Design of L.Q.G. Dissipative Controllers Using Linear Matrix Inequalities

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Abstract

We present an application of the Linear Matrix Inequality approach of robust control to the design of multivariable L.Q.G. controllers verifying various sector conditions that generalizes existing results concerning positive real systems.
The main result is illustrated by considering robust low authority control of a multi-input multi-ouput flexible structure.
The method proposed in this paper provides a controller obtained by solving a single linear matrix inequality, since this a convex optimization problem it can been solved very efficiently using the current available software.
Keywords: dissipativity, robustness, linear matrix inequalities, optimal control, flexible structures

1 Introduction

The contribution of this paper is to show how the framework of Linear Matrix Inequalities (L.M.I.) can be used in order to construct Linear Quadratic Gaussian (L.Q.G.) controllers satisfying specific extended dissipativity conditions. Our first motivation stems from the problem of the robust attitude control of a satellite in presence of non conservative sloshing motions but our method appeared to be also able to perform robust non colocated control of flexible structures. Accordingly our main result is here applied to a non collocated multi-input multi-ouput springs-masses structure.

It is a well established fact that linear quadratic control design may fail to possess elementary robustness properties with respect to parametric uncertainties and that in the output feedback case the the phase and gain margins are no longer those obtained with a full-state feedback assumption.
There exists many contributions trying to remedy to this situation in order to secure both performance and robustness for uncertain systems. Among the more popular we can cite the Loop Tansfer Recovery approach (L.T.R.), the 'overbounding' procedures used in guaranteed-cost design or the various works issued from Kharatonitov's theorem for a more state space oriented and parametric approach.
Concerning robust control of flexible structures a major trend rests on physical considerations by an exploitation of the natural dissipation of these systems.

In particular for passive or positive-real sytems Lozano-Leal-Joshi [1] and Bals-Goh-Grubel [5] have shown how an adequate choice of the weighting and covariance matrices permits the obtention of positive real L.Q.G. controllers.

In this paper the method is extended to design of controllers verifying more general sector conditions in order to be able to consider systems exhibiting possible lack of positivity.
In view of applications to aerospace systems there is a real need for such a generalization. Indeed, for real systems , positive realness is never met exactly and even for very simple mechanical models non positivity typically occurs with 'non colocated systems' or when sensor-actuator dynamics or sloshing motion are taken into account.
We first briefly review some well-known facts concerning positive control.

Positivity

The notion of positivity is inherited from networks theory and is widely used in adaptative and nonlinear control. Positive realness is moreover closely linked to the mechanical concepts of dissipativity and energy: in fact loosely speaking positive real systems can be described as systems that do not generate energy.
An important class of positive systems is given by flexible structures with dual colocated sensor-actuator pairs.
For such systems positive control provides a powerful technique that takes into account phase information

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and can garantee stability in presence of significant modeling uncertainties.
The basic result linking robust and positive controls is given in the next theorem.
In the context of Large Space Structures it has lead naturally to the so-called Low Authority Control-High Authority Control design which is based on the classical scheme of a two level controller cf. Ref. [5] for more details.

Theorem 1.1 (Passivity theorem) *If a strictly proper, strictly positive real controller is used in a negative feedback connection with a proper positive real plant then the closed-loop sytem is asymptotically stable.*

Such a result generally attributed to V. Popov and G. Zames has a unique robustness flavour since stability is guaranteed only by positivity assumptions and in particular it is not affected by the presence of any parametric errors.

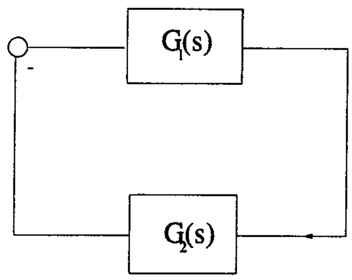

Figure 1: *A general stability result*

2 Extended Dissipativity

Supply Rates and Storage Functions

A natural and important generalization of positivity due to J.C. Willems [2], is now presented following the presentation of Gupta-Joshi [3]:
A system with input u and output y will be said dissipative with respect to the *supply rate* :
$p(y\,,u) = (y^t\,u^t)\begin{pmatrix} Q & N \\ N^t & R \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$
if there exists a *storage function*
$E\,:\,R^n \longrightarrow R_+$ such that:

$$\int_0^T p(y\,,u)\,dt \geq E(x(t)) - E(x_0)$$

$$\forall\, T \geq 0\,,\ \forall u \in L_+^2$$

The following examples show that despite its simplicity, the above framework is in fact sufficiently general to encompass both positive and H_∞ control.

Examples of supply rates

- Positive real systems:
  $p(y\,,u) = 2\,y^t u$
- H_∞-norm bound systems:
  $p(y\,,u) = \gamma^2 u^t u\ - y^t y$
- Sector-bounded systems:
  $p(y\,,u) = \ -ab\,u^t u\ +\ (a\ +\ b)\,y^t u\ -\ y^t y$

Notation: The above sector-bounded systems will be said to belong to *Sect* $[a, b]$, in particular with this convention positive real and H_∞ norm-bound systems belongs respectively to *Sect* $[0, \infty)$ and *Sect* $[-\gamma, \gamma]$

Characterization of Dissipative systems

A very general criterion, whose main interest is however theorical, in order to determine if a dynamical system is dissipative for a given supply rate is the following variational test:

Proposition 2.1 (Willems) *A system Σ with input-output pair $(u, y) \in \mathcal{U} \times \mathcal{Y}$ is dissipative w.r.t. supply rate $p(u\,,y)$ if an only if along all trajectories starting from x_o the function :*

$$S_a(x_o) = \sup_{u,T} - \int_0^T p(u\,,y)\,dt$$

where, $x(0) = x_o$ is well defined that is :
$S_a(x_o) < \infty\,,\ \ \forall x_o \in \mathcal{X}$,
the supremum being taken for all $T \geq 0$ and $u \in \mathcal{U}$.

$S_a(x)$ is called the available storage : it verifies the dissipation inequality and is in fact a lower boung among all possible storage functions.
For a linear dynamical system (A, B, C, D), storage functions are naturally searched as quadratic functions of the state.
The evaluation of the time derivative of $E(x) = \frac{1}{2}\,x^t P x$ along trajectories of the system leads quite directly to the following LMI characterization :

Proposition 2.2 (Willems) *A general dynamical system $(A\,,B\,,C\,,D)$ minimal is dissipative w.r.t.*
$p(y\,,u) = (y^t\,u^t)\begin{pmatrix} Q & N \\ N^t & R \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$ *i.f.f. :*
$\exists\, P\ =\ P^t > 0$ *solution of the following Linear Matrix Inequality :*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^t & \mathcal{C} \end{pmatrix} \geq 0$$

with :
$\mathcal{A} = C^t Q C - (PA + A^t P)$
$\mathcal{B} = C^t(QD + N) - PB$
$\mathcal{C} = R + N^t D + D^t N + D^t Q D$

This result which admits as special cases the well-known Positive and Bounded Real Lemmas has a frequential counterpart:
if $Q \leq 0$, it also equivalent to:

$$\begin{pmatrix} G^*(j\omega) & I \end{pmatrix} \begin{pmatrix} Q & N \\ N^t & R \end{pmatrix} \begin{pmatrix} G(j\omega) \\ I \end{pmatrix} \geq 0$$

A system will be said strictly dissipative if this last inequality is strictly verified.
Note moreover that for SISO systems the preceding frequential inequality has an obvious interpretation in terms of Nyquist plot for each of the three examples of supply rates given, this fact will be exploited in the last section of this paper.
For example one can verify that a transfer function $g(s)$ belongs to *Sect* (a, b) if and only if its frequency response lies inside a circle of diameter $[a, b]$ centered on the real axis.

A General Asymptotic Stability Result

We are now able to state the following important result which generalizes theorem 1.1 and can be found in Gupta-Joshi [3].

Theorem 2.1 *Consider again the feedback connection of Fig.1.*
If Σ_1 is dissipative w.r.t. $p_1(u, y)$ and Σ_2 strictly dissipative w.r.t. $p_2(u, y)$.
Where the supply rates $p_1(\,,\,)$, $p_2(\,,\,)$ verify:

$$\alpha p_1(u, y) + \beta p_2(-y, u) \leq 0$$

for some real constants α and β.
Then the closed-loop sytem is asymptotically stable.

The small gain and passivity theorems can be seen as special cases of theorem 2.1.
Note that this general result does not require in fact any assumption of linearity.

3 Main Result

We will now consider a proper, minimal system (A, B, C) belonging to *Sect* $[-a, \infty)$ with $a > 0$ this means that this non positive system is dissipative with respect to the supply rate: $p(y, u) = 2\,y^t u + 2\,a u^t u$
or equivalently that (A, B, C, aI) is positive real.
In both cases one can verify that the above LMI characterization insures that there exists $P = P^t > 0$, L, W of appropriate dimension such that:

$$\begin{cases} PA + A^t P = -L^t L \\ PB = C^T - L^t W \\ 2\,a I = W^t W \end{cases}$$

Before going further and in order to present our main result we have to recall briefly the basic elements of L.Q.G. control.

Linear Quadratic Gaussian Design

Let be given the following minimal system where v and w are two gaussian white noises.

$$\begin{cases} \dot{x} = Ax + Bu + v \\ y = Cx + w \end{cases}$$

We are searching for a dynamic compensator:

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c + D_c y \end{cases}$$

such that the quadratic performance index is minimized:

$$J = lim_{T \to \infty} \frac{1}{T} \mathcal{E} \int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^t \begin{pmatrix} Q_c & S_c \\ S_c{}^t & R_c \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt$$

with: $\begin{pmatrix} Q_c & S_c \\ S_c{}^t & R_c \end{pmatrix} \geq 0$ and $R_c > 0$

The solution of this problem is well known to be:

$$\begin{aligned} A_c &= A - BK_c - K_f C \\ B_c &= K_f \\ C_c &= K_c \\ D_c &= 0 \end{aligned}$$

where:

$$K_c = R_c^{-1}(P_c B + S_c)^t$$
$$K_f = (P_f C^t + S_f) R_f^{-1}$$

P_c and P_f being respectively the positive solution of the classical control and filtering ARE:

$$PA + A^t P - (PB + S_c) R_c^{-1} (PB + S_c)^t + Q_c = 0$$
$$PA^t + AP - (PC^t + S_f) R_f^{-1} (PC^t + S_f)^t + Q_f = 0$$

Where Q_f, R_f, S_f are the three covariance matrices classically associated to v and w that is:

$$\begin{aligned} Q_f \delta(t - \tau) &= \mathcal{E} v(t)^t v(\tau) \\ R_f \delta(t - \tau) &= \mathcal{E} w(t)^t w(\tau) \\ S_f \delta(t - \tau) &= \mathcal{E} v(t)^t w(\tau) \end{aligned}$$

The following theorem contains our main result whose proof is given in the last section of the paper.

Theorem 3.1 *Suppose that the minimal system (A, B, C) belongs to Sect $[-a, \infty)$, $a > 0$.*
Let the weighting matrices be chosen as follows:
$R_c = R_f \geq 2\,aI$, $Q_c = L^t L + C^t R_c^{-1} C$
$Q_f = P^{-1} Q_c P^{-1}$, $S_c = L^t W$, $S_f = 0$
Then (A_c, B_c, C_c), the corresponding L.Q.G. controller will belong to Sect $[0, \frac{1}{a})$.

Remarks :

- The variance matrices have no longer any statistical meaning and must be considered here as mere design parameters. In this sense the term H_2 control may be more appropriate than LQG control.

- The introduction of fictitious input noises for design purpose is a feature shared with the Loop Transer Recovery (L.T.R.) methodology.

- For positive real systems that is if $a = 0$, we recover previous results contained in [1] and [5] where it is shown,in particular, that the choices: $R_c = R_f > 0,\ Q_c = L^tL + C^tR_c^{-1}C$ $Q_f = P^{-1}Q_cP^{-1}$, $S_c = S_f = 0$ insure positivity of the LQG controller.

- Lastly as it will apparent in the course of the proof proposed below, the choice of the ponderation matrices is not unique and more general choices of these design matrices can be envisaged.

Illustration of theorem 3.1

In order to precise the significance and the scope of our result, we are going to illustrate it on generic model of single-input single-output flexible structure. If the inequality $R_c \geq 2aI$ is clearly a limitation on the controller's authority due to the lack of positivity of the initial plant.
The presence of a cross-term matrix S_c may seem more surprising, however its importance and role can be simply illustrated as follows.
Consider a transfer function with the following residue-pole form:

$$g(s) = \sum_{i=1}^{n} \frac{k_i s}{s^2 + 2\xi_i\omega_i s + \omega_i^2}$$

Suppose that only part of the residues k_i are positive e.g. $k_i < 0$ for $i > n_1$.
In the context of flexible structures equipped with rate sensors such a situation is frequently encountered for non collocated systems or in presence of non conservative forces.
If $g(s)$ denotes the transfer between actuator and rate sensor signal, a negative residue indicates that the slope of the corresponding mode shape at sensor location has the opposite sign of the slope at actuator's (c.f. [7]).
As a result it is well known that a positive real control (e.g. a positive rate feedback) can destabilize the system.
A (modal) realization of $g(s)$ is given by:

$$A = diag\,(A_i)$$

$$A_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & -2\xi_i\omega_i \end{pmatrix}$$

$$B = (|k_1|\ 0\ |k_2|\ 0 \ldots |k_{n-1}|\ 0\ |k_n|\ 0)^t$$

$$C = (1 \quad 0 \quad 1 \quad 0 \cdots -1 \quad 0 \ -1 \quad 0)$$

corresponding to a state :

$$x = (q_1\ \dot{q}_1\ \ldots q_n\ \dot{q}_n)^t$$

Now we know from general theory of observer-based design that the control input will be of the form :

$$u = -K_c\hat{x}$$

$$\text{with: } K_c = R_c^{-1}(B^tP + S_c^t)$$

and $\hat{x}$ the reconstructed state

Here since:

$$\begin{cases} S_c = L^tW \\ PB = C^t - L^tW \end{cases},$$

we have:

$$u = -R_c^{-1}C\hat{x}$$

And thus, we obtain the following feedback law:

$$u = -\kappa_1\dot{\hat{q}}_1 \ \cdots - \kappa_{n_1}\dot{\hat{q}}_{n_1} + \cdots + \kappa_n\dot{\hat{q}}_n$$

Where for simplicity we have set:

$$R_c = diag(\kappa_i{}^{-1})$$

The precise role of the matrix S_c is thus to reverse the sign of the unstably interacting modes in order to prevent a positive (i.e. destabilizing) feedback.

4 Effective Design and Simulation Results

As indicated in Section 2 the examination of the Nyquist plot for SISO systems and/or the resolution of the associated LMI for the multivariable case provides a simple general methodology susceptible to be applied to systems presenting various defects of positivity.
Indeed if the Nyquist plot of a scalar transfer function is included in a circle of diameter [-a,b], $a > 0$ centered on the real axis or if more generally we have : $G(s) \in Sect\,[-a\,,\,b]$.
Then by virtue of the general stability result contained in theorem 2.1 the choice of any: $K(s) \in Sect\,[-\frac{1}{b}, \frac{1}{a}]$ will ensure stability.
Such an approach has obviously close connexions with the multivariable circle criterion.
Note however that in the context of flexible structures the presence of a very high number of lightly damped structural modes leads naturally to consider systems $G(s) \in Sect\,[-a, \infty)$ that is systems described by the hypotheses of theorem 3.1.

4.1 Non Collocated Flexible System

We apply our method to the following multivariable system which is taken from [6] where the following data are given :

$$\begin{cases} m_1 = 2 \ , \ m_3 = 2 \ , \ m2 = 1 \\ k_1 = k_4 = 5 \ , \ k_2 = k_3 = 2 \\ d_1 = 0.5, d_2 = 0.2, d_3 = 0.15, d_4 = 0.45 \end{cases} .$$

Input forces are applied at masses 1 and 3 while the velocities of masses 2 and 3 are measured.

One can verify that the above system belongs to

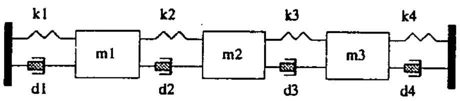

Figure 2: Springs-masses mechanical system

Sect $(-a, b)$ with $a = 0.582$ and $b = 2.662$.
In order to take into account possible neglected higher frequency dynamics we will consider it as belonging to *Sect* $(-a, \infty)$ and apply litterally the analytic formulae given in Section 3.
We take as in [6] the H_2-norm of the closed loop system as a performance measure.
That is we are searching to minimize $\mathcal{P}$ the the root mean square values of the outputs when unit intensity white processes are applied in each of the two input channels.
Using the program Semidefinite Programming SP of Vandenberghe and Boyd as implemented in the package LMITOOL [8].
We obtain : $\mathcal{P} = 0.8088$
This value of the performance index is a little greaterthan the one obtained in [6] with a LQG belonging to *Sect* $(-1/b, -1/a)$ after a trial and error search.
Note moreover that the above value of $\mathcal{P}$ remains slightly better than the one obtained in [6] by another LQG controller designed this time from a H_∞-norm characterization of the system.

5 Conclusions

We have shown how the notion of dissipativity can be used in order to provide controllers achieving both parametric and frequential robustness. This powerful methodology, largely initiated by J.C. Willems, is here developped within the classical framework of L.Q.G. or H_2 control.
Our approach that uses a quantitative characterization of linear systems via the LMIs, sectoricity, and dissipativity concepts allows a systematic robust control design for non positive systems.
We have illustrated it on generic models of flexible structures.
The extension of the above method to the design of generalized dissipative compensators subject to mixed H_2/H_∞ constraints as in Haddad-Bernstein-Wang [9] is currently under investigation.

Appendix

Proof of theorem 3.1.

We have to prove that if (A, B, C, aI) is positive real the controller (A_c, B_c, C_c) in theorem 3.1 belongs to *Sect* $[0, \frac{1}{a})$.
Using the LMI characterization of Section 2 we are searching some $P > 0$ such that:

$$\begin{cases} A_c^tP + PA_c + 2a\, C_c^tC_c < 0 \\ PB_c = C_c^t \end{cases}$$

Under the hypothesis that there exists:
$P = P^t > 0$, L_a such that:

$$\begin{cases} PA + A^tP = -L^tL \\ PB = C^T - L_a^t \end{cases}$$

Where we have set : $L_a^t = L^tW$, $2\,a\,I = W^tW$
It is readily seen that:

$$PB + L_a^t = C^t$$

implies:

$$PB_c = C_c^t$$

that is:

$$P(P_fC^t + S_f)R_f^{-1} = (P_cB + S_c)R_c^{-1}$$

provided: $R_f = R_c \ \ S_c = L_a^t$, $S_f = 0$
and : $P_c = P_f^{-1} = P$
Now the equation:

$$A^tP + PA = -L^tL$$

can be obviously rewritten in the form of the usual control and filtering Riccati equations :

$$PA + A'P - (PB + S_c)R_c^{-1}(PB + S_c)' + Q_c = 0$$
$$P_fA' + AP_f - (P_fC' + S_f)R_f^{-1}(P_fC' + S_f)' + Q_f = 0$$

with:

$$Q_c = L^tL + C^tR_c^{-1}C$$

and

$$Q_f = P^{-1}L^tLP^{-1} + P^{-1}C^tR_f^{-1}CP^{-1}$$

Now we only have to verify that:

$$P(A - BK_c - K_fC) + (A - BK_c - K_fC)^tP + 2\,aK_c^tK_c < 0$$

in order to insure: $(A_c, B_c, C_c) \in$ *Sect* $[0, \frac{1}{a})$
Recalling that:

$$A^tP + PA = -L^tL$$
$$PBK_c = PBR_c^{-1}(PB + S_c)^t \ ,$$
$$PK_f = C^tR_c^{-1}$$

We therefore have:

i) $P(A - K_fC) + (A - K_fC)^tP = -L^tL - 2C^tR^{-1}C$

ii) $PBK_c + K_c^t B^t P$
$= PBR^{-1}(PB + S_c)^t + (PB + S_c)R^{-1}B^t P$
$= (PB+S_c)R^{-1}(PB+S_c)^t - S_c R^{-1} S_c^t + PBR^{-1}B^t P$
$= C^t R^{-1} C - S_c R^{-1} S_c^t + PBR^{-1} B^t P$

iii) $2\,aK_c^t K_c = 2\,aC^t R^{-2} C$

Adding these three terms, we obtain:

$PA_c + A_c^t P + 2\,aC_c^t C_c$
$= -L^t L - 3C^t R^{-1} C + S_c R^{-1} S_c$
$- PBR^{-1} B^t P + 2\,aC^t R^{-2} C$
using: $S_c = L_a^t = L^t W$
it is also equal to:
$-L^t(I - WR^{-1}W^t)L - PBR^{-1}B^t P$
$- C^t(3R^{-1} - 2\,aR^{-2})C$

It is clear that the above quantity will be negative definite provided:
$$\begin{cases} I - WR^{-1}W^t \geq 0 \\ 3R^{-1} - 2\,aR^{-2} \geq 0 \end{cases}$$
Since : $W^t W = 2aI$ it will verified if $R \geq 2\,aI$.
Finally this last condition which exprimes a limitation of the controller authority is also sufficient to insure:
$$\begin{pmatrix} Q_c & S_c \\ S_c{}^t & R_c \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} R_c > 0 \\ Q_c - S_c^t R_c^{-1} S_c \geq 0 \end{cases}$$
since this equivalent to :
$$\begin{cases} R_c > 0 \\ L^t L + C^t R^{-1} C - WR^{-1} W^t \geq 0 \end{cases}$$
Again $R \geq 2\,aI$ suffices to insure that the L.Q.G. problem considered is well-posed.

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