

# State observers for the asymptotic inversion of nonlinear maps, when the reference trajectory is not differentiable

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**Abstract**— In this paper, we deal with the problem of the inversion of nonlinear maps, when the reference trajectory is not differentiable, thus extending previous works. First, we give the definition, in a formal framework, of a problem related to map inversion, which is already present in the literature, for instance in the robotics area and in observer theory: the inverse kinematics problem and the inversion of the nonlinear observability map. Secondly, we give the solution of such a problem by means of an asymptotic observer, for which a structure is proposed, for the state estimation of a time-varying nonlinear system associated with the given map and reference trajectory.

**Keywords**— Asymptotic nonlinear map inversion, observers, non-smooth trajectories, non-smooth impacts.

## I. INTRODUCTION

The problem of the “real-time” inversion of non-linear maps is fundamental in various research fields, e.g., in the asymptotic tracking of reference trajectories [1], in the observer design for discrete-time nonlinear systems [2], [3], and in robotics (such a problem in robotics is termed “inverse kinematics”: see [4], [5]). The real-time inversion of nonlinear maps is the counterpart of the analytic computation of the inverse of a non-linear map, which in some case could be difficult or impossible to be carried out in closed-form.

The problem of the asymptotic inversion of non-linear maps has been considered by the authors [6], [7], [8], under the assumption that the reference trajectory is differentiable a sufficiently high number of times. Aim of this paper is to extend the previous results when in the reference velocity there are instantaneous jumps (and, therefore, it need not to be differentiable).

Such an extension is fundamental in robotics, when the robot must impact with the external environment [9], e.g., for walking robots. As a matter of fact, in such a case the reference trajectory is non differentiable at each impact time.

## II. THE INVERSE KINEMATICS PROBLEM FOR NON-DIFFERENTIABLE TRAJECTORIES

Let  $\mathcal{A} \subset \mathbb{R}^{\kappa}$  be an open and connected domain of a direct kinematics map  $h(\cdot) : \mathcal{A} \rightarrow \mathbb{R}^{\kappa}$ ; assume that  $\mathcal{B} := \langle(\mathcal{A})$  is an open and connected subset of  $\mathbb{R}^{\kappa}$ . Assume that

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map  $h(\cdot)$  is of class  $C^{p+1}$  for a sufficiently high  $p \geq 0$ . Given an initial time  $t_0 \in \mathbb{R}$  and a **reference trajectory**  $y_r(\cdot) : [t_0, +\infty) \rightarrow \mathcal{B}$ , the inverse kinematics problem consists in finding an **inverse reference trajectory**  $q_r(\cdot) : [t_0, +\infty) \rightarrow \mathcal{A}$  such that

$$y_r(t) = h(q_r(t)), \quad \forall t \in [t_0, +\infty). \quad (1)$$

In the sequel, let  $J(q) := \frac{\partial h(q)}{\partial q}$  be the Jacobian matrix of the map  $h(\cdot)$ ; notice that, by the assumption  $h(\cdot) \in C^{p+1}$ ,  $p \geq 0$ , the entries of  $J(q)$  are continuous.

In this paper we make the restrictive assumption that there exist a countable set of times  $\{t_i\}_{i \geq 1}$  such that  $t_{i+1} > t_i$ , and

$$\begin{aligned} \lim_{i \rightarrow +\infty} t_i &= +\infty \\ \dot{y}_r(t_i^-) &\neq \dot{y}_r(t_i^+), \quad \forall i \in \mathbb{N}; \end{aligned}$$

here and in the following we use the shorthand notations  $\varphi(t_i^-) := \lim_{t \rightarrow t_i^-} \varphi(t)$  and  $\varphi(t_i^+) := \lim_{t \rightarrow t_i^+} \varphi(t)$ , when such limits exist.

The times  $t_i, i \in \mathbb{N}$ , are called the **reference impact times** and correspond to sudden jumps in the reference velocity and possibly in the higher order time derivatives: notice that, if  $(q, \dot{q})$  is the state of a mechanical system driven by a non-impulsive control input, they can be actually imposed only if at those times there are non-smooth impacts between the mechanical system and the external environment or between parts of the mechanical system. A similar remark applies also to the case of systems of different nature.

The purpose of this paper is to construct a dynamic system

$$\begin{aligned} \dot{\hat{q}}_r(t) &= f(\hat{q}_r(t), y_r(t), \dot{y}_r(t)), \\ t &\in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\ \hat{q}_r(t^+) &= g(\hat{q}_r(t^-)), \quad i \in \mathbb{N}, \end{aligned} \quad (2) \quad (3)$$

for suitable functions  $f(\cdot, \cdot)$  and  $g(\cdot)$ , such that  $\lim_{t \rightarrow +\infty} \tilde{q}_r(t) = 0$ , where  $\tilde{q}_r(t) := q_r(t) - \hat{q}_r(t)$ , for any sufficiently small initial error  $\tilde{q}_r(t_0)$ , despite the presence of the reference impact times  $t_i$ .

Under the assumption that  $\det(J(q_r(t))) \neq 0, \forall t \in [t_0, +\infty)$ , from (1) we have

$$\dot{q}_r(t) = J^{-1}(q_r(t)) \dot{y}_r(t), \quad \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \quad (4)$$

$$q_r(t_i^+) = q_r(t_i^-), \quad i \in \mathbb{N}. \quad (5)$$

The dynamic system (2) that we propose is the following

$$\begin{aligned} \dot{\hat{q}}_r(t) &= \mu K(\hat{q}_r(t)) (y_r(t) - h(\hat{q}_r(t))) \\ &\quad + J^{-1}(\hat{q}_r(t)) \dot{y}_r(t), \\ \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \end{aligned} \quad (6)$$

$$\hat{q}_r(t_i^+) = \hat{q}_r(t_i^-), \quad i \in \mathbb{N}, \quad (7)$$

where  $K(\cdot)$  is a matrix function to be suitably chosen and  $\mu \in \mathbb{R}, \mu > 0$ ; notice that, such a system to be implemented needs the measure of  $y_r(t)$  and  $\dot{y}_r(t)$ .

The corresponding error dynamics are

$$\begin{aligned} \dot{\tilde{q}}_r(t) &= -\mu K(\hat{q}_r(t)) (y_r(t) - h(\hat{q}_r(t))), \\ \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \end{aligned} \quad (8)$$

$$\tilde{q}_r(t_i^+) = \tilde{q}_r(t_i^-), \quad i \in \mathbb{N}. \quad (9)$$

The stability properties of the error dynamics (8), (9) are stated and proven in the following theorem.

*Theorem 1:* Under the assumption that  $\det(J(q_r(t))) > \varepsilon, \forall t \in [t_0, +\infty)$ , for some  $\varepsilon > 0$ , if  $K(\hat{q}_r(t)) = J^{-1}(\hat{q}_r(t))$  or  $K(\hat{q}_r(t)) = J^T(\hat{q}_r(t))$  then the error dynamics (8), (9) are locally uniformly asymptotically stable.

*Proof:* To prove the theorem consider the quadratic function  $V(\tilde{q}_r, t) = \frac{1}{2}(y_r - h(\hat{q}_r))^T(y_r - h(\hat{q}_r))$ , which, by the assumption  $\det(J(q_r(t))) \neq 0, \forall t \in [t_0, +\infty)$ , is a positive definite function of  $\tilde{q}_r$  in a neighborhood of  $\tilde{q}_r = 0$  (see the Definition 41.1 of positive definite function  $V(q, t)$  given in [10]). As a matter of fact, for each time  $t \in [t_0, +\infty)$  and for each sufficiently small  $\tilde{q}_r$ , there exists  $p_r(t)$  sufficiently close to  $q_r(t)$  such that  $V(\tilde{q}_r, t) = \tilde{q}_r^T J^T(p_r(t)) J(p_r(t)) \tilde{q}_r$ . Since by assumption  $\det(J(q_r(t))) \neq 0, \forall t \in [t_0, +\infty)$ , and  $p_r(t)$  is sufficiently close to  $q_r(t)$ , then there exists a positive constant  $\gamma$  such that  $V(\tilde{q}_r, t) \geq \gamma \|\tilde{q}_r\|^2$  holds for all  $t \geq t_0$  and for all  $\tilde{q}_r$  belonging to a sufficiently small neighborhood of the origin.

Computing the time derivative of  $V$  for  $t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+$ , we have

$$\begin{aligned} \dot{V} &= (y_r - h(\hat{q}_r))^T (\dot{y}_r - J(\hat{q}_r) \dot{\hat{q}}_r) \\ &= -\mu (y_r - h(\hat{q}_r))^T (J(\hat{q}_r) K(\hat{q}_r)) \\ &\quad (y_r - h(\hat{q}_r)), \end{aligned}$$

which, always by the assumption  $\det(J(q_r(t))) > \varepsilon, \forall t \in [t_0, +\infty)$ , and the choices made for matrix  $K(\hat{q}_r(t))$ , is a negative definite function of  $\tilde{q}_r$  in a neighborhood of  $\tilde{q}_r = 0$ . Since  $\dot{V}(t_i^-) = \dot{V}(t_i^+)$ ,  $t = t_i$  is a discontinuity of the first kind which can be removed by taking  $\dot{V}(t_i) = \dot{V}(t_i^-) = \dot{V}(t_i^+)$ . Then, since  $\dot{V}(t)$  is bounded for every  $t \in [t_0, +\infty)$ , by Theorem 41.1 of [10], function  $V$  is decrescent, according to the Definition 41.4 of [10]. This, together with the continuity of  $V(t)$  at the times  $t_i$  (which

is implied by (5) and (7)), shows that for any sufficiently small initial error  $\tilde{q}_r(t_0)$ , we have  $\lim_{t \rightarrow +\infty} V(t) = 0$ , namely  $\lim_{t \rightarrow +\infty} \tilde{q}_r(t) = 0$ , uniformly with respect to the initial time  $t_0$ . In addition, since there exists a positive definite decrescent function with negative definite derivative, then the equilibrium  $\tilde{q}_r = 0$  is locally uniformly asymptotically stable (the reasoning is as in the proof of Theorem 42.4 of [10]). ■

If  $K(\hat{q}_r(t)) = J^{-1}(\hat{q}_r(t))$ , then (6), (7) corresponds to the *Newton algorithm*, whereas if  $K(\hat{q}_r(t)) = J^T(\hat{q}_r(t))$ , then (6), (7) corresponds to the *gradient algorithm*.

The stability properties of the error dynamics are guaranteed for all  $\mu \in \mathbb{R}, \mu > 0$ , but a faster transient can be obtained by taking greater values of  $\mu$ .

A similar theorem can be stated and proven by simply requiring that matrix  $K(\hat{q}_r)$  is such that  $J(\hat{q}_r)K(\hat{q}_r)$  is positive definite and that  $\det(J(\hat{q}_r(t))K(\hat{q}_r(t))) > \hat{\varepsilon}, \forall t \in [t_0, +\infty)$ , for some  $\hat{\varepsilon} > 0$ .

*Example 2:* Consider the map  $h(q) = \frac{1}{3}q^3$ ; then,  $J(q) = q^2$ . Such a simple function has been chosen for some reasons: (1) it has a unique global inverse, (2) it is possible to compute its inverse in closed form,  $h^{-1}(y) = \sqrt[3]{3y}$ ; (3) its Jacobian is null at  $q = 0$ , thus allowing to test the effectiveness of the proposed technique when the inverse reference trajectory is close to singularities.

The proposed algorithm (6), (7) with  $K(q_r) = J^{-1}(q_r)$  (i.e., the Newton algorithm) is

$$\begin{aligned} \dot{\hat{q}}_r(t) &= \mu \frac{1}{\hat{q}_r^2(t)} (y_r(t) - \frac{1}{3}\hat{q}_r^3(t)) + \frac{1}{\hat{q}_r^2(t)} \dot{y}_r(t), \\ \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\ \hat{q}_r(t_i^+) &= \hat{q}_r(t_i^-), \quad i \in \mathbb{N}, \end{aligned}$$

whereas the proposed algorithm (6), (7) with  $K(q_r) = J^T(q_r)$  (i.e., the gradient algorithm) is

$$\begin{aligned} \dot{\hat{q}}_r(t) &= \mu \hat{q}_r^2(t) (y_r(t) - \frac{1}{3}\hat{q}_r^3(t)) + \frac{1}{\hat{q}_r^2(t)} \dot{y}_r(t), \\ \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\ \hat{q}_r(t_i^+) &= \hat{q}_r(t_i^-), \quad i \in \mathbb{N}. \end{aligned}$$

Consider the following reference trajectory described by

$$\begin{aligned} y_r(0) &= 1, \\ \dot{y}_r(t) &= \begin{cases} 1 & \text{if } t \in (2i, 2i+1), \\ -1 & \text{if } t \in (2i+1, 2(i+1)). \end{cases} \end{aligned}$$

Such a reference trajectory is not differentiable at all integer times, whereas it is continuous for all  $t \in \mathbb{R}$ .

The effectiveness of the proposed observer in the case of the Newton algorithm has been tested in simulation, and the simulated results are reported in Figure 1, for  $\mu = 1$ : the inverse reference trajectory  $q_r(t)$  (the continuous line) and its estimate  $\hat{q}_r(t)$  (the dotted line) are reported in position (1,1), whereas the reference trajectory  $y_r(t)$  (the continuous line) and its estimate  $\hat{y}_r(t) := h(\hat{q}_r(t))$  (the dotted line) are reported in position (1,2).

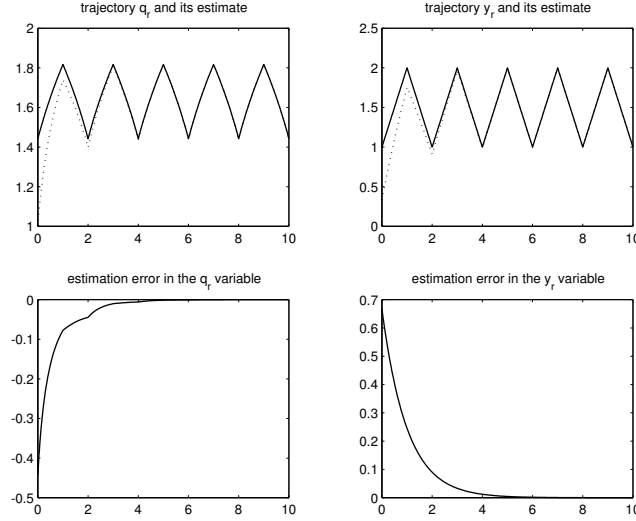


Fig. 1. Simulation results: the case of the Newton algorithm.

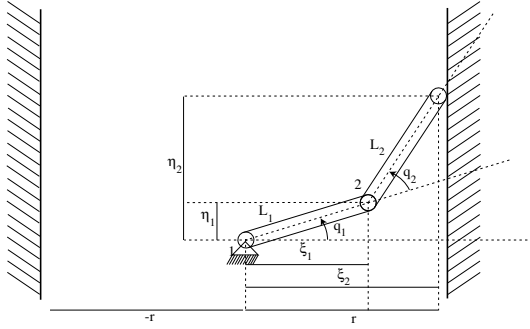


Fig. 3. Two-DOF planar robot arm

The fast convergence can be verified by looking at the estimation errors  $\tilde{q}_r(t) := q_r(t) - \hat{q}_r(t)$  and  $\tilde{y}_r(t) := y_r(t) - \hat{y}_r(t)$ , which are respectively reported in position (2,1) and (2,2).

The effectiveness of the proposed observer in the case of the gradient algorithm has been tested in simulation, and the simulated results are reported in Figure 2 (for  $\mu = 1$ ), using the same notations and conventions adopted in the case of the Newton algorithm.

### III. A TWO-DOF PLANAR ROBOT ARM

Consider the two-DOF planar robot arm depicted in Figure 3. The robot arm is constituted by a base body and two links, which are inter-connected by two rotational joints so to form a planar chain. The two links have respective length  $L_1$  and  $L_2$ . The joint angles  $q_1(t)$  and  $q_2(t)$  are taken as the generalized coordinates, which uniquely describe the configuration of the robot arm in the motion plane.

Two infinitely rigid and massive surfaces are perpendicular to the motion plane at a distance  $r$  from the joint connecting the base body at the first link of the chain (see Figure 3). Assume that  $L_1 < r < L_1 + L_2$ , so that the end-effector (which is also assumed to be infinitely rigid) is

the only part of the robot arm that may collide with the surfaces.

The distance of the robot end-effector from the infinitely rigid and massive surface on the right is

$$r - L_1 \cos(q_1(t)) - L_2 \cos(q_1(t) + q_2(t)); \quad (10)$$

then, the mechanical system is subject to the inequality constraint  $f_r(q(t)) \leq 0$ , where  $f_r(q) := L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) - r$ . The distance of the robot end-effector from the infinitely rigid and massive surface on the left is

$$r + L_1 \cos(q_1(t)) + L_2 \cos(q_1(t) + q_2(t)); \quad (11)$$

then, the mechanical system is subject to the inequality constraint  $f_l(q(t)) \leq 0$ , where  $f_l(q) := -r - L_1 \cos(q_1) - L_2 \cos(q_1 + q_2)$ .

Let  $q(t) := [q_1(t) \quad q_2(t)]^T$  be the vector of the generalized coordinates.

The direct kinematics map is

$$\begin{aligned} \xi &= L_1 \cos(q_1) + L_2 \cos(q_1 + q_2), \\ \eta &= L_1 \sin(q_1) + L_2 \sin(q_1 + q_2), \end{aligned}$$

with  $y := [\xi \quad \eta]^T$  and with the following Jacobian matrix

$$J(q) = \begin{bmatrix} -L_1 \sin q_1 - L_2 \sin(q_1 + q_2) \\ L_1 \cos q_1 + L_2 \cos(q_1 + q_2) \\ -L_2 \sin(q_1 + q_2) \\ L_2 \cos(q_1 + q_2) \end{bmatrix}.$$

At each impact time  $t_i$ , the post-impact velocity can be computed as a linear function of the pre-impact velocity in the following manner:

$$\begin{aligned} \dot{q}(t_i^+) &= Z_r \dot{q}(t_i^-), \\ \dot{q}(t_i^+) &= Z_l \dot{q}(t_i^-), \end{aligned}$$

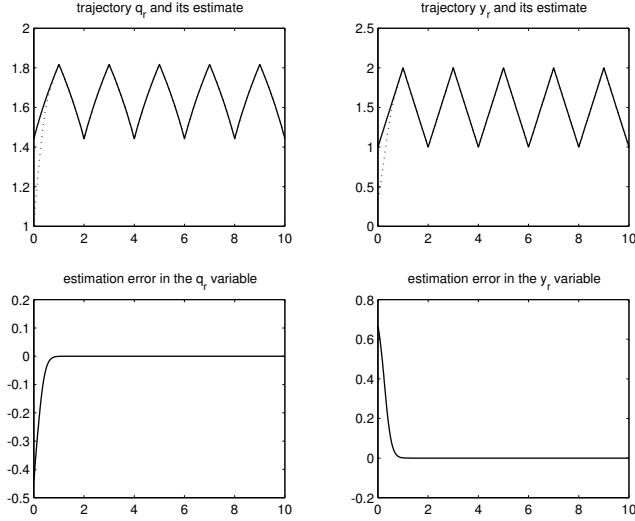


Fig. 2. Simulation results: the case of the gradient algorithm.

depending on the fact that impact is with the surface on the left or with the one on the right, where

$$Z_\ell = I - \frac{1+e}{J_\ell^T B^{-1} J_\ell} B^{-1} J_\ell^T J_\ell,$$

with  $B$  being the generalized inertia matrix of the robot arm,  $e$  the coefficient of restitution and  $J_\ell$  the gradient of  $f_\ell$ ,  $\ell \in \{r, l\}$ .

It is possible to show that at each impact time  $t_i$ , we have

$$\begin{aligned} \xi(\dot{q}(t_i^+)) &= -e \xi(\dot{q}(t_i^-)), \\ \eta(\dot{q}(t_i^+)) &= \eta(\dot{q}(t_i^-)). \end{aligned}$$

Assuming that  $r = 0.5$ , we have considered the following reference trajectory described by

$$\begin{aligned} \xi_r(0) &= 1, \\ \dot{\xi}_r(t) &= \begin{cases} 0.5 & \text{if } t \in (2i, 2i+1), \\ -0.5 & \text{if } t \in (2i+1, 2(i+1)). \end{cases} \\ \eta_r(t) &= 0.7, \end{aligned}$$

which is compatible with the case  $e = 1$ , i.e., with the case of elastic impacts.

This reference trajectory corresponds to asking that the end effector of the robot arm is moved on a segment of a straight-line perpendicular to both surfaces, with a constant velocity in modulus, which changes the sign in correspondence with an impact with one of the two surfaces.

The effectiveness of the proposed observer in the case of the Newton algorithm has been tested in simulation, and the simulated results are reported in Figure 4, for  $\mu = 10$ : the reference trajectory  $y_r(t) := [\xi_r(t) \ \eta_r(t)]^T$  (the continuous lines) and its estimate  $\hat{y}_r(t) := h(\hat{q}_r(t))$  (the dotted lines) are reported in position (1,2), whereas the inverse reference trajectory  $q_r(t)$  is reported in position (1,1) in continuous line.

The fast convergence can be verified by looking at the estimation error  $\tilde{y}_r(t) := y_r(t) - \hat{y}_r(t)$ , which is reported in position (2,1).

The effectiveness of the proposed observer in the case of the gradient algorithm has been tested in simulation, and the simulated results are reported in Figure 5, for  $\mu = 10$  and the same notations and conventions used for the Newton algorithm.

#### IV. CONCLUSIONS AND FUTURE EXTENSIONS

In this paper, we have considered the problem of the inversion of nonlinear maps, when the reference trajectory is not differentiable, thus extending previous works. It is shown that the classical algorithm of Newton and of the gradient, which are special cases of the proposed observer structure, can be actually applied when the reference trajectory is not differentiable at some times.

Such a problem could be extended by defining the following extended direct kinematics map

$$y_{r,e} = h_e(q_{r,e}),$$

where

$$\begin{aligned} y_{r,e} &= \begin{bmatrix} y_r \\ \dot{y}_r \end{bmatrix}, q_{r,e} = \begin{bmatrix} q_r \\ \dot{q}_r \end{bmatrix}, \\ h_e(q_{r,e}) &= \begin{bmatrix} h(q_r) \\ J(q_r) \dot{q}_r \end{bmatrix}. \end{aligned}$$

Given  $y_{r,e}(t)$ , the extended inverse kinematic problem consists in finding  $q_{r,e}(t)$  such that

$$y_{r,e}(t) = h_e(q_{r,e}(t)), \quad \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+. \quad (12)$$

As  $y_r(t)$  is a piecewise-differentiable function of time, with  $t_i$  being the times at which it is not differentiable, the need of avoiding times  $t_i$ ,  $i \in \mathbb{Z}^+$ , in the equality (12) is that  $y_{r,e}(t)$  and  $q_{r,e}(t)$  are not defined at times  $t_i$ . Let

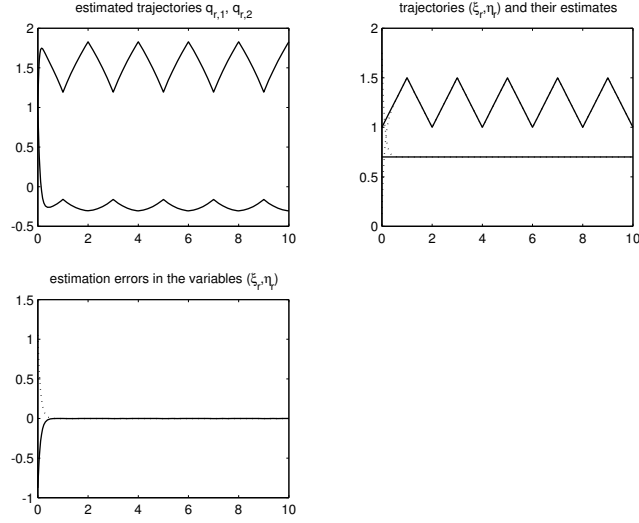


Fig. 4. Simulation results: the case of the Newton algorithm.

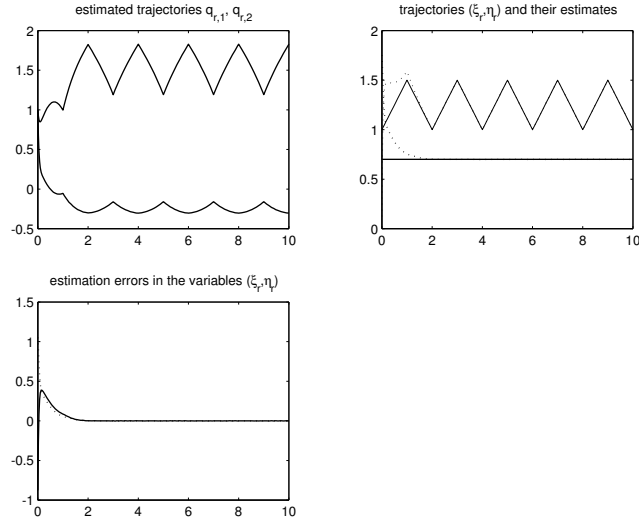


Fig. 5. Simulation results: the case of the gradient algorithm.

$J_e(q_{r,e})$  be the Jacobian matrix of  $h_e(q_{r,e})$ , which has the following form:

$$J_e(q_{r,e}) = \begin{bmatrix} J(q_r) & 0 \\ * & J(q_r) \end{bmatrix}.$$

Then, if  $\det(J(q_r(t))) > \varepsilon, \forall t \in [t_0, +\infty)$ , for some  $\varepsilon > 0$ , then  $\det(J_e(q_{r,e}(t))) > \varepsilon^2, \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+$ .

A future extension of the proposed algorithm is the construction of a dynamic system

$$\begin{aligned} \dot{\hat{q}}_{r,e}(t) &= f_e(\hat{q}_{r,e}(t), t), \quad t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\ \hat{q}_{r,e}(t^+) &= g_e(\hat{q}_{r,e}(t^-), \quad i \in \mathbb{N}, \end{aligned}$$

for suitable functions  $f_e(\cdot, \cdot)$  and  $g_e(\cdot)$ , such that  $\lim_{t \rightarrow +\infty} \tilde{q}_{r,e}(t) = 0$ , where  $\tilde{q}_{r,e}(t) := q_{r,e}(t) - \hat{q}_{r,e}(t)$ , for any

sufficiently small initial error  $\tilde{q}_{r,e}(t_0)$ , despite the presence of the reference impact times  $t_i$ .

Under the assumption that  $\det(J(q_r(t))) \neq 0, \forall t \in [t_0, +\infty)$ , we have

$$\begin{aligned} \dot{q}_{r,e}(t) &= J^{-1}(q_{r,e}(t)) \dot{y}_{r,e}(t), \\ &\quad \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+. \end{aligned} \quad (13)$$

If

$$\begin{aligned} y_r(t_i^+) &= y_r(t_i^-), \quad i \in \mathbb{N}, \\ \dot{y}_r(t_i^+) &= E \dot{y}_r(t_i^-), \quad i \in \mathbb{N}, \end{aligned}$$

for some constant matrix  $E$ , then (for  $i \in \mathbb{N}$ )

$$q_r(t_i^+) = q_r(t_i^-), \quad (14)$$

$$\dot{q}_r(t_i^+) = J^{-1}(q_r(t_i)) E J(q_r(t_i)) \dot{q}_r(t_i^-). \quad (15)$$

Future work will regard the construction of an asymptotic observer for system (13), (14), (15).

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