Nonexpansive maps and option pricing theory

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Abstract

The famous Black-Sholes (BS) and Cox-Ross-Rubinstein (CRR) formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalisations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. In this report, we argue that classical BS and CRR formulas can be actually obtained as a part of the theory of nonexpansive maps, which constitute now one of the most popular object of investigation in (max,+) algebra and its applications to Discrete Event Systems. This framework leads to another type of generalisations of BS and CRR formulas characteraised generally by more rough assumptions on common stocks evolution, which are therefore easier to verify.

1 Introduction

In (max,+)-algebra, the nonexpansive homogeneous maps are defined as the operators $\mathcal{B}$ on, say, $\mathcal{R}^n$, such that $|\mathcal{B}f - \mathcal{B}g| \leq |f-g|$ for all f, g with respect to the uniform norm $|f| = \max\{|f_j|, j = 1, ..., n\}$ and $\mathcal{B}(a+f) = a + \mathcal{B}f$ for any constant a. The theory of such mapings is fast developing right now, see e.g. [4],[5], [7] and references therein. It was proven for instance in [5] that any such map can be presented as the Bellman operator of some (stochastic) game with a value, and in particular, it is constructed by means of extended idempotent algebra with operations $\max, \min, +, \times$. The theory of nonexpansive maps is devoted mostly to the study of the iterates $\mathcal{B}^k$, and their asymptotic behavior as $k \to \infty$, which in turn is governed by the solutions of the "generalised eigenvalue problem" (GEP) $\mathcal{B}f = a + f$, $f \in \mathcal{R}^n, a \in \mathcal{R}$. We shall show now, how this problem appear in the theory of option pricing. We consider first the CRR model and its natural modifications with more rough assumptions on the possible behavior of underlying common stocks, in particular without any probabilistic assumptions, which are usually presented in this model. Then we discuss in this framework the options depending on several common stocks, continuous limit, and the possibility of including additional boundary conditions, which seem to be natural in realistic models.

2 Generalised CRR model

Consider a simplest model of financial market consisting of two securities: the risk-free bonds (or bank account) and common stocks. The prices of the units of these securities, $B = (B_k)$ and $S = (S_k)$ respectively, change in discrete moments of time $k = 0, 1, ...$ according to the recurrent equations

$$B_{k+1} = \rho B_k, \tag{1}$$

where $\rho \geq 1$ is a fixed number, and

$$S_{k+1} = \xi_{k+1} S_k, \tag{2}$$

where ξ_k is an (a priori unknown) sequence taking value in a fixed compact set $M \in \mathcal{R}$. We denote by u and d respectively the exact upper and lower bounds of M (u and d stand for up and down) and suppose that $0 < d < \rho < u$. We shall be interested especially in two cases:

(i) M consists of only two elements, its upper and lower bounds u and d,

(ii) M consists of the whole closed interval $[d, u]$.

No probability assumptions on the sequence ξ_k are specified. Case (i) corresponds to the CRR model and case (ii) stands for the situation when only minimal information on the future evolution of common stocks pricing is available, namely, the rough bounds on its growth per unit of time.

As usual, an investor is supposed to control the growth of his capital in the following way. Let X_{k-1} be his capital at the moment $k-1$. Then the investor choses his portfolio defining the number γ_k of common stock units held in the moment $k-1$. Then one can write

$$X_{k-1} = \gamma_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1}), \tag{3}$$

where the sum in brackets correspond to the part of the capital laid on the bank account (and which will thus increases deterministically). All operations are friction-free. The control parameter γ_k can take all real values (which is a commonly used assumptions in the class of models considered), i.e. short selling and borrowing are allowed. More general situations can be also included in our framework, which we discuss in the last section. In the moment k the value ξ_k becomes known and thus the capital becomes equal to

$$X_k = \gamma_k \xi_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1})\rho. \quad (4)$$

The strategy of the investor is by definition any sequence of numbers $\Gamma = (\gamma_1, ..., \gamma_n)$ such that each γ_j can be chosen using the whole previous information: the sequencies $X_0, ..., X_{j-1}$ and $S_0, ...S_{j-1}$. It is supposed that the investor, selling an option by the price $C = X_0$ should organise the evolution of this capital (using the described procedure) in a way that would allow him to pay to the buyer in the prescribed moment n some premium $f(S_n)$ depending on the price S_n. The function f defines the type of the option under consideration. In the case of the standard European call option, which gives to the buyer the right to buy a unit of the common stock in the prescribed moment of time n by the fixed price K, the function f has the form

$$f(S_n) = \max(S_n - K, 0). \quad (5)$$

Thus the income of the investor will be $X_n - f(S_n)$. The strategy $\gamma_1, ..., \gamma_n$ is called the hedge, if for any sequence $\xi_1, ..., \xi_n$ the investor has the possibility to meet his obligations, i.e. $X_n - f(S_n) \geq 0$. The minimal value of the initial capital X_0 for which the hedge exists will be called here the hedging price C_h of an option. The hedging price C_h is called correct (or fair), if moreover, $X_n - f(S_n) = 0$ for any hedge and any sequence ξ_j. The correctness of the price is equivalent to the impossibility of arbitrage, i.e. of a risk-free premium for the investor. That is why in ideal models of a market one wants the prices to be correct. It was in fact proven in [2] that for case (i) the hedging price C_h exists and is correct. In [2] some additional assumptions on probability distribution of the sequence ξ_k were used and the result was obtained by the martingale theory. Let us show that both for cases (i) and (ii) the hedge exists, is the same for both cases, and is expressed in term of the iterations of a certain nonexpansive map.

When calculating prices, one usually introduces the relative capital Y_k defined by the equation $Y_k = X_k/B_k$. Since the sequence B_k is positive and deterministic, the problem of the maximisation of the value $X_n - f(S_n)$ is equivalent to the maximisation of $Y_n - f(S_n)/B_n$. Consider first the last step of the game. If the relative capital of the investor at moment $n-1$ is equal to $Y_{n-1} = X_{n-1}/B_{n-1}$, then his relative capital at the next moment will be

$$= Y_{n-1} + \gamma_n \frac{S_{n-1}}{B_n}(\xi_n - \rho) - \frac{1}{B_n} f(\xi_n S_{n-1}).$$

Therefore, it is clear that the guaranteed income (in terms of relative capital) in the last step can be written as

$$Y_{n-1} - \frac{1}{B_{n-1}}(\mathcal{B}f)(S_{n-1}), \quad (6)$$

where the Bellman operator $\mathcal{B}$ is defined by the formula

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\xi \in M} [f(\xi z) - \gamma z(\xi - \rho)]. \quad (7)$$

Clearly, $\rho\mathcal{B}$ is a nonexpansive homogeneous map. We suppose further the function f to be nondecreasing and convex (perhaps, not strictly), having in mind the main example, which corresponds to the standard European call option and where this assumption is satisfied. Then the maximum in 7 is evidently attained on the end points of M and thus $(\mathcal{B}f)(z)$ is equal to

$$\frac{1}{\rho} \min_{\gamma} \max\left[f(dz) - \gamma z(d-\rho),\ f(uz) - \gamma z(u-\rho)\right]. \quad (8)$$

One sees directly that for $\gamma \geq \gamma^h$ (resp. $\gamma \leq \gamma^h$), the first term (resp. the second) under max in 8 is maximal, where

$$\gamma^h = \gamma^h(z, [f]) = \frac{f(uz) - f(dz)}{z(u-d)}. \quad (9)$$

It implies that the minimum in 7 is given by $\gamma = \gamma^h$, which yields

$$(\mathcal{B}f)(z) = \frac{1}{\rho}\left[\frac{\rho - d}{u - d} f(uz) + \frac{u - \rho}{u - d} f(dz)\right]. \quad (10)$$

The mapping $\mathcal{B}$ is a linear operator on the space of continuous functions on the positive line that preserves the set of nondecreasing convex functions. Using this property and induction in k one gets that the guaranteed relative income of the investor to the moment of time n is $Y_0 - B_0^{-1}(\mathcal{B}^n f)(S_0)$, and thus his guaranteed income is equal to

$$\rho^n (X_0 - (\mathcal{B}^n f)(S_0)). \quad (11)$$

The hedge strategy (the use of which guarantees him this guaranteed income) is $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$, where each γ_j^h is calculated step by step using formula 9. The minimal value of X_0 for which this income is not negative (and which by definition is the hedge price C_h of the corresponding option contract) is therefore given by the formula

$$C_h = (\mathcal{B}^n f)(S_0). \quad (12)$$

Using 10 one obtains that this C_h is equal to (CRR formula [2]):

$$\frac{1}{\rho^n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d}{u - d}\right)^k \left(\frac{u - \rho}{u - d}\right)^{n-k} f(u^k d^{n-k} S_0), \quad (13)$$

where C_n^k are standard binomial coefficients. One sees therefore that the hedge price of an option is given simply by the iteration of a nonexpansive map.

If the hedge strategy $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$ is used by the investor, then the two terms under max in expression 8 are equal (for each step $j = 1, ..., n$). Therefore, in the case (i) (when the set M consists of only two elements), if $X_0 = C_h$, the resulting income 11 does not depend on the sequence $\xi_1, ..., \xi_n$ and vanishes always, whenever the investor uses his hedge strategy, i.e. the prise C_h is correct in that case (Cox-Ross-Rubinstein theorem).

In general case it is not so anymore. Let us give first the exact formula for the maximum of the possible income (surplus value) of the investor in the general case supposing that he uses his hedge strategy. Copying the previous arguments one sees that this maximal income is given by the formula

$$\rho^n(X_0 - (\tilde{\mathcal{B}}^n f)(S_0)), \tag{14}$$

where

$$(\tilde{\mathcal{B}} f)(z) = \frac{1}{\rho} \min_{\xi \in M} [f(\xi z) - \gamma z(\xi - \rho)]|_{\gamma = \gamma^h}. \tag{15}$$

Thus, in the case of general M, the income of the investor playing with his hedge strategy will belong to the interval with the bounds given by formulas 11 and 14 and the fair price should be fixed somewhere in the interval $[(\tilde{\mathcal{B}}^n f)(S_0), (\mathcal{B}^n f)(S_0)]$, though it would not be more a risk-free price. When no other information is available, it is difficult to say something more precise. Moreover, unlike $\mathcal{B}^n f$, the expression $\tilde{\mathcal{B}}^n f$ is rather difficult to calculate. We present now a reasonable estimate for $\tilde{\mathcal{B}}^n f$ for the case (ii) and then give an approximate formula for a fair price using small additional assumption on common stock price evolution.

For any convex f, the minimum in 15 should be given by some ξ lying in the interval $[d, u]$. To get a simple estimate for this minimum, let us take $\xi = \rho$, which yields $(\tilde{\mathcal{B}}^n f)(z) \leq \rho^{-1} f(\rho z)$ and therefore by induction

$$(\tilde{\mathcal{B}}^n f)(z) \leq \rho^{-n} f(\rho^n z).$$

Looking at the evolution of the capital X_k as at the game of the investor with the nature (γ_k ans ξ_k are their respective controls) one can say that (assuming that the investor uses his hedge stratetgy) the nature plays against the investor, when its controls ξ_k lie near the boundary $[d, u]$ of the set M (then the investor gets his minimal guaranteed income 11) and conversely, it plays for the investor, when its controls ξ_k are in the middle of M, say, near ρ. If it is possible to estimate roughly the probability p that ξ_k would be near the boundaries of M, one can estimate the mean income of the investor (who uses his hedge strategy) by

$$\rho^n(X_0 - ((\mathcal{B}_{mean})^n f)(S_0)),$$

where

$$(\mathcal{B}_{mean} f)(z) = p(\mathcal{B} f)(z) + (1-p)\frac{1}{\rho} f(\rho z)$$

$$= \frac{1}{\rho}\left[p\frac{u-\rho}{u-d} f(dz) + (1-p) f(\rho z) + p\frac{\rho - d}{u-d} f(uz)\right], \tag{16}$$

which gives for fair price the following approximation

$$C = ((\mathcal{B}_{mean})^n f)(S_0). \tag{17}$$

In order to write it explicitly, let us denote by C_k^{ij} the coefficients in the polinomial development

$$(\epsilon_1 + \epsilon_2 + \epsilon_3)^k = \sum_{i+j \leq k} C_k^{ij} \epsilon_1^{k-i-j} \epsilon_2^i \epsilon_3^j.$$

Then for 17 one gets the following representation:

$$((\mathcal{B}_{mean})^n f)(S_0) = \frac{1}{\rho^n} \sum_{i+j \leq n} C_n^{ij} \left(p\frac{u-\rho}{u-d}\right)^{n-i-j}$$

$$\times (1-p)^i \left(p\frac{\rho - d}{u-d}\right)^j f(d^{n-i-j} \rho^i u^j S_0). \tag{18}$$

3 Option contracts on several common stocks

Suppose now there is a number, say I, of common stocks whose prices $S_k^i, i \in I, k = 0, 1, ...,$ satisfy the recurrent equations $S_k^i = \xi_k^i S_{k-1}^i$, where ξ_j^i take values in compact sets M_i with bounds d_i and u_i respectively. The investor controls his capital by chosing in each moment of time $k-1$ his portfolio consisting of γ_k^i units of common stocks of the type i, the rest of the capital being laid on the risk-free bank account. His capital X_k at the next time k becomes therefore

$$\gamma_k^1 \xi_k^1 S_{k-1}^1 + ... + \gamma_k^I \xi_k^I S_{k-1}^I$$

$$+ \rho(X_{k-1} - \gamma_k^1 S_{k-1}^1 - ... - \gamma_k^I S_{k-1}^I).$$

The premium to the buyer of the option at a fixed time n will be now $f(S_n^1, ..., S_n^I)$, where f is a given nondecreasing convex continuous function on the positive octant $\mathcal{R}_+^n$. For instance, the analog of the standard European option is given by the function $f(z_1, ..., z_I)$ of the form

$$\max(\max(0, z_1 - K_1), ..., \max(0, z_I - K_I)), \tag{19}$$

which describes the option contract that permits to the buyer to perchase one unit of the common stocks belonging to any type $1, ..., I$ by his choice.

To simplify formulas, we reduce ourselves to the case of two types of common stocks, i.e. to the case $I = 2$. The arguments similar to those of section 2 will give a similar formula to the guaranteed relative income of the investor in the last step of the game starting from the relative capital Y_{n-1} at the time $n-1$, namely

$$Y_{n-1} - \frac{1}{B_{n-1}}(\mathcal{B}f)(S_{n-1}^1, S_{n-1}^2),$$

where the Bellman operator $\mathcal{B}$ has the form

$$(\mathcal{B}f)(z_1, z_2) = \frac{1}{\rho} \min_{\gamma^1,\gamma^2} \max_{\xi^1 \in M_1, \xi^2 \in M_2} [f(\xi^1 z_1, \xi^2 z_2)$$

$$-\gamma^1 z_1(\xi^1 - \rho) - \gamma^2 z_2(\xi^2 - \rho)]. \quad (20)$$

In order to give an explicit formula for this operator (similar to 10), one should make additional assumptions on the function f. We say that a nondecreasing function f on $\mathcal{R}_+^2$ is nice, if the expression

$$f(d_1 z_1, u_2 z_2) + f(u_1 z_1, d_2 z_2) -$$

$$f(d_1 z_1, d_2 z_2) - f(u_1 z_1, u_2 z_2)$$

is nonnegative everywhere. One easily sees, for instance, that any function of the form $f(z_1, z_2) = \max(f_1(z_1), f_2(z_2))$ is nice for any nondecreasing functions f_1, f_2 and any numbers $d_i < u_i$, $i = 1, 2$, and in particular, function 19 is nice. Clear the nice functions constitute a linear space and the set of continuous nondecreasing convex nice functions is a convex subset in this space, which we denote NS (nice set). Furthermore, let

$$\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}. \quad (21)$$

Lemma. *If $f \in NS$ and $\kappa \geq 0$, then*

$$(\mathcal{B}f)(z_1, z_2) = \frac{\kappa}{\rho} f(d_1 z_1, d_2 z_2)$$

$$+\frac{1}{\rho}\left[\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2)\right]$$

and the γ^{h1}, γ^{h2} giving minimum in 20 are equal to

$$\gamma^{h1} = \frac{f(u_1 z_1, d_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_1(u_1 - d_1)},$$

$$\gamma^{h2} = \frac{f(d_1 z_1, u_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}.$$

If $\kappa \leq 0$ (and again $f \in NS$), then

$$(\mathcal{B}f)(z_1, z_2) = \frac{|\kappa|}{\rho} f(u_1 z_1, u_2 z_2)$$

$$+\frac{1}{\rho}\left[\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2)\right],$$

$$\gamma^{h1} = \frac{f(u_1 z_1, u_2 z_2) - f(d_1 z_1, u_2 z_2)}{z_1(u_1 - d_1)},$$

$$\gamma^{h2} = \frac{f(u_1 z_1, u_2 z_2) - f(u_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}.$$

The proof of this lemma uses only elementary manipulations. It follows that the operator $\mathcal{B}$ preserves NS and by the same induction as in the previous section one proves that if the premium is defined by a function $f \in NS$, then the hedge price for the option contract exists and is equal to

$$C_h = (\mathcal{B}^n f)(S_0^1, S_0^2). \quad (22)$$

One can write down a more explicit expression (analogous to 13). For instance, for the simplest case $\kappa = 0$,

$$C_h = \frac{1}{\rho^n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d_1}{u_1 - d_1}\right)^k$$

$$\times \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(d_1^{n-k} u_1^k z_1, d_2^k u_2^{n-k} z_2). \quad (23)$$

For the most important particular case, when the function f is of form 19, one can rewrite 23 more explicitly [6].

Formula 23 for C_h is very similar to 13. However, even if each M_i consists of only two points, this hedge price will be not correct. As in the previous section, one can represent the maximal income of the investor who uses his hedge strategy by the formula

$$\rho^n (X_0 - (\tilde{\mathcal{B}}^n f)(S_0^1, S_0^2))$$

with

$$(\tilde{\mathcal{B}}f)(z_1, z_2) = \frac{1}{\rho} \min_{\xi^1 \in M_1} \min_{\xi^2 \in M_2} [f(\xi^1 z_1, \xi^2 z_2)$$

$$-\gamma^1 z_1(\xi^1 - \rho) - \gamma^2 z_2(\xi^2 - \rho)]|_{\gamma^1 = \gamma^{h1}, \gamma^2 = \gamma^{h2}}.$$

To estimate the fair price of the option, let us follow the arguments of the previous section. Suppose that one can estimate the probability p of the numbers ξ_k^i to be near the boundaries of the corresponding sets M_i (the case when this probability is different for each type of common stocks can be evidently covered in the same way). Then one gets for the fair price the following estimate

$$C \approx ((\mathcal{B}_{mean})^n f)(S_0^1, S_0^2), \quad (24)$$

where (when supposing $\kappa = 0$ as above)

$$(\mathcal{B}_{mean} f)(z_1, z_2) = \frac{1-p}{\rho} f(\rho z_1, \rho z_2)$$

$$+\frac{p}{\rho}\left[\frac{\rho-d_1}{u_1-d_1}f(u_1z_1,d_2z_2)+\frac{\rho-d_2}{u_2-d_2}f(d_1z_1,u_2z_2)\right].$$

The explicit formula for 24 is similar to 18.

4 Continuous-time limit

As was shown in [2], the binomial CRR formula for option prices 13 tends to the Black-Sholes formula under apropriate limit procedure. Similar limits can be obtained for the formulas of the previous section. Following our metodology we make it in a simplest way ruling out all probability theory. The only "trace" of the geometric Brownian motion model of Black-Sholes will be the assumption that the logarithm of the relative growth of the stock prices is proportional to $\sqrt{\tau}$ for small intervals of time τ. More exactly, if τ is the time between the successive evaluations of common stock prices, then the bounds d_i, u_i of M_i are given by the formulas

$$\log u_i = \sigma_i\sqrt{\tau}+\mu_i\tau, \quad \log d_i = -\sigma_i\sqrt{\tau}+\mu_i\tau, \quad (25)$$

where the coefficients $\mu_i > 0$ stand for the systematic growth and the coefficients σ_i (so called volatilities) stand for "random oscillations". Moreover, as usual, $\log\rho$ is proportional to τ, i.e. $\log\rho = r\tau$ for some constant $r \geq 1$. Let $\mathcal{B}_\tau$ denote the corresponding operator 20, and $\mathcal{B}^t = \lim_{n\to\infty}\mathcal{B}_\tau^n$, where $\tau = t/n$. Under these assumptions, the calculation of the coefficient κ from 21 for small τ yields

$$\kappa = \frac{1}{2}\left(\frac{\sigma_1+\sigma_2}{2}+\frac{\mu_1-r}{\sigma_1}+\frac{\mu_2-r}{\sigma_2}\right)\sqrt{\tau}+O(\tau^{3/2}),$$

and aking the limit as $n\to\infty$ one easily gets (see e.g. [6]) for details) that the function $F(t,z_1,z_2) = (\mathcal{B}^t)f(z_1,z_2)$, satisfies the equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}\sigma_1^2z_1^2\frac{\partial^2 F}{\partial z_1^2}+\frac{1}{2}\sigma_2^2z_2^2\frac{\partial^2 F}{\partial z_2^2}+r(z_1\frac{\partial F}{\partial z_1}+z_2\frac{\partial F}{\partial z_2}-F) \quad (26)$$

with initial condition $F(0,z_1,z_2) = f(z_1,z_2)$. Rewriting this equation in terms of the function R defined by the formula

$$F(t,z_1,z_2) = e^{-rt}R(t, rt+\log z_1, rt+\log z_2)$$

one gets the standard heat equation

$$\frac{\partial R}{\partial t} = \frac{1}{2}\sigma_1^2\left(\frac{\partial^2 R}{\partial p_1^2}-\frac{\partial R}{\partial p_1}\right)+\frac{1}{2}\sigma_2^2\left(\frac{\partial^2 R}{\partial p_2^2}-\frac{\partial R}{\partial p_2}\right).$$

Consequantly, one can write the solution of the Cauchy problem for 26 explicitly, which yields the two-dimensional version of the Black-Sholes formula for hedging option price in continuous time:

$$C_h = \frac{e^{-rt}}{2\pi}\int_{\mathcal{R}^2} f(g_1,g_2)\exp\{-(u_1^2+u_2^2)/2\}\,du_1du_2,$$

where

$$g_j = S_0^j\exp\{u_j\sigma_j\sqrt{t}+(r-\sigma_j^2/2)t\}, \quad j=1,2.$$

The same procedure for the continuous limit of 13 gives the standard Black-Sholes formula

$$C_h = e^{-rt}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp\{-u^2/2\}$$
$$\times f(S_0\exp\{u\sigma\sqrt{t}+(r-\sigma^2/2)t\})\,du.$$

When the volatility σ is unknown and only its upper bound (which will be denoted by the same σ) is available, the hedge price C_h is surely not correct. The continuous limit of the estimates 17 and 24 of the option prices for corresponding discrete models can be found in the same way as above. Let us return for simplicitly to the most important case of only one type of common stocks. Using similar limit procedure for operator 16, one finds (in the same seting, i.e. under assumptions 25 on the stocks prices evolution) for the function

$$F(t,z) = (\mathcal{B}_{mean}^t f)(z) = \lim_{n\to\infty}(\mathcal{B}_{mean}^n(\tau)f)(z),$$

with $\tau = \frac{t}{n}$, the standard Black-Sholes equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}p\sigma^2z^2\frac{\partial^2 F}{\partial z^2}+rz\frac{\partial F}{\partial z}-F,$$

but with volatility $\sqrt{p}\sigma$ instead of σ. Thus the estimate for the fair price is given by the Black-Sholes formula with volatility $\sqrt{p}\sigma$. The same holds for the corresponding limit of 24.

5 Discussion and problems

It was shown by many investigations that the basic assumptions of the Black-Sholes model, especially the constancy of the volatility, are not fulfilled for real securities (see e.g. discussion in [1]). Recently there appeared many models, where this volatility is described by different types of stochastic processes, see e.g. [9], [11] and references there. The introduction of new processes make it always more difficult to estimate their (always changing) parameters on real financial markets. On the other hand, neither of these models seems to be able to cover all practical situations. This makes it natural just to admit that the volatility is unknown and perhaps only some bounds on it are available. This asumption leads inevitably to the risk-free premium on the hedging strategies (arbitrage situations). This idea was discussed in [8], where the Black-Sholes model with unknown volatility from a fixed convex set was investigated and fair prices were estimated through the solutions of rather difficult nonlinear partial differential equations. In the present paper, simple explicit estimates are given for fair option prices starting from a discrete model with similar assumptions.

We have always supposed (which is a commonly used assumption) in our models that the number of stock units γ, which an investor choses in every moment of time, is arbitrary (no restrictions are posed, this number can even be negative). However, in reality, the boundaries on possible values of γ seem to exist either due to the general boundary on the existing common stock units (one should suppose then that $\gamma \leq \gamma_0$ for some fixed γ_0), or due to the bounds on the possibilities of an investor to make (friction-free) borrowing (one should suppose then the restrictions of the type $\gamma_k \leq X_k/S_k$, say, when no borrowing is allowed). In both cases, one proves the existence of hedge strategies and the formula of type 11 for the hedging price by the same arguments, but the formula for the corresponding Bellman operator $\mathcal{B}$ would be different from 10 and the calculation of its iterations is a rather complicated task. Surely, one can use and develop different numerical methods. However, since the Bellman operator $\mathcal{B}$ is a nonexpansive homogeneous mapping in any case, one can expect that the theory of such mappings would allow to calculate some reasonable approximations to the iterations $\mathcal{B}^n$ for large n. More precisely, as was mentioned in the introduction, the asymptotic formulas for the iterations of nonexpansive maps are obtained in terms of the solutions of the GEP and therefore the problem under consideration is essentially reduced to the following question: *under what reasonable assumptions on the parameters of the model (say, the range of γ_k), the solution of GEP for the corresponding operator $\mathcal{B}$ exists, or/and when it is unique?* The same question arises in other natural realistic complications of the model, for instance, when the exchange of market securities is not friction free.

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