Eigenvalue assignment with spectral condition number optimization by static output feedback

Luciano Carotenuto

D.E.I.S.,
Università degli Studi della Calabria
Via P. Bucci, Cubo 42-C
87036 Rende (Cs), Italy
Phone: +39.0984.494723
Fax : 39.094.494713
Email: carotenuto@unical.it

Giuseppe Franzè

D.E.I.S.,
Università degli Studi della Calabria
Via P. Bucci, Cubo 42-C
87036 Rende (Cs), Italy
Phone: +39.094.494728
Fax : 39.094.494713
Email: franze@deis.unical.it

***Abstract*— This paper presents a method for the robust assignment of eigenvalues by static output feedback, based on the minimization of the spectral condition number of the closed -loop dynamic matrix. The procedure relies on an assignment method recently proposed by the authors, that enables to explicitly parameterize the set of the feedback gain matrices that assign the prescribed closed-loop spectrum, and on the column-scaling properties of eigenvector matrices.**

Keywords: Eigenvalue assignment, Spectral condition number, Robust control, Genetic algorithms.

I. Introduction

In this paper we propose an original procedure to solve the *static output feedback eigenvalue assignment problem* (**SOFEAP**) *with optimization of the spectral condition number*. In the control area it has been considered since a long time by several authors, but mainly with reference to state feedback (**SFEAP**) for which many solution algorithms have been proposed (see f.i. [1],[2] and references therein). Output feedback is dealt with in refs. [3] and [4]. The aim is to find an output feedback gain matrix that minimizes the spectral condition number of the closed loop dynamic matrix, subject to the constraint that its spectrum is a specified symmetric set of the complex plane: then we are faced with a constrained optimization which requires the solution of two "difficult" problems:

- the assignment of the eigenvalues of a LTI system by static output feedback;
- the computation of the spectral condition number of a matrix.

As to the first problem let $A(n \times n), B(n \times m), C(p \times n)$ the triple of matrices characterizing the LTI system, with the usual assumptions of minimality and of full rank for B and C. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be the desired spectrum. A solution of the assignment problem is a real matrix $K(m \times p)$ such that the spectrum $\sigma(A + BKC) = \Lambda$. The results from [5] ensure that the eigenvalue assignment has a real solution if $mp > n$, generically with respect to the choice of the triple (A, B, C) and of the spectrum Λ. This result definitively improves the old sufficient condition of Kimura [6] $m+p > n$, and is sharp, because it is known that if $mp = n$ (f.i. $m = p = 2, n = 4$) the real solvability is not generic. The condition $mp > n$ means that the number of unknowns (the entries of K) is greater than the number of constraints (the equations that impose the eigenvalues or the coefficients of the characteristic polynomial). By a continuity argument if a real solution exists it must belong to an $r = mp - n$ dimensional manifold of real solutions, where a feedback matrix can be chosen as to satisfy further requirements. Although we have now a clear picture of the conditions that ensure generic assignability of the eigenvalues by static output feedback, there not exists a generally accepted algorithm for the numerical computation of a solution, and less an efficient way to represent the set of all solution matrices. This is due to the intrinsic non-linearity of the SOFEAP, which makes it qualitatively different from SFEAP. Moreover some relatively simple algorithms that have been proposed work only in case the Kimura sufficient condition $m + p > n$ holds. If it is necessary not only to assign the eigenvalues, but also to satisfy further requirements, for instance to optimize some performance index, it is extremely important how the solution set is parameterized. An appropriate parameterization allows to transform a constrained optimization problem into an unconstrained one, whose decision variables are the parameters left free by the assignment. As to the spectral condition number ([7],[8]) of a matrix M, denoted as $\kappa_{2S}(M)$, let us consider for simplicity only the case of M diagonalizable, with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$: then $\kappa_{2S}(M)$ is the smallest condition number of an eigenvector matrix T of M. Finding the spectral condition number of a diagonalizable matrix is by no means trivial, because it is required to choose optimally the columns of T, within the respective one dimensional subspaces, or equivalently to find the optimal scaling of the columns of T. This justifies the interest in the literature ([9],[10] and refs. therein) for lower and upper bounds to $\kappa_{2S}(M)$. The relevance of the spectral condition number of M arises from its connection to the sensitivity of the eigenvalues

of M to perturbations. Several versions of this connection are in the literature, and we report in this form: let E be a matrix,$\mu_1, \ldots, \mu_n$ the eigenvalues of $M + E$; if the disks $D_i = \{\mu : |\lambda_i - \mu| < \kappa_{2S}(M)\|E\|_2\}$ are disjoint, then the eigenvalues of $M + E$ can be ordered in such a way that $|\lambda_i - \mu_i| < \kappa_{2S}(M)\|E\|_2$, $i = 1, 2, \ldots, n$. The above inequality can also be read as: if the norm of the perturbation is bounded by $\|E\|_2 < \rho/\kappa_{2S}(M)$, then the eigenvalues of the perturbed matrix $M+E$ belong to the disks (assumed disjoint) with center in λ_i and radius ρ. When we put $M = A+BKC$, we clearly see that SFEAP and SOFEAP can be regarded as special cases of the robust pole placement problem for uncertain systems: given an uncertain system, find a control law such that the closed loop system has poles in a prescribed region for all suitably characterized perturbations. Some other approaches to the robust pole placement are in [11]-[13]. The procedure we propose is based on a recent formulation of the SOFEAP [14]. It allows to parameterize explicitly the family of feedback gain matrices assigning a given spectrum: the resulting algorithm efficiently assigns the eigenvalues, once the free parameters have been fixed. The optimization is performed using a genetic algorithm (GA) [15]. As well known, GA's are general-purpose search algorithms, based on the mechanics of natural evolution, that have been often used in system analysis an control problems (see for instance [16]-[17]). This choice is due to the ease of implementation and to the widely recognized ability of GA's to quickly locate near-optimal solutions of problems with non-convex, multi-modal objective functions. A GA seems appropriate to the problem here considered, because of the lack of a-priori information about the properties of the spectral condition number as a function of the gain matrix.

II. Eigenvalue assignment algorithm

In order to reduce the notational burden, it is assumed that the closed loop eigenvalues are distinct and do not belong to the spectrum of A. Moreover let $m \leq p$. The theoretical basis, as well as the general formulation where the above assumptions are removed, can be found in [14]. Let

- $A = V^T \text{diag}(\pi_1, \ldots, \pi_n) U$;
- $\Omega = [\omega_1, \ldots, \omega_n]$ the closed-loop eigenvector matrix;
- $\Gamma = U^{-1}\Omega$;
- $\{i_1, \ldots, i_m\}$ a subset of $\{1, \ldots, n\}$;
- V_m^T the matrix made of rows $\{i_1, \ldots, i_m\}$ of V^T;
- $z_k = \text{diag}\left[(\lambda_k - \pi_{i_1}), \ldots, (\lambda_k - \pi_{i_m})\right] \begin{bmatrix} \gamma_{k i_1} \\ \vdots \\ \gamma_{k i_m} \end{bmatrix}$, $k = 1, \ldots, p$.

Then, all the gain matrices K which assign the eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_p\}$ of $A + BKC$ are given by:

$$K = (V_m^T B)^{-1} \left[z_1 | \ldots | z_p\right] \left[C\omega_1 | \ldots | C\omega_p\right]^{-1}. \quad (1)$$

where $\omega_1, \ldots, \omega_p$ are chosen arbitrarily in the so-called characteristic subspaces:

$$\omega_k = (\lambda_k I - A)^{-1} \left[\theta_{k1} \ \ldots \ \theta_{k(m-1)} \ 1\right]^T, \quad k = 1, \ldots, p. \quad (2)$$

The remaining n-p eigenvalues are assigned by the reduced observer method [14] which leads to solve a set of $n - p$ multivariate polynomial equations in the $p(m-1)$ variables

$$\Theta_p = [\theta_{11}, \ldots, \theta_{1(m-1)}, \theta_{21}, \ldots, \theta_{p1}, \ldots, \theta_{p(m-1)}].$$

Then, the "conceptual" assignment algorithm can be stated as:

AL1 -

- Fix $r = mp - n$ parameters $\underline{\Theta}_R = [[\Theta_p]_{i_1}, [\Theta_p]_{i_2}, \ldots, [\Theta_p]_{i_r}]$;
- Solve the $n - p$ multivariate polynomial equations that assign the eigenvalues of the reduced-order observer, with respect to the unknowns $[\Theta_p]_j, j = 1, 2, \ldots, mp-p$, $j \neq i_1, i_2, \ldots, i_r$;
- Compute $\omega_1, \omega_2, \ldots, \omega_p$ from eq. (2);
- Compute K from eq. (1).

Let $\underline{\Theta}_R^*$ denote a specific value of the parameter vector to which the above described operations associate a real K^*: the operations performed to find K are such that there exists a neigborhood of $\underline{\Theta}_R^*$ where we can define a continuous function $\underline{\Theta}_R \rightarrow K(\underline{\Theta}_R)$. In what follows we shall assume that the parameter vector $\underline{\Theta}_R$ belongs to a compact set $\mathbf{S}_r$ of $\mathcal{R}^r$, where the function $K(\underline{\Theta}_R)$ is continuous.

III. Parameterization of the eigenvector matrix

As pointed out in the introduction, the spectral condition number of a diagonalizable matrix M is obtained by searching in the set of all matrices which diagonalize M that one with minimal condition number:

$$\kappa_{2S}(M) = \min_{T \in \mathbf{G}} \|T\|_2 \|T^{-1}\|_2,$$
$$\mathbf{G} = \{T \,|\, T^{-1} M T = diag(\lambda_1, \ldots, \lambda_n)\}.$$

To perform the minimization in practice, it is necessary to characterize the set $\mathbf{G}$. To this purpose we make the further assumption that the eigenvalues are distinct: it is well known that in this case all the matrices diagonalizing M can be written as

$$T = T_0 \text{diag}(\alpha_1, \ldots, \alpha_n) = \alpha_n T_0 \text{diag}(\beta_1, \ldots, \beta_{n-1}, 1),$$

where T_0 is any matrix whose columns are eigenvectors associated to respective eigenvalues. Then we have, putting $\underline{\beta} = \{\beta_1, \ldots, \beta_{n-1}\}$,:

$$\begin{aligned} \|T\|_2 \|T^{-1}\|_2 = \ & |\alpha_n| \, \|T_0 \, \text{diag}(\underline{\beta}, 1)\|_2 \\ & \cdot |\alpha_n^{-1}| \, \|\text{diag}(\underline{\beta}, 1)^{-1} T_0^{-1}\|_2 \end{aligned}$$

and finally

$$\begin{aligned} & \kappa_{2S}(M) = \\ & \min_{\underline{\beta} \in \mathcal{R}^{n-1}} \|T_0 \text{diag}(\underline{\beta}, 1)\|_2 \ \|\text{diag}(\underline{\beta}, 1)^{-1} T_0^{-1}\|_2 \end{aligned} \quad (3)$$

In the above formula T_0 is arbitrary, but particular choices can be advantageous from the computational point of view. An interesting result is in [10]: the condition number of the matrix $\overline{T}$ whose columns are the eigenvectors of M, with unity 2-norm, say $\kappa_{2N}(M) = \|\overline{T}\|_2 \, \|\overline{T}^{-1}\|_2$, provides a "near-minimal" estimate of $\kappa_{2S}(M)$, and in any case

$\kappa_{2N}(M)/\sqrt{n} \leq \kappa_{2S}(M) < \kappa_{2N}(M)$. Then it is obvious to have $T_0 = \overline{T}$.

IV. Problem formulations and solution algorithms

The abstract formulation of the SOFEAP with optimization of the spectral condition number of the closed-loop dynamic matrix can be given as follows: let $\mathcal{F}_\Lambda$ be the set of real feedback matrices assigning the given spectrum Λ: $\mathcal{F}_\Lambda = \{K \in \mathcal{R}^{m\times p} | sp(A+BKC) = \Lambda\}$; let $G(K)$ be the set of matrices diagonalizing $A_{CL} = A + BKC : G(K) = \{\Omega \in \mathcal{C}^{n\times n} | \Omega^{-1}(A+BKC)\Omega = \Lambda\}$; then

PB1A - *Find* K^* *such that*

$$\begin{aligned}&\kappa_{2S}(A+BK^*C) = \min_{\Omega\in G(K^*)} \|\Omega\|_2\|\Omega^{-1}\|_2 \\ &\leq \kappa_{2S}(A+BKC) \\ &= \min_{\Omega\in G(K)} \|\Omega\|_2\|\Omega^{-1}\|_2, \forall K \in \mathcal{F}_\Lambda.\end{aligned}$$

The abstract problem **PB1A** is converted into a computational procedure using the parameterizations of the sets defined above. Indeed for each $\underline{\Theta}_R \in \mathbf{S}_r$, a gain matrix $K(\underline{\Theta}_R)$ performing the assignment of the spectrum is obtained. Given $K(\underline{\Theta}_R)$, a whole family of closed loop eigenvector matrices exists: recalling the discussion of section 3, we select the eigenvector matrix with columns of unit 2-norm which is uniquely defined as the solution of the equation:

$$[A + BK(\underline{\Theta}_R)C]\Omega = \Omega\Lambda, \;\; \|\omega_i\|_2 = 1, \; i = 1, \ldots, n. \quad (4)$$

This matrix is clearly a function of $\underline{\Theta}_R$ through the composite map: $\underline{\Theta}_R \rightarrow K(\underline{\Theta}_R) \rightarrow \overline{\Omega}(K(\underline{\Theta}_R)) = \overline{\Omega}(\underline{\Theta}_R)$. Finally the spectral condition number associated to $\underline{\Theta}_R$ is obtained by optimally scaling the columns of $\overline{\Omega}$, according to formula (3):

$$\begin{aligned}&\chi(\underline{\Theta}_R) = \kappa_{2S}(A + BK(\underline{\Theta}_R)C)) = \\ &\min_{\underline{\beta}\in\mathcal{R}^{n-1}} \|\overline{\Omega}(\underline{\Theta}_R)\text{diag}(\underline{\beta}, 1)\|_2 \|\text{diag}(\underline{\beta}, 1)^{-1}\overline{\Omega}(\underline{\Theta}_R)^{-1}\|_2.\end{aligned} \quad (5)$$

The new version of **PB1A** is then:

PB1B - *Find* $\underline{\Theta}_R^* \in \mathbf{S}_r$ *such that*

$$\chi(\underline{\Theta}_R^*) \leq \chi(\underline{\Theta}_R), \;\; \forall \underline{\Theta}_R \in \mathbf{S}_r,$$

where $\overline{\Omega}(\underline{\Theta}_R)$ *is defined by equation (4),*$\chi(\underline{\Theta}_R)$ *is defined by eq. (5) and* $K(\underline{\Theta}_R)$ *is the solution of the assignment algorithm.*

The statement **PB1B** is conceptually very clear and it suggests the following solution procedure, where the improvement of the current estimate is made using the techniques of genetic optimization:

AL2 -

Initialization:

- *Fix the number of generations* N_{t1}, N_{t2}, *and the values of the genetic operators: crossover rate* p_c *and mutation probability* p_m;
- *Fix a hyper-rectangle* $\mathbf{S}_r \subset \mathcal{R}^r$;
- *Fix a hyper-rectangle* $\mathbf{S}_{n-1} \subset \mathcal{R}^{n-1}$;

Outer optimization:

1.1 *Generate an initial population of* N *random* s *bit strings uniformly distributed (chromosomes) that represent* N *arbitrary values of the parameter vector* $\underline{\Theta}_R \in \mathbf{S}_r$;

1.2 *For each chromosome:*

1.2.1 *Compute by* **AL1** *the matrix* $K(\underline{\Theta}_R)$ *such that* $sp(A + BK(\underline{\Theta}_R)C) = \Lambda$;

1.2.2 *Compute the eigenvector matrix* $\overline{\Omega}(\underline{\Theta}_R)$ *of* $A + BK(\underline{\Theta}_R)C$, *with columns of unit 2-norm;*

1.2.3 *Compute the spectral condition number* $\chi(\underline{\Theta}_R)$ *(fitness function) by the* **Inner optimization**.

1.3 *Reproduce a new generation by using the parameters* p_c *and* p_m *of the genetic operator;*

1.4 *Check the actual number of generations: if* $N_{act} < N_{t1}$, *then go to 1.2, otherwise*

1.5 *Compute the optimal* $K^* = K(\underline{\Theta}_R^*)$ *and the corresponding optimal* $\kappa_{2S}^* = \kappa_{2S}(A + BK^*C)$.

Inner optimization:

2.1 *Generate an initial population of* N *random* s *bit strings uniformly distributed (chromosomes) that represent* N *arbitrary values of the parameter vector* $\underline{\beta} \in \mathbf{S}_{n-1}$;

2.2 *For each chromosome compute the condition number of the eigenvector matrix* $\overline{\Omega}(\underline{\Theta}_R)\text{diag}(\underline{\beta}, 1)$;

2.3 *Reproduce a new generation by using the parameters* p_c *and* p_m *of the genetic operator;*

2.4 *Check the actual number of generations: if* $N_{act} < N_{t2}$, *then go to 2.2, otherwise*

2.5 *Compute the optimal scaling* β^* *and the corresponding* $\chi(\underline{\Theta}_R) = \kappa_{2S}(A + BK(\overline{\Theta}_R)C)$.

The actual implementation of **AL2** is a formidable task, because two nested optimization problems must be solved. Indeed each evaluation of the objective function of the outer optimization requires:

- **a-** the solution of the assignment problem: it is efficiently performed by the algorithm outlined in **AL1**, which makes use of symbolic computation in all steps, except the final solution of the algebraic equations;
- **b-** the solution of a minimization problem to find the optimal scaling of the columns of the eigenvector matrix, which gives the exact estimate of the spectral condition number.

A dramatic simplification arises when one accepts a suboptimal solution: keeping into account that the condition number of the eigenvector matrix with columns of unit 2-norm gives a good estimate of the spectral condition number, we may solve the problem:

PB2 - *Find* $\underline{\Theta}_R^+ \in \mathbf{S}_r$ *such that*

$$\chi_N(\underline{\Theta}_R^+) \leq \chi_N(\underline{\Theta}_R), \;\; \forall \underline{\Theta}_R \in \mathbf{S}_r,$$

$$\chi_N(\underline{\Theta}_R) = \|\overline{\Omega}(\underline{\Theta}_R)\|_2 \|\overline{\Omega}(\underline{\Theta}_R)^{-1}\|_2,$$

where $\overline{\Omega}(\underline{\Theta}_R)$ *is defined by equation (4) and* $K(\underline{\Theta}_R)$ *is the solution of the assignment algorithm.*

A single optimization allows to compute a-posteriori the spectral condition number corresponding to the computed gain

matrix.

AL3 -

Initialization:

- *Fix the number of generations N_{t1}, N_{t2}, and the values of the genetic operators: crossover rate p_c and mutation probability p_m;*
- *Fix a hyper-rectangle $\mathbf{S}_r \subset \mathcal{R}^r$;*
- *Fix a hyper-rectangle $\mathbf{S}_{n-1} \subset \mathcal{R}^{n-1}$;*

Phase 1:

1.1 *Generate an initial population of N random s bit strings uniformly distributed (chromosomes) that represent N arbitrary values of the parameter vector $\underline{\Theta}_R \in \mathbf{S}_r$;*

1.2 *For each chromosome:*

1.2.1 *Compute by* **AL1** *the matrix $K(\underline{\Theta}_R)$ such that $sp(A + BK(\underline{\Theta}_R)C) = \Lambda$;*

1.2.2 *Compute the eigenvector matrix $\overline{\Omega}(\underline{\Theta}_R)$ of $A + BK(\underline{\Theta}_R)C$, with columns of unit 2-norm;*

1.2.3 *Compute the upper bound $\chi_N(\underline{\Theta}_R)$ (fitness function) of the spectral condition number;*

1.3 *Reproduce a new generation by using the parameters p_c and p_m of the genetic operator;*

1.4 *Check the actual number of generations: if $N_{act} < N_{t1}$, then go to 1.2, otherwise*

1.5 *Compute the minimizer $K^+ = K(\underline{\Theta}_R^+)$ of $\chi_N(\underline{\Theta}_R)$;*

Phase 2:

1.6 *Generate an initial population of N random s bit strings uniformly distributed (chromosomes) that represent N arbitrary values of the parameter vector $\underline{\beta} \in \mathbf{S}_{n-1}$;*

1.7 *For each chromosome compute the condition number of the eigenvector matrix $\overline{\Omega}(\underline{\Theta}_R^+)\text{diag}(\underline{\beta}, 1)$;*

1.8 *Reproduce a new generation by using the parameters p_c and p_m of the genetic operator;*

1.9 *Check the actual number of generations: if $N_{act} < N_{t2}$, then go to 1.7, otherwise*

1.10 *Compute the optimal scaling β^+ and the corresponding $\chi(\underline{\Theta}_R^+) = \kappa_{2S}(A + BK(\overline{\underline{\Theta}}_R^+)C)$.*

In this way a value of K is found, which is only suboptimal, because it minimizes an upper bound of the closed loop spectral condition number, but we may expect that it is not very far from the optimal one.

Remark 1. It is not easy to impose the bounds on the parameters which define the compact sets $\mathbf{S}_r$ and $\mathbf{S}_{n-1}$ where the search is performed. Indeed the parameters on which the gain matrix depends have not a direct physical interpretation: it would be easier to impose bounds on the entries of K, which are related to the control effort and/or technological constraints, but a straightforward way to translate bounds on K into bounds on $\underline{\Theta}_R$ is not available.

V. Numerical experiments

Example 1. The first experiment aims at comparing the optimal solution we get by solving problem **PB1B** and the suboptimal one, solution of **PB2**, in the simplest case where an analytical solution of the assignment problem can be obtained, namely when $n = 3, m = p = 2$. The procedure is outlined with the further simplifying assumptions: $\lambda_1, \lambda_2, \lambda_3$, distinct, real, and different from the eigenvalues of A. Then all the matrices K that assign λ_1, λ_2 are solution of the equation

$$K \begin{bmatrix} \theta_1 & \theta_2 \\ 1 & 1 \end{bmatrix} = C[\omega_1\ \omega_2] = \left[G(\lambda_1) \begin{bmatrix} \theta_1 \\ 1 \end{bmatrix}\ G(\lambda_2) \begin{bmatrix} \theta_2 \\ 1 \end{bmatrix} \right], \tag{6}$$

where $\omega_i = (\lambda_i I - A)^{-1} B \, [\theta_i\, 1]^T$, $i = 1, 2$, and obviously $G(s) = C(sI - A)^{-1}B$; ; the reduced order observer is one-dimensional and the equation that assigns its eigenvalue at λ_3, can be written as

$$\alpha - [\phi_{11}\theta_1 + \phi_{12}\theta_2 + \phi_{22}]$$
$$\left[G(\lambda_1) \begin{bmatrix} \theta_1 \\ 1 \end{bmatrix} G(\lambda_2) \begin{bmatrix} \theta_2 \\ 1 \end{bmatrix} \right]^{-1} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \lambda_3$$

with $\alpha, \phi_{ij}, \eta_i$, scalars that depend on the entries of A, B, C; performing the inverse and the products we finally obtain a bilinear equation in the variables θ_1, θ_2

$$\rho_0 + \rho_1\theta_1 + \rho_2\theta_2 + \rho_{12}\theta_1\theta_2 = 0. \tag{7}$$

Every pair (θ_1, θ_2) that satisfies equation (7) when replaced into (6) gives a matrix K that performs the desired assignment, provided that $C[\omega_1\omega_2]$ is nonsingular. Since all coefficients can be pre-computed, it is easy to generate a family of feedback matrices by making one parameter, f.i. θ_1, to vary in a closed interval. It is worth noting that eq.(7) defines a hyperbola in the plane (θ_1, θ_2). Then the set of pairs (θ_1, θ_2) that satisfies eq.(7) can be parameterized in several ways: other parameterization than the most obvious one may be preferable from the numerical point of view.

The assignment equations (6) and (7) have been used for the triple taken from [18]:

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 3 \end{bmatrix},\ C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 9 \end{bmatrix}$$

and $\Lambda = \{-6, -3, -1\}$. The optimal gain matrix has been computed by **AL2** as

$$K^* = \begin{bmatrix} 3.78563020831415 & -1.16097605468718 \\ -1.08022427556643 & 0.07445757115369 \end{bmatrix},$$

with $\kappa_{2S}(A + BK^*C) = 6.12692035050912$.

The sub-optimal gain matrix, computed by **AL3** is:

$$K^+ = \begin{bmatrix} 2.84317751622931 & -1.40982936611884 \\ -0.85419424702995 & 0.09880829719553 \end{bmatrix},$$

with $\kappa_{2N}(A + BK^+C) = 6.40337542107424$ and $\kappa_{2S}(A + BK^+C) = 6.36755007720606$.

The difference between the optimal end suboptimal spectral condition number is about 3.9%, which is certainly acceptable

in practical applications. For example consider the perturbed closed loop matrix $A+BK^*C+E$: the maximum norm of E for which the eigenvalues are within disks of radii $\rho\,(\rho<2)$, centered at the nominal ones, is estimated as

$$\|E\|_2 < \rho/6.12692035050912 = 0.16321413414766\,\rho,$$

whereas using K^+ we obtain the estimate

$$\|E\|_2 < 0.15704627178037\,\rho.$$

To enforce this result the procedure has been repeated with ten different matrices A, with random elements $N(0,1)$. The results (Table1) show again that the suboptimal estimate is always very close to the optimal one, and encourage to use the algorithm **AL3** for more complex cases, where **AL2** would require very large time.

Table 1		
$\kappa_{2S}(A+BK^*C)$	$\kappa_{2N}(A+BK^+C)$	$\kappa_{2S}(A+BK^+C)$
2.425189	2.873416	2.648092
160.415389	172.071742	160.422904
48.935601	52.857450	48.947343
18.996753	20.066506	19.001966
3.450668	3.531213	3.463153
32.919385	34.790274	32.920470
4.819572	5.144766	4.911961
76.881799	85.442528	76.881832
192.975997	213.094960	193.008422
58.500085	64.726706	58.505319

Example 2. Experiment 2 considers the same system as in [3], described by the triple:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The set of nominal closed loop eigenvalues is $\Lambda = \{-1,-2,-3,-4\}$. The set is divided into the subsets $\Lambda_p = \{-1,-2,-3\}$ and $\Lambda_{n-p} = \{-4\}$. Recalling that the set Λ_p is used for defining the structure of the feedback gain matrix K according to eq.(1), the characteristic subspaces to which p closed-loop eigenvectors belong are given by

$$\begin{aligned} \omega_1 &= (A+I)^{-1}\,[\theta_{11}\,1]^T, \\ \omega_2 &= (A+2*I)^{-1}\,[\theta_{21}\,1]^T, \\ \omega_3 &= (A+3*I)^{-1}\,[\theta_{31}\,1]^T. \end{aligned} \tag{8}$$

The vector $\underline{\Theta}_R = [\theta_{11}\,\theta_{31}]^T$ is taken as the vector of decision variables for the solution of problem **PB2** through algorithm **AL3**. The fourth eigenvalue is assigned by solving the (unique because $n-p=1$) equation of the reduced-order observer in the unknown variable θ_{21}. Note that also in this case such equation can be solved by linear operations. The result is

$$\theta_{11}^+ = -6.47165094640053e-01,$$
$$\theta_{31}^+ = -1.61274762376301e+00,$$
$$\theta_{21}^+ = -1.28539712774863e+00,$$

$$K^{+^T} = \begin{bmatrix} -4.700000000000000e+01 & 1.955662821438330e+01 \\ 1.109413810469629e+01 & -4.104620553862129e+00 \\ 2.388418560624693e+01 & -1.100000000000000e+01 \end{bmatrix}.$$

The condition number of the matrix of normalized eigenvectors is

$$\kappa_{2N}(A+BK+C) = 582.6627.$$

The second phase of the optimization leads to estimate the spectral condition number as

$$\kappa_{2S}(A+BK+C) = 499.7107,$$

with parameters given by:

$$\beta_1^+ = 3.10726678974605e-01,$$
$$\beta_2^+ = 8.89218327239468e-01,$$
$$\beta_3^+ = 1.29971394192083e+00.$$

From the comparisons of Table 2 it turns out that the proposed method obtains a value of the condition number very close to the best one reported in the literature, but realizes an extremely precise assignment, with accuracy comparable with the machine precision.

Table 2		
Method	κ_2	Float precision of eigenvalues
AL3 Algorithm	499.7107	15
The gradient flow approach [3]	499.19	9
Chu, Nichols and Kausky'method [4]	778.2	1

Example 3. Experiment 3 considers the triple

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 7 & 5 & 1 & 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Note that in this case $(n=5,\ m=2,\ p=3)$ the Kimura condition $m+p>n$ does not hold, and that two quadratic equations must be solved for assignment: up our knowledge, no example of optimization of the spectral condition with such an assignment problem is reported in literature.
The spectrum of A is:$\{-3.91881, 3.68691, 1.4166,$ $0.407643 \pm i1.90912\}$; the desired spectrum is $\Lambda = \{-1,-2,-3,-4,-5\}$. The characteristic subspaces are again given by eqs.(8), but in this case one parameter is free for

optimization (say θ_{21}), and the other come from the equations that assign $\Lambda_{n-p} = \{-4, -5\}$. The result is:

$$\theta_{11}^{+} = -9.723665820384297e-01,$$
$$\theta_{31}^{+} = -2.746333565922710e-03,$$
$$\theta_{21}^{+} = -9.048169248050119e-01,$$

$$K^{+} = \begin{bmatrix} 2.746047543385010e+00 & -5.459040616307200e-01 \\ -2.044893305535538e+01 & 5.307982809683310e+00 \\ -2.637687970362707e+01 & 7.463915414834160e+00 \end{bmatrix}.$$

The condition number of the matrix of normalized eigenvectors is

$$\kappa_{2N}(A + BK + C) = 2545.586.$$

The second phase of the optimization leads to estimate the spectral condition number as

$$\kappa_{2S}(A + BK^{+}C) = 2304.7622,$$

with parameters given by:

$$\beta_1^{+} = 3.6881857107508e-01$$
$$\beta_2^{+} = 6.8285262840866e-01$$
$$\beta_3^{+} = 7.823907990978e-01$$
$$\beta_4^{+} = 1.0302853309343e+00.$$

VI. Conclusion

The main contribution of the paper is to show that an appropriate parameterization of the feedback matrix that assigns the eigenvalues at prescribed points of the complex plane, allows to optimize the closed loop system with respect to some other criterion. Moreover the problem can be solved by a general purpose global optimizer such as a Genetic Algorithm. We have considered the spectral condition number, as an index of the robustness of the closed loop eigenvalues to unstructured perturbations: other performance indices, related to robustness, control effort etc. can be considered as well.

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