the local controllability of Roesser model implies its local reachability*, Bull. Pol. Acad. Techn. Sci., vol. 41, 1994.

[9] T. Kaczorek, *U–reachability and U-controllability of 2–D Roesser model*, Bull. Pol. Acad. Techn. Sci., vol. 43, No 1, 1995, pp.31–37.

[10] T. Kaczorek and J. Klamka, *Minimum energy control of 2–D linear systems with variable coefficients*, Int. J. Control, vol. 44, No 3, 1986, pp. 645–650.

[11] T. Kaczorek and J. Klamka, *Minimum energy control for general model of 2–D linear systems*, Int. J. Control, vol. 47, No 5, 1988, pp. 1555–1562.

[12] J. Klamka, *M–dimensional nonstationary linear discrete systems in Banach spaces*, Proc. 12 World IMACS Congress, Paris, 1988, vol. 4, pp. 31–33.

[13] J. Klamka, *Constrained controllability of 2–D linear systems*, Proc. 12 World IMACS Congress, Paris, 1988, vol. 2, pp. 166–169.

[14] J. Klamka, *Complete controllability of singular 2–D systems*, Proc. 13 IMACS World Congress, Dublin, 1991, pp. 1839–1840.

[15] J. Klamka,*Minimum energy control of singular 2–D linear systems with variable coefficients*, Prooc. IMACS Symp. Lille 1991, vol. 2, pp. 155–159.

[16] J. Klamka, *Minimum energy control problem for general linear 2–D systems in Hilbert spaces*, Proc. IEEE Symp. Crete, 1993.

[17] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic Publ., Dordrecht, 1991.

[18] J. Klamka, *Constrained Controllability of Discrrete 2–D Linear Systems*, Proc. IMACS Intern. Symp. Signal Processing, Robotics and Neural Networks, April 25–27, 1994, Lille, pp. 166–169.

[19] P.R. Roesser, *A discrete state–space model for linear image processing*, IEEE Trans. Autom. Contr. 1975, vol. AC–20, No 1, pp. 1–10.

[20] M.E. Valcher and E. Fornasini, *State Models and Asymptotic Behaviour of 2D Positive Systems*, IMA Journal of Mathematical Control & Information, No 12, 1995, pp. 17–36.