On the determination of moving sensor policies for parameter estimation of distributed systems

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Abstract— **An activation strategy of pointwise discrete scanning sensors used for estimating unknown parameters in models described by partial differential equations is addressed. In contrast to the common approach based on parameter-space criteria, attention is paid here to a criterion in output space, which is of interest if the purpose of parameter estimation is to accurately predict system outputs. The performance index thus constructed constitutes the average variance of the predicted response and is called the V-optimality criterion. The setting examined here may correspond to situations where one has many sensors and activates only some of them during a given time interval, or alternatively, has several sensors which are mobile. The proposed approach being a non-trivial extension of Fedorov's idea of directly constrained design measures consists in imposing constraints on the sensor density in a given spatial domain. One of the main results is an efficient iterative procedure whose each step reduces to replacing less informative sensors with the ones which furnish more informaton about the parameters. This planning algorithm is verified through a numerical example on a two-dimensional heat equation.**

Keywords— **Distributed-parameter systems, parameter estimation, sensor location, experimental design.**

I. Introduction

SINCE for distributed parameter systems it is impossible to observe their states over the entire spatial domain, the question arises of where to locate discrete sensors so as to accurately estimate the unknown system parameters. Both researchers and practitioners do not doubt that making use of sensors placed in an 'intelligent' manner may lead to dramatic gains in the achievable accuracy of the resulting parameter estimates, so efficient sensor location strategies are highly desirable. In turn, the complexity of the sensor location problem implies that there are very few sensor placement methods which are readily applicable to practical situations. What is more, they are not well known among researchers. This generates keen interest in the potential results, as the motivations to study the sensor location problem stem from practical engineering issues. Optimization of air quality monitoring networks is among the most interesting ones. One of the tasks of environmental protection systems is to provide expected levels of pollutant concentrations. But to produce such a forecast, a smog prediction model is necessary which is usually chosen in the form of an advection-diffusion partial-differential equation. Its calibration requires parameter estimation, e.g. the unknown spatially-varying turbulent diffusivity tensor should be identfied based on the measurements from monitoring stations Since measurement transducers are usually rather costly and their number is limited, we are faced with the problem of how to optimize their locations in order to obtain the most precise model. Other stimulating applications include, among other things, groundwater modelling, recovery of valuable minerals and hydrocarbon from underground permeable reservoirs, gathering measurement data for calibration of mathematical models used in meteorology and oceanography, automated inspection in static and active hazardous environments where trial-and-error sensor planning cannot be used (e.g. in nuclear power plants), or emerging smart material systems.

The sensor placement problem was attacked from various angles, but the results communicated by most authors are limited to the selection of stationary sensor positions (for reviews, see [1], [2], [3]). An intuitively clear generalization is to apply sensors which are capable of continuously tracking points providing at a given time moment best information about the parameters (such a strategy is usually called continuous scanning). However, communications in this field are rather limited. Rafajłowicz [4] considers the determinant of the Fisher Information Matrix (FIM) associated with the parameters to be estimated as a measure of the identification accuracy and looks for an optimal time-dependent measure, rather than for the trajectories themselves. On the other hand, Uciński [2], [19], [3], [20], [18], apart from generalizations of Rafajłowicz's results, develops some computational algorithms based on the FIM. The problem is then reduced to a state-constrained optimal-control one for which solutions are obtained via gradient techniques capable of handling various constraints imposed on sensor motions.

Apart from mobile sensors, discrete scanning devices can also be used in applications. The observation system comprises then multiple sensors whose positions are already specified and it is desired to activate only a subset of them during a given time interval while the other sensors remain dormant [7]. A reason for not using all the available sensors could be the reduction of the observation system complexity and the cost of operation and maintenance [8]. Such a scanning strategy of taking measurements can be also interpreted in terms of several sensors which are mobile but the time necessary for taking measurements may be neglected. The problem has received little attention yet (though some trials have been conducted in a related context of state estimation, see e.g. [9]). A first attempt to fill this gap was reported in [21], where the idea of the so-called clusterization-free designs set forth in [10] was extended, based on Fedorov's idea of replication-free designs [11], [12], [13] which have emerged relatively late in the context of spatial statistics (see the monograph [5]). In spite of its somewhat abstract assumptions and the inherent

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combinatorial nature of the sensor scheduling problem, the resulting algorithm of exchange type is very easy to implement, extremely fast and efficient.

As opposed to most existing aproaches, where sensor locations are determined in experiments performed for the most accurate determination of parameter values which may have some physical significance, in the present paper we consider the reliability of model predictions. In many applications, especially when a control scheme is to be built, the accuracy of model predictions is more important than the accuracy of model parameters, because the ultimate objective in modelling is the prediction or forecast of the system states [5]. The topic was discussed to some extent in [6], but without connection to constructive solution methods. This failing was the main motivation for the study [18] undertaken in order to extend sensor motion planning techniques set forth in [2], [3] based on optimal control techniques. In this note, we consider the discrete scanning sensor scheduling problem for the most accurate model prediction and show how the results obtained in [21] can be appropriately extended.

II. Scanning Problem for Optimal Prediction

As our fundamental state system we consider the scalar (possibly non-linear) distributed system

$$\frac{\partial y}{\partial t} = \mathcal{F}\left(x, t, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_2^2}, \theta\right), \quad x \in \Omega, \quad t \in T \tag{1}$$

with initial and boundary conditions of the general form

$$y(x,0) = y_0(x), \quad x \in \Omega \tag{2}$$

$$\mathcal{E}(x,t,y,\theta) = 0, \quad x \in \partial\Omega, \quad t \in T \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a fixed, bounded, open set with sufficiently smooth boundary $\partial\Omega$, the points of which will be denoted by $x = (x_1, x_2)$, $\mathcal{F}$ and $\mathcal{E}$ are some known functions, y_0 is a given initial state, t stands for time, $T = [0, t_f]$, $y = y(x,t)$ signifies the state variable with values in $\mathbb{R}$. $\mathcal{F}$ may include terms accounting for given *a-priori* forcing inputs. The system evolves from $t = 0$ to $t = t_f$, the period over which observations are available.

Note that the system state at a spatial point $x \in \Omega$ and time instant $t \in T$ depends on an unknown constant parameter vector θ to be estimated using observations of the system. Consequently, in order to stress this dependence, we will use the notation $y(x,t;\theta)$.

In what follows, we form an arbitrary partition on the time interval T by choosing points $0 < t_1 < t_2 < \cdots < t_K = t_f$ defining subintervals $T_k = [t_{k-1}, t_k]$, $k = 1, \ldots, K$. We then consider N moving sensors which possibly will be changing their locations at the beginning of every time subinterval but will be remaining stationary for the duration of each of the subintervals. In other words, the measurement process can be formally represented as

$$z^j(t) = y(x_k^j, t; \theta) + \varepsilon^j(t), \quad t \in T_k \tag{4}$$

for $j = 1, \ldots, N$ and $k = 1, \ldots, K$, where $z^j(t)$ is the scalar output, $x_k^j \in X$ stands for the location of the j-th sensor on the subinterval T_k, X signifies the part of Ω where the measurements can be made, and $\varepsilon^j(\,\cdot\,)$ denotes the measurement noise. It is customary to assume that the noise is zero-mean, Gaussian, uncorrelated in both time and space [11], [1].

We assume that the parameter estimate $\hat{\theta}$, defined as the solution to the usual output least-squares formulation of the parameter estimation problem, is to provide a basis for prediction of certain variables depending on spatial location and/or time. Since in general the conditions applied for prediction may differ from the conditions of the experiment, the prediction equations need not be the same as the state equations, nor need the variables to be predicted coincide with the state y. Let the solution to the prediction problem in context be a scalar quantity $q = q(x,t;\theta)$. We are interested in selecting the sensors' configurations in such a way as to maximize the accuracy of q in a given compact spatio-temporal domain $\mathcal{Q} = \mathcal{X} \times \mathcal{T}$. Clearly, in order to compare different configurations, a quantitative measure of the 'goodness' of particular configurations is required. A logical approach is to choose a measure related to the expected accuracy of prediction.

For a given $(x,t) \in \mathcal{Q}$, the variance of q obtained by a first-order expansion around a preliminary estimate θ^0 of θ has the form

$$\begin{aligned} \operatorname{var}(q(x,t;\hat{\theta})) &= \mathrm{E}\left((q(x,t;\theta) - q(x,t;\hat{\theta}))^2\right) \\ &\approx \left(\nabla_\theta q(x,t;\theta^0)\right)^{\mathrm{T}} \operatorname{cov}(\hat{\theta}) \nabla_\theta q(x,t;\theta^0) \end{aligned} \tag{5}$$

where we write $\nabla_\theta q$ for the gradient of q with respect to θ. It is customary to choose θ^0 as a nominal value of θ or a result of a preliminary experiment. As for $\operatorname{cov}(\hat{\theta})$, under some assumptions it can be approximated by the inverse of the *Fisher Information Matrix* (FIM) whose normalized version can be written down as [4]

$$M = \frac{1}{N} \sum_{k=1}^{K} \sum_{j=1}^{N} \Upsilon_k(x_k^j), \tag{6}$$

where

$$\Upsilon_k(x) = \frac{1}{t_f} \int_{T_k} g(x,t) g^{\mathrm{T}}(x,t)\, \mathrm{d}t, \tag{7}$$

$$g(x,t) = \nabla_\theta y(x,t;\theta)\big|_{\theta=\theta^0}, \tag{8}$$

θ^0 being a prior estimate to the unknown parameter vector θ [3]. Such a formulation is generally accepted in optimum experimental design for DPS's, since the inverse of the FIM constitutes, up to a constant multiplier, the Cramér-Rao lower bound on the covariance matrix of any unbiased estimator of θ [22].

Consequently, we get

$$\operatorname{var}(q(x,t;\hat{\theta})) \sim \left(\nabla_\theta q(x,t;\theta^0)\right)^{\mathrm{T}} M^{-1} \nabla_\theta q(x,t;\theta^0) \tag{9}$$

A criterion may now be set up such that the 'optimal' sensor configurations x_k^j minimize $\operatorname{var}(q(x,t;\hat{\theta}))$ over $\mathcal{Q}$. Based

on the suggestions of (Fedorov and Hackl, 1997, p.25), in the sequel the following V-optimality criterion is considered:

$$\Psi[M] = \iint_{\mathcal{Q}} \operatorname{var}(q(x,t;\hat{\theta}))\,\mathrm{d}x\,\mathrm{d}t = \operatorname{trace}\left\{CM^{-1}\right\} \quad (10)$$

where

$$C = \iint_{\mathcal{Q}} \left(\nabla_\theta q(x,t;\theta^0)\right)\left(\nabla_\theta q(x,t;\theta^0)\right)^{\mathrm{T}}\,\mathrm{d}x\,\mathrm{d}t \quad (11)$$

Sensor positions which guarantee the best accuracy of the least-squares estimates of θ are then found by choosing x_k^j, $j = 1,\ldots,N$, $k = 1,\ldots,K$ so as to minimize Ψ.

The assumption of independent measurements made by different sensors implies that we admit of replicated measurements, i.e. some values x_k^j may appear several times in the optimal solution (this is an unavoidable consequence of independent measurements). Consequently, it is sensible to distinguish only the components of the sequence $x_k^1,\ldots,x_k^N$ which are different and, if there are $\ell(k)$ such components, to relabel them as $x_k^1,\ldots,x_k^{\ell(k)}$ while introducing $r_k^1,\ldots,r_k^{\ell(k)}$ as the corresponding numbers of replications. The redefined x_k^i's are said to be the *design* or *support* points. The collection of variables

$$\xi_k^N = \begin{Bmatrix} x_k^1, & x_k^2, & \ldots, & x_k^{\ell(k)} \\ p_k^1, & p_k^2, & \ldots, & p_k^{\ell(k)} \end{Bmatrix}, \quad (12)$$

where $p_k^i = r_k^i/N$, $N = \sum_{i=1}^{\ell(k)} r_k^i$, is called the *exact design* of the experiment on the subinterval T_k. The proportion p_k^i of observations performed at x_k^i can be considered as the percentage of experimental effort spent at that point.

On account of the above remarks, we rewrite the FIM in the form

$$M(\xi^N) = M(\xi_1^N,\ldots,\xi_K^N) = \sum_{k=1}^{K}\sum_{i=1}^{\ell(k)} p_k^i \Upsilon_k(x_k^i). \quad (13)$$

Here the p_k^i's are rational numbers, since both r_k^i's and N are integers. Removing this constraint by assuming that they can be any real numbers of the interval $[0,1]$ such that $\sum_{i=1}^{\ell(k)} p_k^i = 1$, we may think of the designs as discrete probability distributions on X. But if so, we may attempt to take one more step to widen the class of admissible designs a bit further, i.e. to all probability measures ξ_k over X which are absolutely continuous with respect to the Lebesgue measure and satisfy by definition the condition

$$\int_X \xi_k(\mathrm{d}x) = 1, \quad k = 1,\ldots,K. \quad (14)$$

Such an extension of the design concept allows us to replace (13) by

$$M(\xi) = \sum_{k=1}^{K} \int_X \Upsilon_k(x)\,\xi_k(\mathrm{d}x), \quad (15)$$

where

$$\xi = (\xi_1,\ldots,\xi_K) \quad (16)$$

and the integration in (14) and (15) is to be understood in the Stieltjes-Lebesgue sense. This leads to the so-called *continuous* designs which constitute the basis of the modern theory of optimal experiments [11], [22]. It turns out that, in spite of its slightly abstract form, such an approach drastically simplifies the design.

Then we may redefine an optimal design as a solution to the optimization problem

$$\xi^\star = \arg\min_{\xi\in\Xi} \Psi[M(\xi)], \quad (17)$$

where Ξ denotes the set of all designs of the form (16).

In the remainder of this paper we shall make the following assumptions:

(A1) X is compact,

(A2) $g(\,\cdot\,,\,\cdot\,)$ is continuous,

Moreover, the following properties can be easily proved:

(P1) Ψ is convex,

(P2) If $M_1 \le M_2$, then $\Psi(M_1) \ge \Psi(M_2)$,

(P3) There exists a finite real q such that

$$\left\{\xi : \Psi[M(\xi)] \le q < \infty\right\} = \Xi(q) \neq \emptyset,$$

(P4) For any $\xi \in \Xi(q)$ and $\bar{\xi} \in \Xi$, we have

$$\begin{aligned}\Psi[M(\xi) + \lambda(M(\bar{\xi}) - M(\xi))] &\\ = \Psi[M(\xi)] + \lambda \sum_{k=1}^{K} \int_X \psi_k(x,\xi)\,\bar{\xi}(\mathrm{d}x) &\\ + o(\lambda;\xi,\bar{\xi}), &\end{aligned} \quad (18)$$

where $\lim_{\lambda\downarrow 0} o(\lambda;\xi,\bar{\xi})/\lambda = 0$.

As regards the notation in (P2), we adopt that of the Loewner ordering of symmetric matrices, i.e. $M_1 \le M_2$ iff $M_2 - M_1$ is non-negative definite. Note that (P4) simply amounts to the existence of the directional derivative whose form must be on one hand specific, but on the other hand, such a condition is not particularly restrictive. In fact, we obtain

$$\psi_k(x,\xi) = \operatorname{trace}\Big[\overset{\circ}{\Psi}(\xi)\Upsilon_k(x)\Big] - \frac{1}{K}\operatorname{trace}\Big[\overset{\circ}{\Psi}(\xi)M(\xi)\Big], \quad (19)$$

where

$$\overset{\circ}{\Psi}(\xi) = \left.\frac{\partial\Psi(M)}{\partial M}\right|_{M=M(\xi)} = -M^{-1}(\xi)CM^{-1}(\xi).$$

III. Problem Reformulation Using the Notion of Clusterization-Free Designs

Independent observations are convenient from a theoretical point of view, but they can hardly be justified when in an optimal solution several sensors are to take measurements near one another (this phenomenon is called *sensor clusterization*). Indeed, in the spatial data collection schemes there is usually no possibility of replicated measurements, i.e. different sensors cannot take measurements at

one point without influencing one another. Anyway, several sensors situated in the close vicinity of one another usually do not give more information than a single sensor.

In order to avoid such clustered sensor configurations, we implement the idea of operating on the density of sensors (i.e. the number of sensors per unit area), rather than on the sensors' locations, which is justified for a sufficiently large total number of sensors N. In contrast to the designs discussed in the previous section, however, we impose the crucial restriction that the density of sensor allocation must not exceed some prescribed level. This amounts to the condition

$$\xi_k(\mathrm{d}x) \le \omega(\mathrm{d}x), \quad k = 1, \ldots, K, \tag{20}$$

where $\omega(\mathrm{d}x)$ signifies the maximal possible 'number' of sensors per $\mathrm{d}x$ [11] such that

$$\int_X \omega(\mathrm{d}x) \ge 1. \tag{21}$$

Consequently, we are faced with the following optimization problem: Find

$$\xi^\star = \arg\min_{\xi\in\Xi} \Psi[M(\xi)] \tag{22}$$

subject to

$$\xi_k(\mathrm{d}x) \le \omega(\mathrm{d}x), \quad k = 1, \ldots, K. \tag{23}$$

The design $\xi^\star$ above is then said to be a (Ψ, ω)-*optimal design* on the analogy of the definition introduced in [11] in the context of directly constrained design measures.

Apart from Assumptions (A1) and (A2), a proper mathematical formulation calls for the following proviso:
(A3) $\omega(\mathrm{d}x)$ is atomless, i.e. for any $\Delta X \subset X$ there exists a $\Delta X' \subset \Delta X$ such that

$$\int_{\Delta X'} \omega(\mathrm{d}x) < \int_{\Delta X} \omega(\mathrm{d}x). \tag{24}$$

In what follows, we write $\bar{\Xi} \subset \Xi$ for the collection of all the design measures (16) which satisfy the requirement

$$\xi_k(\Delta X) = \begin{cases} \omega(\Delta X) & \text{for } \Delta X \subset \operatorname{supp}\xi_k, \\ 0 & \text{for } \Delta X \subset X \setminus \operatorname{supp}\xi_k, \end{cases} \tag{25}$$

$k = 1, \ldots, K$. Given a design ξ, we will say that the function $\psi_k(\,\cdot\,, \xi)$ defined by (19) separates sets X_1 and X_2 with respect to $\omega(\mathrm{d}x)$ if for any two sets $\Delta X_1 \subset X_1$ and $\Delta X_2 \subset X_2$ with equal non-zero measures we have

$$\int_{\Delta X_1} \psi_k(x, \xi)\, \omega(\mathrm{d}x) \le \int_{\Delta X_2} \psi_k(x, \xi)\, \omega(\mathrm{d}x). \tag{26}$$

We can now formulate the main result which provides a characterization of (Ψ, ω)-optimal designs.

Theorem 1: Let Assumptions (A1)–(A3) hold. Then:
(i) There exists an optimal design $\xi^\star \in \bar{\Xi}$, and
(ii) A necessary and sufficient condition for $\xi^\star \in \bar{\Xi}$ to be (Ψ, ω)-optimal is that $\psi_k(\,\cdot\,, \xi^\star)$ separates $X_k^\star = \operatorname{supp}\xi_k^\star$ and its complement $X \setminus X_k^\star$ with respect to the measure $\omega(dx)$, $k = 1, \ldots, K$.

This constitutes a fairly straightforward generalization of Theorem 4.3.1 of [11, p. 63], also see [12], and the main ideas of the proof given therein are retained here.

IV. Sensor Scheduling

From a practical point of view, Theorem 1 means that at all the support points of an optimal design component $\xi_k^\star$ the mapping $\psi_k(\,\cdot\,, \xi^\star)$ should be less than anywhere else, i.e. preferably $\operatorname{supp}\xi_k^\star$ should coincide with minimum points of $\psi_k(\,\cdot\,, \xi^\star)$, which amounts to allocating observations to the points at which we know least of all about the system response.

If we were able to construct a design with this property, then it would be qualified as an optimal design. This conclusion forms a basis for numerical algorithms of constructing solutions to the problem under consideration.

As regards the interpretation of the resultant optimal designs (provided that we are in a position to calculate at least their approximations), one possibility is to partition X into subdomains ΔX_i of relatively small areas and then, on the subinterval T_k, to allocate to each of them the number

$$N_k^\star(\Delta X_i) = \left\lceil N \int_{\Delta X_i} \xi_k^\star(\mathrm{d}x) \right\rceil \tag{27}$$

of sensors whose positions may coincide with nodes of some uniform grid (here $\lceil\zeta\rceil$ is the smallest integer greater than or equal to ζ). This grid can consist e.g. of points at which scanning sensors may be located, which will be exploited in what follows.

Clearly, unless the considered design problem is quite simple, we must employ a numerical algorithm to make the outlined conceptions useful. Since $\xi_k^\star(\mathrm{d}x)$ should be non-zero in the areas where $\psi_k(\,\cdot\,, \xi^\star)$ takes on a smaller value, the central idea is to move some measure from areas with higher values of $\psi_k(\,\cdot\,, \xi^n)$ to those with smaller values, as we expect that such a procedure will improve ξ^n. This is embodied by the iterative algorithm presented below:

Algorithm for sensor scheduling

Step 1. Guess an initial design $\xi^0 \in \bar{\Xi}$. Set $n = 0$.
Step 2. For $k = 1, \ldots, K$ separately set $X_1^n(k) = \operatorname{supp}\xi_k^n$ and $X_2^n(k) = X \setminus X_1^n(k)$. Determine

$$x_1^n(k) = \arg\max_{x\in X_1^n(k)} \psi_k(x, \xi^n),$$

$$x_2^n(k) = \arg\min_{x\in X_2^n(k)} \psi_k(x, \xi^n).$$

If $\psi_k(x_1^n(k), \xi^n) > \psi_k(x_2^n(k), \xi^n) + \eta$, where $\eta \ll 1$, then find two sets $S_1^n(k) \subset X_1^n(k)$ and $S_2^n(k) \subset X_2^n(k)$ such that $x_1^n(k) \in S_1^n(k)$, $x_2^n(k) \in S_2^n(k)$ and

$$\int_{S_1^n(k)} \omega(\mathrm{d}x) = \int_{S_2^n(k)} \omega(\mathrm{d}x) = \alpha_n$$

(i.e. the measures of $S_1^n(k)$ and $S_2^n(k)$ must be identical) for some $\alpha_n > 0$. Otherwise, set $S_1^n(k) = S_2^n(k) = \emptyset$. If $\psi_k(x_1^n(k), \xi^n) < \psi_k(x_2^n(k), \xi^n) + \eta$ for all $k = 1, \ldots, K$, then STOP.

Step 3. Construct ξ^{n+1} such that

$$\begin{aligned}\operatorname{supp}\xi_k^{n+1} &= X_1^{n+1}(k)\\ &= (X_1^n(k)\setminus S_1^n(k))\cup S_2^n(k).\end{aligned}$$

for $k=1,\ldots,K$. Increment n and to go Step 2.

Convergence is guaranteed if the sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the conditions

$$\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty, \tag{28}$$

which is established in much the same way as in [13].

Within the framework of sensor placement, we usually have $\omega(\mathrm{d}x)=\varrho(x)\mathrm{d}x$, where ϱ is a density function. But in this situation we may restrict our attention to constant ϱ's (indeed, in any case we can perform an appropriate change of coordinates). Moreover, while implementing the algorithm on a computer, all integrals are replaced by sums over some regular grid elements. Analogously, the sets X, $X_1^n(k)$, $X_2^n(k)$, $S_1^n(k)$ and $S_2^n(k)$ then simply consist of grid elements (or potential sensor locations). Consequently, the above iterative procedure may be considered as an exchange-type algorithm with the additional constraint that every grid element must not contain more than one supporting point and the weights of all supporting points are equal to $1/N$. In practice, α_n is usually fixed and, what is more, one-point exchanges are most often adopted, i.e. $S_1^n(k)=\{x_1^n(k)\}$ and $S_2^n(k)=\{x_2^n(k)\}$, which substantially simplifies implementation. Let us note, however, that convergence to an optimal design is assured only for decreasing α_n's and hence some oscillations in $\Psi[M(\xi^n)]$ may sometimes be observed. A denser spatial grid usually constitutes a remedy for this predicament [5].

V. Numerical Example

Consider the two-dimensional diffusion equation

$$\begin{aligned}\frac{\partial y(x,t)}{\partial t} &= \frac{\partial}{\partial x_1}\left(\kappa(x)\frac{\partial y(x,t)}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa(x)\frac{\partial y(x,t)}{\partial x_2}\right)\\ &\quad+20\exp\left(-50(x_1-t)^2\right)\quad \text{on } \Omega\times T=(0,1)^3,\end{aligned} \tag{29}$$

subject to homogeneous initial and boundary conditions, where

$$\kappa(x)=\theta_1+\theta_2x_1+\theta_3x_2,$$
$$\theta_1^0=0.1,\quad \theta_2^0=-0.05,\quad \theta_3^0=0.2$$

θ_1^0, θ_2^0 and θ_3^0 being treated as nominal and known to the experimenter prior to the identification itself. The forcing term in (29) imitates a line source whose support is constantly oriented along the x_2-axis and moves with constant speed to the right. Our purpose is to estimate y inside the circle $\mathcal{X}$ with centre $(0.5,0.5)$ and radius 0.2 during the time horizon $\mathcal{T}=T$. In the course of the experiment, the sensors must be placed outside $\mathcal{X}$.

In order to numerically solve the measurement location problem, a computer program was written in Lahey Fujitsu Fortran 95 v.5.6 using a PC with Pentium IV running Windows 2000. The state and sensitivity equations (cf. [2]) were first solved using the finite-element method on an even grid (with 15 divisions along each space axis, and 30 divisions of the time interval). The sensitivity coefficients were then interpolated via tri-cubic spline interpolation and the corresponding spline parameters stored in computer memory.

The problem of locating $N=90$ scanning sensors was considered. For that purpose, a (20×20)-point uniform grid was introduced to approximate the design space, from which only points from the outside of $\mathcal{X}$ were selected as feasible. The initial design was generated by randomly selecting its support points. The sensors were allowed to take measurements over the time intervals $T_k=[(k-1)/20,k/20]$, $k=1,\ldots,20$. The resulting optimal solution is shown in Fig. 1, where dots represent the grid points (these were potential sites where the sensors could be placed, but at most one sensor at one point) and open circles indicate the actual sensor positions.

In order to calculate a V-optimal design, a simple one-point correction algorithm was employed ($\eta=10^{-2}$) which produced the solution after only 79 iterations (practically, within several seconds).

As regards the forcing term in our model, it approximates the action of a line source whose support is constantly oriented along the x_2-axis and moves with constant speed from the left to the right boundary of Ω. This is reflected by the consecutive configurations of scanning sensors which also advance to the right.

VI. Conclusion

This paper considers the optimal sensor location problem in an experiment performed for prediction of certain variables depending on spatial location and/or time. An efficient iterative algorithm capable of handling the adopted V-optimum design criterion has been developed. Thereby, it is possible to enforce a wide variety of constraints imposed on sensor configurations. The general formulation applies to a large class of systems. Research is conducted on extending this approach to include robust or Bayesian designs and applications in fault detection schemes.

Acknowledgments

This research was supported by the State Committee for Scientific Research under Grant No. 7 T11A 023 20.

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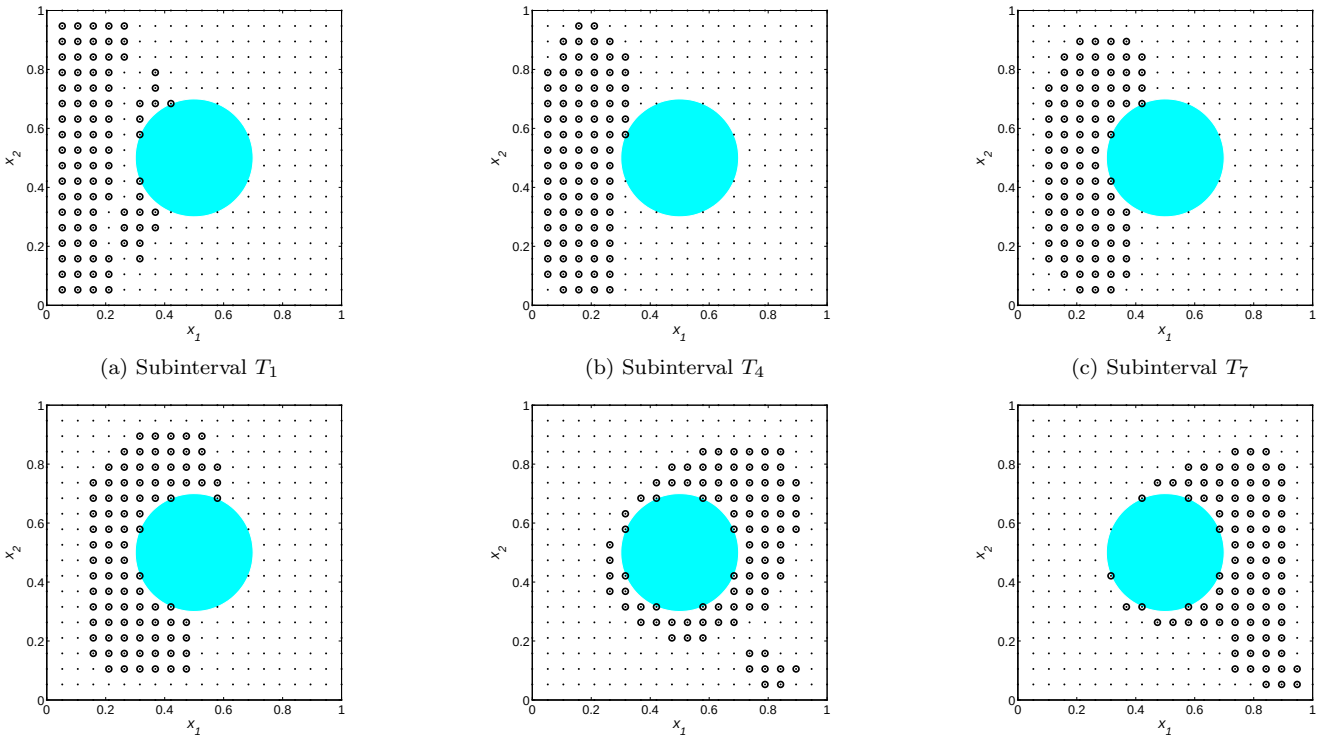

(a) Subinterval T_1 (b) Subinterval T_4 (c) Subinterval T_7

(d) Subinterval T_{10} (e) Subinterval T_{17} (f) Subinterval T_{20}

Fig. 1. Consecutive sensor configurations.

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