OBSERVERS FOR SYSTEMS WITH IMPLICIT OUTPUT

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Abstract

In this paper we prove the convergence of Luenberger-type observers for systems with implicitly defined outputs.

1 Introduction

In this paper we consider a system whose output $y(t)$ is defined as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ B(y(t))x(t) &= p(t) \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$, $p \in R^l$ and $B(y)$ is a continuous matrix function, $B(y) \in C(R^m, R^{l\times n})$. The system is assumed to be Lyapunov stable but not asymptotically stable. The information about the system one can measure is the functions $y(t)$ and $p(t)$. The problem we consider is to construct a dynamical observer for $x(t)$. This system is quite similar to a linear time-varying system if one considers $p(\cdot)$ as output instead. The difference is that $B(\cdot)$ varies with $y(\cdot)$ here. As far as we know, even for linear time-varying systems, constructing an observer is far from trivial.

Systems with implicit output functions appear in a number of applications, an important one is *dynamic vision*. In order to motivate the study of systems defined by (1), we explain here the typical problems in dynamic vision.

Dynamic vision is the discipline that studies the inverse problem of recovering information on the scene from a *sequence* of images. The main paradigm of dynamic vision is the estimate of the motion of the camera and of the 3-D structure of the scene. Let us simplify the problem by assuming that the motion of the objects being viewed is *rigid*. The "structure" of the scene is represented by a number of point-features whose coordinates in the ambient space are $x \doteq [x_1\ x_2\ x_3]^T$; $v = [v_1\ v_2\ v_3]^T$ indicates the relative translational velocity of the object with respect to the viewer frame and $\omega := [\omega_1\ \omega_2\ \omega_3]^T$is the rotational velocity vector also expressed in the viewer frame. The coordinates of the *projection* of a point feature $P = [x_1\ x_2\ x_3]^T$onto the *image plane* (perpendicular to the x_3 axis) assumed at a conventional distance $f = 1$ from the origin, are

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \doteq \begin{pmatrix} \frac{x_1}{x_3} \\ \frac{x_2}{x_3} \end{pmatrix},$$

so that we can write a nonlinear dynamical model having the position of the point in the ambient plane as the state, and the projection as the measured output:

$$\begin{aligned} \dot{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \frac{x_1}{x_3} \\ \frac{x_2}{x_3} \end{pmatrix} \qquad x_3 \neq 0 \end{aligned} \tag{2}$$

which is in the form

$$\begin{aligned} \dot{x} &= f(x), \qquad x(t_0) = x_0 \in R^3 \\ B(y)x &= p(t) \end{aligned} \tag{3}$$

where

$$\begin{aligned} f(x) &= \Omega x + v \\ \Omega &\doteq \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ B(y) &= \begin{pmatrix} -1 & 0 & y_1 \\ 0 & -1 & y_2 \end{pmatrix} \\ p(t) &= 0 \end{aligned} \tag{4}$$

Estimating the structure of the scene and the motion of the camera is equivalent, respectively, to estimating the state and identifying the parameters of the above model. Note that the observation equation is nonlinear and it is *implicit* in the output.

The dynamic vision model described above seems different than (1). One can, however, easily convert the problem into the form of (1) by using the coordinate change:

$$\bar{x}(t) = x(t) - \int_0^t e^{\Omega(t-s)} v(s) ds$$

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Then, the dynamic vision model falls into the class described by (1) where

$$\begin{array}{rcl} f(\bar{x}) & = & \Omega\bar{x} \\ B(y) & = & \begin{pmatrix} -1 & 0 & y_1 \\ 0 & -1 & y_2 \end{pmatrix} \\ p & = & -B(y)\int_0^t \exp(\Omega(t-s))v(s)\,ds. \end{array} \tag{5}$$

The paper is organized as follows. In section 2 we review some stability results on time varying linear systems. In section 3, we present the main results and in section 4 we give the proof.

2 Preliminaries

In this section we review some stability results on time varying linear systems, which will be needed later on. Most of the results can be found, for example, in [1, 2].

Our notations are as follows:

$|\cdot|_p$ — L_p-norm,

$\|A\|$ — the operator norm of a matrix A.

Lemma 2.1 *Let B be $n\times n$-matrix such that the linear system of differential equations*

$$\dot{x} = Bx, \quad 0 \le t < \infty, \quad x(t) \in R^n \tag{6}$$

is Lyapunov stable.

Then there exists a positive-definite $n \times m$-matrix P such that

$$PB + B^*P \le 0. \tag{7}$$

Furthermore, consider the spectrum $\sigma(B)$ of B and its "asymptotically stable" $\sigma_-(B) := \{\lambda \in \sigma(B) : Re\,\lambda < 0\}$ and "neutral" $\sigma_0(B) := \{\lambda \in \sigma(B) : Re\,\lambda = 0\}$ parts. Introduce the corresponding invariant subspaces $L_- \sim \sigma_-(B)$ and $L_0 \sim \sigma_0(B)$ of the matrix B. Let $\pi_- : R^n \to L_-$ and $\pi_0 : R^n \to L_0$ be the projectors related to the direct sum $R^n = L_- \oplus L_0$. Then, for any $\kappa > 0$, there exists a positive definite $n\times n$-matrix P such that

$$PB + B^*P \le -\kappa\pi_-^*\pi_-. \tag{8}$$

Lemma 2.2 *Let $K(\cdot) \in L_\infty(0,+\infty)$ be $n \times n$-matrix function. Consider the system*

$$\dot{z} = K(t)z. \tag{9}$$

Assume that

$$\int_{t_0}^{\infty} |z(t)|^2\,dt \le c^2\,|z(t_0)|^2 \tag{10}$$

for any solution $z(\cdot)$ of (9) and any $t_0 \ge 0$ with the constant $c > 0$ being independent of $z(\cdot)$ and t_0.

Then and only then,

$$|z(t)| \le b|z(t_0)|e^{-r(t-t_0)}$$

for some $b > 0,\ r > 0$.

Lemma 2.3 *Let $K(\cdot) \in L_\infty(0,+\infty)$ be $n\times n$-matrix function. Suppose*

$$\dot{z} = K(t)z \tag{11}$$

is exponentially stable. Then, there exists a positive real $\delta > 0$ such that any system

$$\dot{z} = \widetilde{K}(t)z \tag{12}$$

with

$$\widetilde{K}(\cdot) \in L_\infty(0,+\infty), \quad \left|\widetilde{K}(\cdot) - K(\cdot)\right|_\infty < \delta \tag{13}$$

is asymptotically stable and

$$|z(\tau)| \le ce^{-\rho(t-\tau)}|z(\tau)| \tag{14}$$

for any $t \ge \tau \ge 0$ and any solution $z(\cdot)$ of the system (12). Here the constants $c > 0$ and $\rho > 0$ are independent of $\widetilde{K}(\cdot), z(\cdot), t, \tau$

Lemma 2.4 *Let the hypotheses of Lemma 2.3 be valid. Consider the matrix-function $\widetilde{K}(\cdot) \in L_\infty(0,+\infty)$ satisfying (13) and a vector-function $f(\cdot) \in L_\infty((0,+\infty) \to R^n)$.*

Any solution $z(\cdot)$ of the equation

$$\dot{z} = \widetilde{K}(t)z + f(t) \tag{15}$$

is bounded on $(0,+\infty)$ and satisfies the estimation

$$|z(t)| \le a|f(\cdot)|_\infty + ce^{-\rho t}|z(0)| \qquad \forall t \ge 0 \tag{16}$$

where the constants $a > 0, c > 0, \rho > 0$ are independent of $f(\cdot), z(\cdot)$ and $\widetilde{K}(\cdot)$ (provided that (13) is true).

Lemma 2.5 *Let $\phi(\cdot) \in L_2([t_0,+\infty) \to R^n)$ and $\theta > 0$. Denote*

$$\overline{\phi}(t) := \int_0^\theta \phi(t+s)\,ds \quad \forall t \ge t_0. \tag{17}$$

Then $\overline{\phi}(\cdot) \in L_2([t_0,+\infty) \to R^n)$ and

$$\left|\overline{\phi}(\cdot)\right|_2 \le \theta|\phi(\cdot)|_2. \tag{18}$$

Proof. By using the Cauchy-Schwartz inequality, we get

$$\left|\overline{\phi}(t)\right| \le \sqrt{\theta}\left(\int_t^{t+\theta} |\phi(s)|^2\,ds\right)^{1/2}.$$

Hence,

$$\int_{t_0}^{\infty} \left|\overline{\phi}(t)\right|^2\,dt \le \theta\int_{t_0}^{\infty} dt \int_t^{t+\theta} |\phi(s)|^2\,ds =$$

$$= \theta\int_{t_0}^{\infty} |\phi(s)|^2\,ds \int_{s-\theta}^{s} dt = \theta^2|\phi(\cdot)|_2^2.$$

Thus, $\overline{\phi}(\cdot) \in L_2$ and (18) is true.

3 Luenberger-type observer

Now let us consider system (1). The problem is to identify the current state $x(t)$.

Consider a positive-definite $n \times n$-matrix P that satisfy (8) with some $\kappa > 0$. In this paper we investigate the following Luenberger-type observer

$$\frac{d\widehat{x}}{dt} = A\widehat{x} - P^{-1}B(y(t))^* \left[B(y(t))\widehat{x} - p(t)\right]. \quad (19)$$

Here $\widehat{x}$ is the estimate of the current state x. Here and throughout, the asterisk stands for transposition.

In the following Theorems 3.1 and 3.2, we shall show that the above estimate tends asymptotically to the real state and also that this property is stable in a sense. The main assumption we need here is some kind of observability of system (1). More specifically what we need here is the observability of the invariant subspace L_0 of the matrix A associated with the purely imaginary part of its spectrum. This condition is formulated as follows.

Assumption 3.1 *Consider the spectrum $\sigma(A)$ of the matrix A and its "asymptotically stable" $\sigma_-(A) := \{\lambda \in \sigma(A) : Re\,\lambda < 0\}$ and "neutral" $\sigma_0(A) := \{\lambda \in \sigma(A) : Re\,\lambda = 0\}$ parts. Introduce the corresponding invariant subspaces $L_- \sim \sigma_-(A)$ and $L_0 \sim \sigma_0(A)$ of the matrix A. Consider a basis $\xi_1, \ldots, \xi_q \subset L_0$ of the subspace L_0 and introduce $n \times q$-matrix $\Xi := (\xi_1, \ldots, \xi_q)$ whose columns are the above vectors.*

There exist two positive reals $T > 0$ and $\varepsilon > 0$ such that, for any $t \geq 0$,

$$\int_0^T \Xi^* e^{A^*\tau} B(y(t+\tau))^* B(y(t+\tau)) e^{A\tau}\Xi \, d\tau \geq \varepsilon I_q. \quad (20)$$

The properties of the observer under consideration are reflected by the following theorems.

Theorem 3.1 *Let $y(\cdot) \in L_\infty(0, +\infty)$ and let Assumption 3.1 be fulfilled.*

Then the error $z(t) := x(t) - \widehat{x}(t)$ exponentially tends to the zero

$$|z(t)| \leq ce^{-\rho t}|z(0)| \qquad \forall t \geq 0.$$

Here the constants $c > 0$ and $\rho > 0$ are independent of $t, x(\cdot)$ and $\widehat{x}(\cdot)$.

The proof of both this theorem and the next one will be given in section 4.

Note now that usually we have at our disposal only approximate values of $A, y(t)$ and $p(t)$. Namely, we know $n \times n$-matrix $\widetilde{A}$, a vector-valued function $\widetilde{y}(\cdot) \in R^s$ and a function $\widetilde{p}(\cdot) \in R$ whose meaning is the following

$$\widetilde{A} \approx A, \quad \widetilde{y}(t) \approx y(t), \quad \widetilde{p}(t) \approx p. \quad (21)$$

Likewise, we can usually compute P only approximately to produce a matrix $\widetilde{P} \approx P$. So the actual observer looks as follows

$$\frac{d\widehat{x}}{dt} = \widetilde{A}\widehat{x} - \widetilde{P}^{-1}B(\widetilde{y}(t))^* \left[B(\widetilde{y}(t))\widehat{x} - \widetilde{p}(t)\right]. \quad (22)$$

The next theorem describes the properties of this actual observer.

Theorem 3.2 *Let the hypotheses of Theorem 3.1 be valid. Suppose that all the errors are sufficiently small, i.e.,*

$$\|A - \widetilde{A}\| \leq \delta, \|y(\cdot) - \widetilde{y}(\cdot)\|_\infty \leq \delta,$$

$$\|p - \widetilde{p}(\cdot)\|_\infty \leq \delta, \|P - \widetilde{P}\| \leq \delta$$

where $\delta > 0$ is sufficiently small.

Then the error $z(t) := x(t) - \widehat{x}(t)$ satisfies the estimation

$$\begin{aligned} &|z(t)| \leq ce^{-\rho t}|z(0)| + \\ &+q\left(\|A - \widetilde{A}\| + \|y(\cdot) - \widetilde{y}(\cdot)\|_\infty\right)|x(0)| + \\ &+q\left(\|p - \widetilde{p}(\cdot)\|_\infty + \|P - \widetilde{P}\|\right)|x(0)|. \end{aligned} \quad (23)$$

Here the constants $\rho > 0, c > 0, q > 0$ are independent of $t, x(\cdot), \widehat{x}(\cdot), \widetilde{A}, \widetilde{y}(\cdot), \widetilde{d}(\cdot)$, and $\widetilde{P}$ but depend on $A, y(\cdot), p, B(\cdot)$, and δ.

4 Proofs of the main results

In this section we give proofs of Theorem3.1 and Theorem3.2.

Proof of Theorem3.1. By (1) and (19), we have

$$\begin{aligned} \dot{z} &= Az - P^{-1}B(y(t))^* B(y(t))z(t) && (24) \\ &= \left(A - P^{-1}B(y(t))^* B(y(t))\right) z(t). && (25) \end{aligned}$$

Then,

$$\frac{d}{dt}(z^* Pz) = z^*(PA + A^*P)z - 2z^* B(y(t))^* B(y(t))z$$

$$\leq\leq -\kappa|\pi_- z|^2 - 2|B(y(t))z|^2.$$

So, for any two instants $t \geq t_0 \geq 0$,

$$z(t)^* Pz(t) \leq z(t_0)^* Pz(t_0) - \kappa \int_{t_0}^t |\pi_- z(s)|^2 \, ds -$$

$$2\int_{t_0}^t |B(y(s))z(s)|^2 \, ds,$$

$$z(t)^* Pz(t) \leq z(t_0)^* Pz(t_0) \leq \|P\||z(t_0)|^2,$$

$$\kappa \int_{t_0}^{t} |\pi_- z(s)|^2 \, ds \leq \|P\| |z(t_0)|^2,$$

$$2 \int_{t_0}^{t} |B(y(s)) z(s)|^2 \, ds, \leq \|P\| |z(t_0)|^2.$$

Since the matrix P is positive definite $a^* P a \geq m|a|^2$, we have

$$m|z(t)|^2 \leq z(t)^* P z(t) \leq \|P\| |z(t_0)|^2,$$

$$|z(t)| \leq \sqrt{\frac{\|P\|}{m}} |z(t_0)|, \tag{26}$$

$$\int_{t_0}^{\infty} |B(y(s)) z(s)|^2 \, ds \leq \frac{\|P\|}{2} |z(t_0)|^2, \tag{27}$$

$$\int_{t_0}^{\infty} |z_-(s)|^2 \, ds \leq \frac{\|P\|}{\kappa} |z(t_0)|^2 \tag{28}$$

where we denote $z_-(t) := \pi_- z(t), z_0(t) := \pi_0 z(t)$ and the projectors π_-, π_0 were introduced in Lemma 2.1.

By virtue of (24),

$$\dot{z}_0 = \pi_0 \dot{z} = \pi_0 A z - \pi_0 P^{-1} B(y(t))^* B(y(t)) z \tag{29}$$

$$= A z_0 - \underbrace{\pi_0 P^{-1} B(y(t))^* \overbrace{B(y(t)) z(t)}^{\eta}}_{\zeta(\cdot)}. \tag{30}$$

Further the symbol $|\cdot|_2$ will be used to denote the L_2-norm over the interval $[t_0, \infty)$ while $|\cdot|_\infty$ will denote the L_∞-norm over $[0, \infty)$. In accordance with (27), $|\eta(\cdot)|_2 \leq \sqrt{\frac{\|P\|}{2}} |z(t_0)|$. So

$$|\zeta(\cdot)|_2 \leq \|\pi_0\| \|P^{-1}\| K_B |y(\cdot)|_\infty |\eta(\cdot)|_2 \tag{31}$$

$$\leq \underbrace{\|\pi_0\| \|P^{-1}\| K_B |y(\cdot)|_\infty \sqrt{\frac{\|P\|}{2}}}_{b} |z(t_0)|. \tag{32}$$

Due to (29), we have

$$z_0(t+s) = e^{As} z_0(t) + \underbrace{\int_t^{t+s} e^{A(t+s-\theta)} \zeta(\theta) \, d\theta}_{\varphi_s(t)}. \tag{33}$$

Let $t, s \geq 0, s \leq T$ where T is the constant from Assumption 3.1. Since the matrix A is Lyapunov stable, $\|e^{A\tau}\| \leq \alpha < \infty$ for all $\tau \geq 0$. So, in (33),

$$|\varphi_s(t)| \leq \alpha \underbrace{\int_t^{t+T} |\zeta(\theta)| \, d\theta}_{\chi(t)} \tag{34}$$

where, by Lemma 2.5 and (31),

$$|\chi(\cdot)|_2 \leq \alpha \sqrt{T} |\zeta(\cdot)|_2 \leq \alpha b \sqrt{T} |z(t_0)|. \tag{35}$$

In the light of (27) and (28), we see that

$$\begin{aligned} &\underbrace{|B(y(\cdot)) z_0(\cdot)|_2}_{\varphi(\cdot)} \\ &= |B(y(\cdot)) z(\cdot) - B(y(\cdot)) z_-(\cdot)|_2 \\ &\leq |B(y(\cdot)) z(\cdot)|_2 + |B(y(\cdot)) z_-(\cdot)|_2 \\ &\leq \sqrt{\frac{\|P\|}{2}} |z(t_0)| + |y(\cdot)|_\infty K_B |z_-(\cdot)|_2 \\ &\leq \underbrace{\sqrt{\|P\|} \left(\frac{1}{\sqrt{2}} + \frac{|y(\cdot)|_\infty K_B}{\sqrt{\kappa}} \right)}_{f} |z(t_0)|. \end{aligned} \tag{36}$$

Hence

$$\begin{aligned} &\int_{t_0}^{\infty} dt \int_0^T |\varphi(t+s)|^2 \, ds \\ &= \int_{t_0}^{\infty} dt \int_t^{t+T} |\varphi(\theta)|^2 \, d\theta \\ &= \int_{t_0}^{\infty} |\varphi(\theta)|^2 \, d\theta \int_{\max\{t_0, \theta - T\}}^{\theta} dt \\ &\leq T \int_{t_0}^{\infty} |\varphi(\theta)|^2 \, d\theta \\ &\leq T f^2 |z(t_0)|^2. \end{aligned} \tag{37}$$

On the other hand, invoking (33) and (34), we get

$$\begin{aligned} &\int_0^T \left| B(y(t+s)) e^{As} z_0(t) \right|^2 \, ds \\ &= \int_0^T |B(y(t+s)) (z_0(t+s) - \varphi_s(t))|^2 \, ds \\ &\leq 2 \int_0^T |B(y(t+s)) z_0(t+s)|^2 \, ds + \\ &\quad +2 \int_0^T |B(y(t+s)) \varphi_s(t)|^2 \, ds \\ &\leq 2 \int_0^T |\varphi(t+s)|^2 \, ds + 2 |y(\cdot)|_\infty^2 \|E\|^2 \chi(t)^2 \end{aligned} \tag{38}$$

In view of this, it follows from (35) and (37) that

$$\begin{aligned} &\int_{t_0}^{\infty} dt \int_0^T \left| B(y(t+s)) e^{As} z_0(t) \right|^2 \, ds \\ &leq \quad 2T \left(f^2 + |y(\cdot)|_\infty^2 K_B^2 \alpha^2 b^2 \right) |z(t_0)|^2. \end{aligned} \tag{39}$$

Now we recall that $z_0(t) \in L_0$ and, in Assumption 3.1, $\Xi = (\xi_1, \ldots, \xi_q)$ where $\xi_1, \ldots, \xi_q$ is a basis of L_0. Denote by $\zeta_1, \ldots, \zeta_q$ the adjoint basis: $\zeta_j^* \xi_i = 0$, if $i \neq j$, and $\zeta_i^* \xi_i = 1$. Introduce the $n \times q$-matrix $\Upsilon := (\zeta_1, \ldots, \zeta_q)$. Then $\Xi \Upsilon^* \widehat{z} = \widehat{z}$ for any $\widehat{z} \in L_0$. In particular,

$$z_0(t) = \Xi \Upsilon^* z_0(t) \qquad \forall t \geq 0. \tag{40}$$

So

$$\int_{t_0}^{\infty} dt \int_0^T \left| B(y(t+s)) e^{As} z_0(t) \right|^2 \, ds$$

$$
\begin{aligned}
&= \int_{t_0}^{\infty} dt \int_0^T \left|B(y(t+s))e^{As}\Xi\Upsilon^* z_0(t)\right|^2 ds \\
&= \int_{t_0}^{\infty} (z_0(t)^* \Upsilon y_\xi \Upsilon^* z_0(t))dt. \qquad (41)
\end{aligned}
$$

where $y_\xi = \int_0^T \Xi^* e^{A^* s} B(y(t+s))^* B(y(t+s)) e^{As} \Xi ds$. Then (20) and (42) yield

$$
\int_{t_0}^{\infty} dt \int_0^T \left|B(y(t+s))e^{As} z_0(t)\right|^2 \, ds \geq
$$

$$
\varepsilon \int_{t_0}^{\infty} |\Upsilon^* z_0(t)|^2 \, dt \geq \frac{\varepsilon}{\|\Xi\|+1} \int_{t_0}^{\infty} |z_0(t)|^2 \, dt.
$$

By invoking (41), we get

$$
\int_{t_0}^{\infty} |z_0(t)|^2 \, dt
$$

$$
\leq \frac{2T(\|\Xi\|+1)}{\varepsilon} \left(f^2 + |y(\cdot)|_\infty^2 K_B^2 \alpha^2 b^2\right) |z(t_0)|^2.
$$

Coupling this estimate with (28) and taking into account the decomposition $z(t) = z_0(t) + z_-(t)$, we see that

$$
\int_{t_0}^{\infty} |z(t)|^2 \, dt \leq 2 \int_{t_0}^{\infty} |z_-(t)|^2 \, dt + 2 \int_{t_0}^{\infty} |z_0(t)|^2 \, dt
$$

$$
\leq \underbrace{2\left(\frac{\|P\|}{\kappa} + \frac{2T(\|\Xi\|+1)}{\varepsilon}(f^2 + |y(\cdot)|_\infty^2 K_B^2 \alpha^2 b^2)\right)}_{c^2} |z(t_0)|^2.
$$

Here the constant c is evidently independent of $z(\cdot)$ and $t \geq t_0 \geq 0$. This means that the system (24) satisfies the hypotheses of Lemma 2.2. In other words, the assumptions of Lemma 2.2 are fulfilled with respect to the matrix-function.

$$
K(t) := A - P^{-1} B(y(t))^* B(y(t)). \qquad (42)
$$

This completes the proof.

Proof of Theorem 3.2. On the basis of (19) and (22), we have

$$
\begin{aligned}
\dot{z} &= \dot{x} - \dot{\widehat{x}} \\
&= Ax - \widetilde{A}\widehat{x} + \widetilde{P}^{-1} B(\widetilde{y}(t))^* \left[B(\widetilde{y}(t))\widehat{x} - B(y(t))x\right] \\
&= \underbrace{\left(\widetilde{A} - \widetilde{P}^{-1} B(\widetilde{y}(t))^* B(\widetilde{y}(t))\right)}_{\widetilde{K}(t)} z + \\
&+ \underbrace{(A - \widetilde{A})x + \widetilde{P}^{-1} B(\widetilde{y}(t))^* \left(B(\widetilde{y}(t)) - B(y(t))\right) x}_{f(t)}.
\end{aligned}
$$

Here $|\widetilde{K}(\cdot) - K(\cdot)|_\infty \approx 0$ provided that, in (3.2), the real $\delta > 0$ is sufficiently small. By applying Lemmata 2.2-2.4, we get

$$
|z(t)| \leq ce^{-\rho t}|z(0)| + a|f(\cdot)|_\infty.
$$

The evident estimation of $|f(\cdot)|_\infty$ completes the proof.

References

[1] R. Brockett, *Finite Dimensional Linear Systems*, John Wiley and Sons, New York, 1970.

[2] W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[3] J. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, American Mathematical Society, Providence, 1974.

[4] M. Jankovic, B.K. Ghosh, Visually guided ranging from observations of points, lines and curves via an identifier based nonlinear observer, *Systems Control Lett. 25 (1995)*, no. 1, 63–73.

[5] S. Soatto, R. Frezza, and P. Perona, Motion Estimation via Dynamic Vision, *IEEE Trans. Aut. Control 41 (1996)*, no. 3.