Lattice ℓ-step Ahead Output Uncertainty Predictor for Auto-Regressive Systems

George Nikolakopoulos *and* Anthony Tzes

Abstract—**In this article, an ℓ-step ahead output uncertainty prediction algorithm for linear discrete Auto-Regressive (AR) systems is presented. The novelty in the suggested scheme stems from the utilization of a lattice filter for the system description. Subject to the a priori bounds of the lattice-filter's reflection coefficients and the operating limits of the exciting signal, the set all the feasible predicted output values is computed. This computation is recursive over the length of the ℓ-step ahead time window and the number of stages of the lattice filter. Simulation studies are used to investigate the efficacy of the suggested scheme.**

Keywords—**Lattice filter, ℓ-step ahead prediction, prediction uncertainty bounds**

I. Introduction

Time series prediction of linear discrete systems [1], has matured over the past years with the numbers of publications too large to be cited inhere. Certain algorithms [2] exhibit some degree of conservatism due to the volume of the *a priori* required knowledge regarding the system model. The system's structural description (transversal, cascade, parallel, lattice) and order, the transmission delay, and the degree of stability are amongst those parameters that need proper attention in the algorithm's initialization process.

The majority of research has focused on the transversal structure [3] with fixed parameters reflecting the typical IIR and FIR-parametrization of the system. The system's parameters are assumed to be known and the output of the process is predicted over an ℓ-step ahead prediction horizon. Recursive schemes and computationally efficient algorithms have appeared [4], while certain branches of research have focused in the algorithm's robustness and accuracy [5–7] due to the numerical quantization of the system's parameters. Issues related to the parameters' numerical accuracy (i.e., quantized values from sampling procedures) as well as roundoff errors and numerical conditioning associated with the algorithms' implementation (i.e. using fixed-point rather than floating-point operations) have been discussed extensively in the literature [8, 9].

In comparison to the transversal filter, a lattice filter is modular, and has better robustness and numerical conditioning properties compared to the classical transversal structure [10]. The stability of the lattice filter can be inferred by simple monitoring of the magnitude of its reflection coefficients. However, despite the relevant merits that are embedded in a filter employing a lattice structure, there has been scant attention in the utilization of lattice filters for prediction purposes [11].

It should be noted that most of the forecasting algorithms regard the model description as a singleton, where the model's parameters have fixed known values. Subsequently, the typical prediction requirement amounts to the computation of a single future system output value that minimizes a certain cost functional.

For the case of uncertain systems, rather than predicting distinct future output values, the problem of forecasting the intervals within which the prospective output will vary is considered to be of increased complexity. This problem has been partially addressed in the research domain of set membership identification (SMI) [12, 13]. In SMI, the objective is to identify a closed-set (ellipsoid, polytope, orthotope) within which the model's parameters vary subject to the uncertainty of the model. In most applications, there is an uncertainty embedded in the model's transversal coefficients inferred by the noise contaminating the measurements. The prediction of the uncertainty intervals within which the output is constrained subject to the uncertainty of the system has been discussed in recent research efforts [14, 15], where the underlying stable model was of transversal structure with known order.

The presented work in this article contributes in the prediction area of the system's output bounds (uncertainty). The novelty of the suggested algorithm comprises in the usage of a model with lattice rather than a transversal structure, with the uncertainty of the predicted output stemming from: a) the partial knowledge of the model's reflection coefficients, and b) the bounds of the noise contaminating the measurements.

The problem of describing the output uncertainty for an uncertain lattice filter is addressed in the sequel, followed by the prediction problem in Section III. Simulation studies to explore the applicability of the proposed algorithm are presented in Section IV, followed by concluding remarks.

II. Output Uncertainty Prediction Problem Statement

Consider the following AR-discrete stable system of order M, as shown in Figure 1 with the ensuing nominal transversal parameterization:

$$y(n) = \sum_{i=1}^{M} a_i^\circ y(n-i) + u(n-1) \, , \tag{1}$$

where a_i° are the nominal AR-parameters, and $y(n)$ and $u(n)$ correspond to the system's output and input respectively at the n–th time instant.

Let the uncertainty embedded in the system with a nominal transfer function from (1)

$$H^\circ(z) = \frac{z^{-1}}{1 - \sum_{i=1}^{M} a_i^\circ z^{-i}} = \frac{z^{-1}}{1 + D_M\left(z^{-1}\right)}$$

can be cast in a multiplicative $H(z) = H^\circ \times (1 + \Delta_m)$ term, where the M-dimensional AR-coefficient vector $\vec{a} =$

This work was supported by U. of Patras Research Foundation (Karatheodori Research Project #2782). The authors are with the Electrical and Computer Engineering Department, University of Patras, Rio 26500, Greece. E-mail:gnikolak(tzes)@ee.upatras.gr

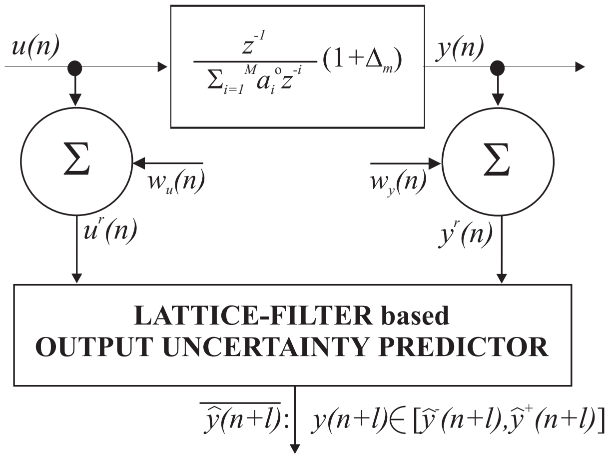

Fig. 1. Output uncertainty prediction problem structure

$[a_1, \ldots, a_M]^T$ is in an orthotope centered around its nominal value $\vec{a}^\circ = [a_1^\circ, \ldots, a_M^\circ]^T$, or

$$\vec{a} \in \vec{a}^\circ \times \left\{1 \pm [\Delta a_1, \ldots, \Delta a_M]^T\right\} .$$

Assume that the measured input $u^r(n)$ and output $y^r(n)$ signals are contaminated with noise of known bounds, or $u^r(n) = u(n) + w_u(n)$ and $y^r(n) = y(n) + w_y(n)$, where $|w_u(n)| < W_u(n)$, $|w_y(n)| < W_y(n)$ with $W_u(n)$ and $W_y(n)$ known *a priori*. Due to these contaminating noise sources, it is equivalent to state that the actual system input and output sequences can be determined at each sample with a predetermined accuracy, or

$$u(n) \in [u^r(n) - W_u(n), u^r(n) + W_u(n)] = \left[u^-(n), u^+(n)\right] ,$$
$$y(n) \in [y^r(n) - W_y(n), y^r(n) + W_y(n)] = \left[y^-(n), y^+(n)\right] .$$

The system's equivalent nominal lattice parametrization [16], amounts to the formation of a sequence of primitive "butterfly" blocks placed in a cascade configuration, as shown in Figure 2.

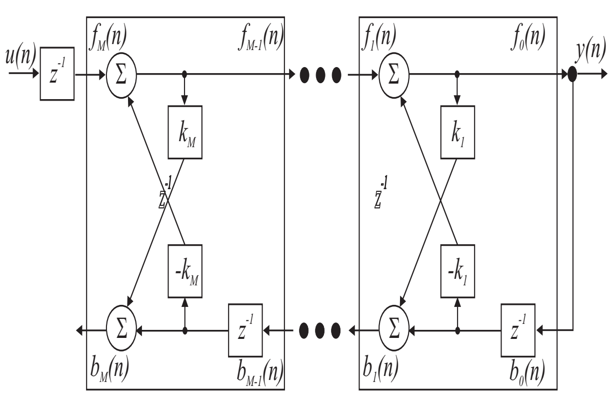

Fig. 2. Lattice–filter structural configuration of AR–system

Classical two–port circuit theory can be used to analyze the behavior of this lattice structure [17]. Based on this observation, each stage is considered as a two-input / two output system with an admittance description

$$\begin{bmatrix} f_{m-1}^\circ(n) \\ b_m^\circ(n) \end{bmatrix} = \begin{bmatrix} 1 & -k_m^\circ z^{-1} \\ k_m^\circ & \left(1 - (k_m^\circ)^2\right) z^{-1} \end{bmatrix} \begin{bmatrix} f_m^\circ(n) \\ b_{m-1}^\circ(n) \end{bmatrix} \quad (2)$$

where $f_m^\circ(n)$ $(b_m^\circ(n))$ is the nominal forward (backward) prediction error of the mth-stage, and z^{-1} corresponds to the delay operator. The k_m°, $m = 1, \ldots, M$ parameters, known as the nominal systems reflection coefficients, characterize the behavior of each stage. The system's stability is guaranteed through the restriction of the reflection coefficients within a hypercube $\{k_1 \times k_2 \times \ldots k_M\} \in \{[-1,1] \times [-1,1] \times \ldots [-1,1]\} \subset \Re^M$.

The lattice-structure's boundary conditions are:

$$f_M^\circ(n) = u(n-1), \text{ and } f_0^\circ(n) = b_0^\circ(n).$$

The input-output nominal system description can be computed based on the observation that

$$y(n) = f_0^\circ(n) = f_M^\circ(n) - \sum_{m=1}^{M} k_m^\circ b_{m-1}^\circ(n-1) . \quad (3)$$

In a similar manner, working with the lattice parameterization, the uncertainty about the system can be reflected through the reflection coefficient vector $\vec{k} = [k_1, \ldots, k_M]^T$, or

$$\begin{aligned} \vec{k} &\in [k_1^\circ, \ldots, k_M^\circ]^T \times \left\{1 \pm [\Delta k_1, \ldots, \Delta k_M]^T\right\} \\ &= \vec{k}^\circ \times \left\{1 \pm [\Delta k_1, \ldots, \Delta k_M]^T\right\} . \end{aligned} \quad (4)$$

The link between the direct and lattice form parameterizations can be found from the following relationships shown in Table I. It should be noted that an orthotope reflection coefficient uncertainty is not mapped at an equivalent AR-parametric uncertainty of the same orthotopic-shape.

TABLE I
LATTICE TO AR–REALIZATION

$D_1(z) = k_1$
Recursive Computation – For AR-order $m = 2, \ldots, M$.
Let $D_m(z) = d_0 + d_1 z + \ldots + d_m z^m$.
$D_m^*(z) = d_{m-1}^* z^{m-1} + \ldots + d_0^*$
$D_m(z) = z\left[D_{m-1}(z) + k_m D_m^*(z)\right] + k_m$

In order to facilitate the mathematical formulation, consider the following notation concerning the interval arithmetic of a fixed, yet uncertain bounded quantity. The notation $\bar{x}$ indicates that x takes values over an interval restricted by $x \in [x^-, x^+]$; the width of this interval is defined as $w(\bar{x}) \stackrel{\triangle}{=} x^+ - x^-$. Furthermore, classical interval arithmetic theory [18] states that the addition and multiplication operations over two interval quantities $\bar{x}_1$ and $\bar{x}_2$ are defined as

$$\begin{aligned} \bar{x} &= \bar{x}_1 + \bar{x}_2 \in \left[x_1^- + x_2^-, x_1^+ + x_2^+\right] \\ \bar{x} &= \bar{x}_1 \cdot \bar{x}_2 \in \end{aligned}$$
$$\left[\min\left(x_1^- x_2^-, x_1^- x_2^+, x_1^+ x_2^-, x_1^+ x_2^+\right), \max\left(x_1^- x_2^-, x_1^- x_2^+, x_1^+ x_2^-, x_1^+ x_2^+\right)\right] .$$

In this research effort, the objective is to compute the ℓ–step ahead interval of the system's output $\overline{\hat{y}(n+\ell)}$, subject to:

- the knowledge regarding the system's interval inputs $\overline{u(i)} \in [u^-(i), u^+(i)]$, $i = n+\ell-1, n+\ell-2, \ldots$,
- the system's interval output $\overline{y(n-1)} \in [y^-(n-1), y^+(n-1)]$, and
- the system's uncertainty indicated by the bounds of the reflection coefficients $k_m \in [k_m^-, k_m^+]$.

Inhere, it is emphasized that the predicted output interval must contain the actual future system response, or

$$y(n+\ell) \in \overline{\hat{y}(n+\ell)} = \left[\hat{y}^-(n+\ell), \hat{y}^+(n+\ell)\right] .$$

III. ℓ-STEP AHEAD OUTPUT PREDICTED UNCERTAINTY

From the single–stage input/output description from (2), the expressions relating the interval outputs of each stage at the nth sample, can be computed recursively as

$$\begin{aligned} \overline{f_{m-1}(n)} &\in \left[f_{m-1}^-, f_{m-1}^+\right] = \overline{f_m(n)} - \overline{k_m b_{m-1}(n-1)} \\ \overline{b_m(n)} &\in \left[b_m^-(n), b_m^+(n)\right] = \overline{b_{m-1}(n-1)} + \overline{k_m f_{m-1}(n)}, \end{aligned} \quad (5)$$

where the m-index of each stage varies recursively within $m = M, \ldots, 1$.

The boundaries of the quantities $\overline{f_{m-1}(n)}$ and $\overline{b_m(n)}$ can be found, based on the interval arithmetic properties as

$$\begin{aligned} f_{m-1}^+(n) &= f_m^+(n) - \min\Big(k_m^- b_{m-1}^+(n-1), k_m^- b_{m-1}^-(n-1), \\ &\qquad k_m^+ b_{m-1}^-(n-1), k_m^+ b_{m-1}^-(n-1)\Big) \\ f_{m-1}^-(n) &= f_m^-(n) - \max\Big(k_m^- b_{m-1}^+(n-1), k_m^- b_{m-1}^-(n-1), \\ &\qquad k_m^+ b_{m-1}^-(n-1), k_m^+ b_{m-1}^-(n-1)\Big) \\ b_m^+(n) &= b_{m-1}^+(n-1) + \max\Big(k_m^- f_{m-1}^+(n), k_m^- f_{m-1}^-(n), \\ &\qquad k_m^+ f_{m-1}^-(n), k_m^+ b_{m-1}^-(n)\Big) \\ b_m^-(n) &= b_{m-1}^-(n-1) + \min\Big(k_m^- f_{m-1}^+(n), k_m^- f_{m-1}^-(n), \\ &\qquad k_m^+ f_{m-1}^-(n), k_m^+ b_{m-1}^-(n)\Big) \end{aligned} \quad (6)$$

This recursive procedure is computed over the order $m = M, \ldots, 1$ of the lattice stages, and is initialized with:

- $f_M^+(n) = u^+(n-1)$ and $f_M^-(n) = u^-(n-1)$ for the case of a partially known exciting input, and
- $b_0^+(n-1) = y^+(n-1)$ and $b_0^-(n-1) = y^-(n-1)$.

At the end of the first recursion in time, the predicted output uncertain interval can be inferred from $\overline{\hat{y}(n)} = \overline{f_0(n)} = \overline{b_0(n)}$. The aforementioned recursion, shown in (6) is repeated for $n+1, \ldots, n+\ell$ with the backward substitution of the reflection coefficient-intervals from the previous recursion. In a similar manner, the boundary conditions are:

- $f_M^+(n+i) = u^+(n+i-1)$, $f_M^-(n+i) = u^-(n+i-1)$ and
- $b_0^+(n+i-1) = \hat{y}^+(n+i-1)$ and $b_0^-(n+i-1) = \hat{y}^-(n+i-1)$

for $i = 1, \ldots, \ell$.

The predicted output uncertainty interval $\overline{\hat{y}(n+\ell)}$ is computed after ℓ-iterations of the recursive set of equations in (6). Is should be noted that during this computation, the set of all intermediate predicted output intervals are inferred $\overline{\hat{y}(n)}, \overline{\hat{y}(n+1)}, \ldots, \overline{\hat{y}(n+\ell)}$.

The uncertainty of the predicted output, as expressed by the width of the predicted interval, can be computed as

$$w\left(\overline{y(n+\ell)}\right) = w\left(\overline{u(n+\ell-1)}\right) + w\left(\overline{k_m b_{m-1}(n+\ell-1)}\right) .$$

This quantity is a monotonic function with respect to the prediction horizon, or $w\left(\overline{\hat{y}(n+i)}\right) < w\left(\overline{\hat{y}(n+j)}\right)$, for $i < j$.

IV. SIMULATION STUDIES

For simulation purposes, we consider a second-order AR–process with its poles located at $-0.76 \pm 0.5678\iota$; its corresponding k_i° reflection coefficients are set at $[k_1^\circ, k_2^\circ] = [0.8, 0.9]$. The system is excited with a zero mean uniform white noise bounded in the interval $[-1, 1]$. There is a multiplicative 10% uncertainty associated with the reflection coefficients, or $k_1 \in [0.72, 0.88]$ and $k_2 \in [0.81, 0.99]$.In Figure 3 we present the mapping of the orthotopic reflection uncertainty to its equivalent uncertainty regarding the system's pole–locations.

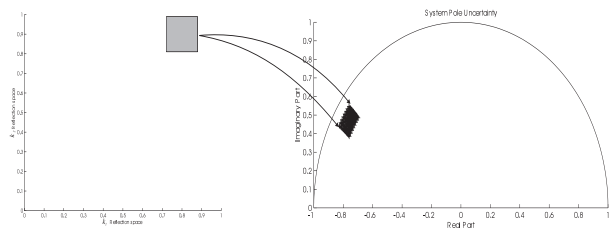

Fig. 3. Reflection coefficient to pole uncertainty mapping

The input and output data streams are quantized and subsequently fed to the prediction routine. The uncertainty regarding the knowledge of the measured quantities stems from the quantization process, or $w_u(n) = w_y(n) = \frac{U_{\min}^{\max}}{2^Q}$, where Q corresponds to the accuracy (# of bits) of the quantizer and $U_{\min}^{\max}$ is the dynamic range of the quantizer unit; in the ensuing simulations $U_{\min}^{\max} = 10$. The previous recursive algorithm is used to compute the predicted interval output $\overline{\hat{y}(n+\ell)}$, as shown in Figure 4.

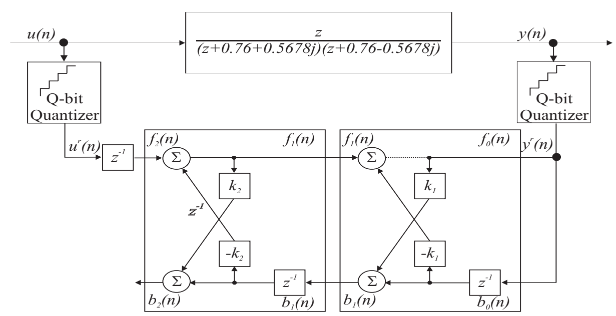

Fig. 4. Lattice-based output interval prediction

The results for the one-step ($\ell = 1$) ahead prediction uncertainty intervals appear in Figure 5. In this Figure's top (bottom) portion, the output $y(n)$ and the bounds $\overline{\hat{y}(n+1)}$ are presented for the case of a $Q = 4$ (12)–bit quantization process. As expected, the actual output $y(n+1)$ is restricted within the bounds of the predicted uncertainty, or $y(n+1) \in [\hat{y}^-(n+1), \hat{y}^+(n+1)]$.

Similarly, the larger measuring uncertainty $(w_u(n), w_y(n))$ emanating from the usage of a less accurate quantizer, results in a higher prediction uncertainty. This is clearly indicated by a visible comparison of the bounds of the predicted output interval between the top ($Q = 4$) and the bottom ($Q = 12$) case.

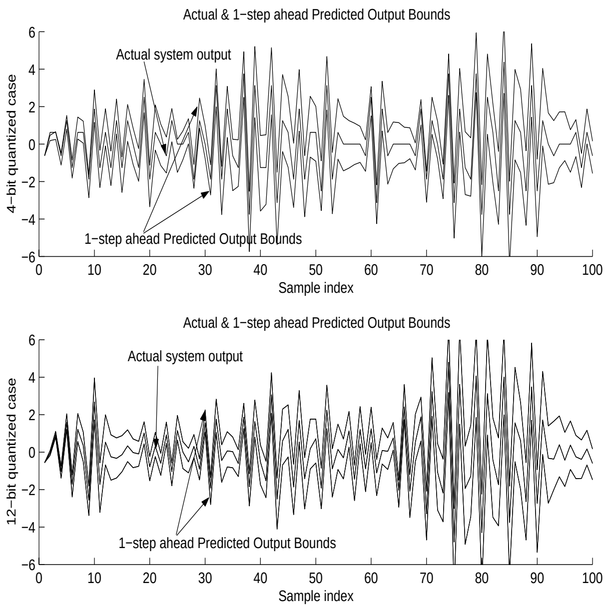

Fig. 5. One-Step Ahead Predicted Output Uncertainty Bounds

In Figure 6, we present the system's predicted output sequences for a one-step and a two-step ahead prediction horizon for the case of the 12-bit quantized data stream. The bounds for the latter case are significantly larger than the ones for the shorter horizon, or $w\left(\overline{\hat{y}(n+1)}\right) < w\left(\overline{\hat{y}(n+2)}\right)$.

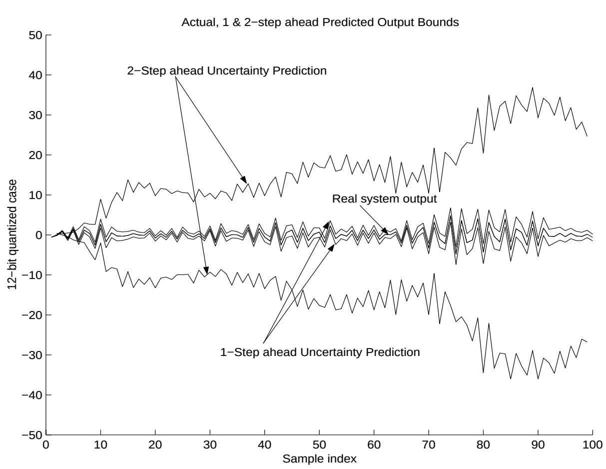

Fig. 6. 1- & 2–Steps Ahead Predicted Output Uncertainty Bounds

V. Conclusions

The main contribution of this paper is the design of output uncertainty predictors, based on lattice filters in a recursive scheme, for linear AR–models of known order. These predictors instead of predicting a future value of the output, predict the confidence interval that includes the nominal future value of the system output, based on the a priori known parameter uncertainty intervals of the reflection coefficients and the bounds of the system driving signal. The presented algorithm is recursively computing the bounds for the output of each separate lattice stage, and finally the bounds for the last stage that equals the system output. It is shown that the width of the confidence intervals increases with respect to the prediction horizon.

Extensions of this work to ARMA-systems of unknown order are underway. Furthermore, the adaptive version of this scheme for identification of the reflection coefficients' bounds is expected to fulfill the gap in the SMI-area of systems expressed by a lattice filter configuration.

References

[1] P. Wellstead and M. Zarrop, *Self–Tuning Systems: Control and Signal Processing*. John Wiley & Sons, Inc, 1991.

[2] J. Makhoul, "Linear prediction: A tutorial review," *Proceedings of the IEEE*, vol. 63, pp. 501–580, 1975.

[3] C. Papaodysseus, G. Carayannis, E. Koukoutsis, and E. Kayafas, "Comparing LS FIR Filtering and ℓ-step Ahead Linear Prediction," *IEEE Transactions on Signal Processing*, vol. 41, pp. 768–780, Feb. 1993.

[4] A. Carini and E. Mumolo, "A Numerically Stable Fast RLS Algorithm for Adaptive Filtering and Prediction Based on the UD Factorization," *IEEE Transactions on Signal Processing*, vol. 47, pp. 2309–2313, Aug. 1999.

[5] M. Milanese and R. Tempo, "Optimal algorithm theory for robust estimation and prediction," *IEEE Transactions on Automatic Control*, vol. 30, pp. 730–738, 1985.

[6] Y.-A. Kao and S.-G. Chen, "Design of finite–word–length FIR filters with least–squares error," *Signal Processing*, vol. 80, pp. 1591–1596, Aug. 2000.

[7] A. Dempster and M. Macleod, "Variable wordlengths in filters with least–squares error criterion," *Electronics Letters*, vol. 33, pp. 201–202, Jan. 1997.

[8] L. Smith and M. Henderson, "Roundoff noise reduction in cascade realizations of FIR digital filters," *IEEE Transactions on Signal Processing*, vol. 48, no. 4, pp. 1196–1200, 2000.

[9] K. Ralev and P. Bauer, "Realization of block floating–point digital filters and application to block implementations," *IEEE Transactions on Signal Processing*, vol. 47, pp. 1076–1086, Apr. 1999.

[10] S. S. Haykin, *Adaptive Filter Theory*. Englewood Cliffs, NJ: Prentice–Hall, 2001.

[11] B. Friedlander, "Lattice Filters for Adaptive Processing," *Proceedings of the IEEE*, vol. 70, pp. 829–867, Aug. 1982.

[12] J. Deller, "Set membership identification in digital signal processing," *IEEE ASSP Magazine*, vol. 6, pp. 4–20, Oct. 1989.

[13] E. Fogel, "System identification via membership set constraints with energy constrained noise," *IEEE Transactions on Automatic Control*, vol. 24, pp. 752–758, Oct. 1979.

[14] A. Tzes and K. Le, "Weighted Minimum Uncertainty Prediction Control," *Automatica*, vol. 32, pp. 793–795, May 1996.

[15] A. Tzes and K. Le, "Sufficient Stability Conditions For The Weighted Minimum Uncertainty Prediction Controller," *Automatica*, pp. 2273–2275, Dec. 1997.

[16] B. Friedlander, "Lattice implementation of some recursive parameter-estimation algorithms," *International Journal of Control*, vol. 37, pp. 661–684, Jan. 1983.

[17] D. H. S. Mitra and R. Sherwood, "New methods of digital ladder realization," *IEEE Transactions on Audio and Electroacoustics*, vol. 21, pp. 485–491, Aug. 1973.

[18] R. Moore, *Methods and Applications of Interval Analysis*. Philadelphia: SIAM Bookmart, 1979.