Numerical Stability of the Green-Galerkin Thermal Control Method in Infinite-Dimensional Dynamic Systems

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***Abstract*—The Green-Galerkin method has been shown to successfully address the problem of thermal control in distributed-parameter heat conduction systems. This article investigates the effect of altering the iterative time step on the convergence of the solution generated by the aforementioned method when applied to multi-dimensional problems. It is proved that regardless of the variations in numerical processing, the iterative technique is able to solve the inverse heat conduction problem encountered in thermal processing of solids. Furthermore, it is shown that the method is able to generate heat input functions for as many heating locations chosen to exist on the surface of a solid body. In all cases studied, FEA simulations are conducted to validate the analytical solution obtained by the proposed method.**

***Index Terms*—Distributed parameter, thermal control.**

I. Introduction

ATTAINING optimal product features, such as resistance to oxidation, corrosion and wear, is of primary importance in the field of thermal manufacturing. Such features are defined by the resulting metallurgical structure of the materials constituting the manufactured part. In turn, the temperature field that needs to be achieved during the process in order to obtain the desired metallurgical structure is dictated by material transformation diagrams. Hence, modeling and control of the temperature fields achieved in thermal processes of manufacturing interest has been greatly emphasized in current research and development efforts, both in the engineering academia and manufacturing industry. However, despite the many advances in the field of thermal controls, distributed parameter systems theory has not yet resolved the fundamental question of controllability of the internal temperature field generated within a three-dimensional solid body, when heating exists strictly on its accessible two-dimensional surface. Consequently the need for a method that can provide solutions to the inverse heat conduction problem in infinite-dimensional dynamic systems is of great value in thermal processing of materials [6].

In previous work, an open-loop control strategy was developed to address the thermal controllability problem. The method was based on Galerkin optimization of an energy index employing Green's functions. The numerical algorithm developed from the deconvolution of the spatial and temporal components of the optimization technique, was employed in previous work to solve both the one- and two- dimensional inverse heat conduction problem and was proved effective for various temperature profiles of increasing complexity [1,3]. Furthermore, an optimization case study was performed, to investigate the convergence of the solution generated by the iterative technique when the time step and duration of processing time in the one-dimensional scenario are varied [2]. In all situations, the results prove the method to be successful in generating the required surface heating function(s) (one/two-dimensional problem) necessary to achieve the desired temperature field during the process. However, the method's stability in the multi-dimensional problem when the iterative time step employed in the numerical technique is altered remains undetermined. This article, therefore, addresses the aforementioned issue, and furthermore, investigates the ability of the method to generate heat input functions for any number of locations chosen on the surface of the solid.

II. Mathematical Model Development: A Control-Volume Approach

A desired temperature distribution can be achieved within the volume of a solid by directly applying heat inputs at internal parts. Such a process, provided by technological means (microwave, chemical/nuclear methods), is expressed by multi-dimensional heat transfer as

$$Q_V(P;t) = \rho c \frac{\partial T_d}{\partial t}(P;t) - \nabla\left(K \nabla T_d(P;t)\right) + h\left(T_d(R;t) - T_\infty\right) \quad (1)$$

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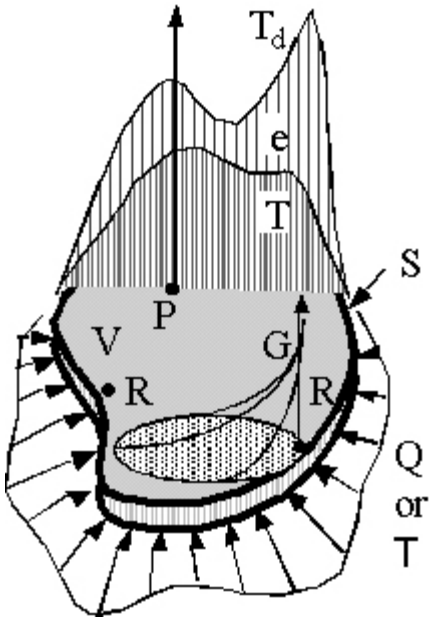

Fig. 1. The control volume.

where ρ is the material density, *c* the specific heat, *K* the thermal conductivity, *h* the convection coefficient, and T_∞ the ambient temperature of the surroundings. The convective heat transfer mechanism is constrained to exist at points along the surface, labeled *R* in figure 1, while radiative heat transfer is ignored. Furthermore, $Q_V(P;t)$ denotes the heat inputs occurring in time at inner points of the control volume, labeled *P* in the figure. Finally, $T_d(P/R;t)$ represents the desired temperature field as a function of space and time.

Hence, thermal control is enabled and could exist at all instances of time when volume heating is employed. However, when implementing strictly surface heating, inputs exist along points on the surface, *R*, but not at inner points *P*.

On a closer look at the problem now, three questions arise, namely:

- Does such a distribution $Q_S(R;t)$, which can create a thermal field at the proper instant in time such that $T(P;t) \approx T_d(P;t)$ exist?(Controllability Question)
- Assuming its existence, how can this surface heat input function $Q_S(R;t)$ be obtained? (Open-Loop Control)
- What desired distributions $T_d(P;t)$ can be obtained exactly by all feasible heat input distributions $Q_S(R;t)$? (Controllable Subspace)

III. A Green-Galerkin Controllability Method

The Galerkin optimization method, which is in wide use in finite element analysis, aids in answering the controllability question. This method deals with obtaining approximate solutions to problems that are highly complex to solve for exact solutions. This is achieved by assuming a solution composed of functions that satisfy all the specified boundary conditions. The assumed solution varies along the surface and within the volume via interpolation functions. However, since this solution is just an approximation, it will not satisfy exactly the governing equations. As a result, there exists an error, labeled the *residual*. Hence, the method optimizes the volume and surface residuals, by making them minimal at all points of the solution domain. This can be stated as

$$\int_V I_V * R_V dV + \int_S I_S * R_S dS = 0 \qquad (2)$$

where I_V and I_S denote the interpolation function within the volume and along the surface respectively, R_V the volume residual, and R_S, the residual along the surface.

This concept is employed to solve for the surface heating function necessary to obtain the temperature distribution closest to the desired one. This is performed by setting the volume residual to be the fictitious heat input required in time within the volume, to generate the exact desired temperature profile inside the solid at the desired time, namely

$$R_V(P;t) = Q_V(P;t) \qquad (3)$$

and for Neumann boundary conditions, the boundary residual is expressed as the difference between the heat flux conducted from the surface point where heating is said to exist, and the heat flow that is actually applied, that is

$$R_S(R;t) = K\frac{\partial T_d(R;t)}{\partial n} - Q_S(R;t) \qquad (4)$$

where *n* represents the inward normal from the boundary. Employing the Galerkin methodology, namely equation (2), to minimize the residuals as defined, it can then be stated that for every surface point R' and time τ ,

$$\int_t^\infty \left[\int_V G(P,R',\tau - t) * R_V(P;t) dV + \int_S G(R,R',\tau - t) * R_S(R';t) dS \right] d\tau = 0 \qquad (5)$$

where $G(P/R,R',\tau - t)$ is the Green's function describing the conduction of heat inside/on the surface of a body of chosen geometry and boundary conditions. This *G* signifies the temperature developed at point *P* / *R* and time τ , due to an impulsive unit heat input (1J) at location R' and time t. At the same time, this denotes the interpolation function necessary for the Galerkin optimization.

Upon direct substitution of the residuals into equation (5), the resulting form that is obtained is

$$\int_t^\infty \int_S G(R,R',\tau - t) * Q_S(R';t) dS d\tau = \int_t^\infty \left[\int_V G(P,R',\tau - t) *$$

$$Q_V(P;t)dV + \int_S G(R,R',\tau - t) * K \frac{\partial T_d(R';t)}{\partial n} dS \Big] d\tau \quad (6)$$

and the term $C\ (R';t)$ will be used to refer to the known, right-hand side of equation (6). Therefore, the required heat flux Q_S on the surface can be determined by deconvolution from equation (6). If $Q_S(R';t)$ occurs in stepwise increments of duration Δt within the range of time $0 \le t \le T$, a simple technique for deconvolving equation (6) backwards in time can be proposed and implemented. The technique creates an iterative process, starting from the time of application of the final input distribution, $t = T$, moving backwards in steps of value Δt, up until the time of the first input, namely $t = 0$. The iterative technique originating from the above-developed mathematical model is as follows:

- $t = T$:

$$Q_S(R';T) = \frac{C(R';T)}{G(R',R',T-T) * \Delta t} \quad (7)$$

- $t = T - \Delta t$:

$$\frac{C(R';T-\Delta t)}{\Delta t} = \int_S G(R,R',\Delta t) * Q_S(R';T) dS + G\left[R',R',(T-\Delta t)-(T-\Delta t)\right] * Q_S(R';T-\Delta t) \quad (8)$$

The unknown term to be determined in equation (8) is $Q_S(R';T-\Delta t)$. The iterative technique proceeds in the same manner backwards in time, adding the previously calculated terms to the known parameters, successively up to time $t = 0$, thus solving for the distributed, time varying, surface heat input function for all selected surface locations R'.

IV. The Two-Dimensional Problem

The first step to mathematically model a heat conduction problem is to determine the appropriate Green's function. This is dependent on the part geometry and boundary conditions. In the particular case, a square geometry of length $2L$ represents the two-dimensional plate under study. The same heat input function exists along all four sides of the square. In addition, on each side, the thermal loading is symmetric with respect to the midpoint of the side. This is explained in figure 2 below.

Applying the thermal loads as shown in figure 2 allows the use of symmetry, which results in the schematic of figure 3. In this figure (fig. 3) the inputs are symmetric with respect to the dotted diagonal line. It is possible to study one of the two triangular shapes that result from figure 3, however, the figure as a whole makes it easier to create the surface plots shown in the simulations below.

TABLE I

Properties of chosen steel material.

Length of Quadrant (L)	0.05 meters
Density (ρ)	7830 kg/m^3
Thermal Conductivity (K)	64 W/(m K)
Specific Heat (c)	434 J/(kg K)

The Green's function for the derived model (figure 3) is obtained by the method of images [4,5] and is found to be:

$$G(x,y,x',y',\tau,t) = \frac{1}{\rho c} * \frac{1}{4\pi\alpha(\tau - t)} * \sum_{n=-\infty}^{\infty} \Bigg[\exp\left[-\frac{(2nL + x - x')^2 + (2nL + y - y')^2}{4\alpha(\tau - t)}\right] + \exp\left[-\frac{(2nL + x - x')^2 + (2nL + y + y')^2}{4\alpha(\tau - t)}\right] + \exp\left[-\frac{(2nL + x + x')^2 + (2nL + y - y')^2}{4\alpha(\tau - t)}\right] + \exp\left[-\frac{(2nL + x + x')^2 + (2nL + y + y')^2}{4\alpha(\tau - t)}\right] \Bigg] \quad (9)$$

where the variables x' and y' represent the location/coordinates along the x and y-axes where each heat input occurs. Also, α, is the diffusivity of the material in (m^2/s), defined as $\alpha = K/(\rho c)$.

In order to express the temperature evolution on the entire surface of figure 3 as a result of thermal applications on the boundary, the superposition of the Green's functions

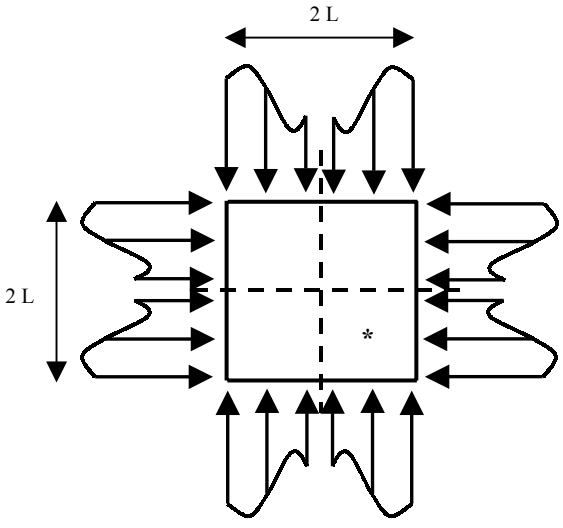

Fig. 2. Description of symmetric thermal loading.

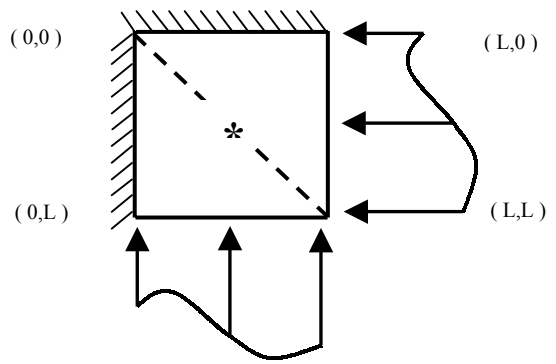

Fig. 3. Model resulting from the symmetry of the loading.

originating both from the horizontal and vertical loading is required. Of note, the model uses the assumption of homogeneous, isotropic, time invariant properties and no phase transformations.

The temperature profile investigated involves having the boundaries of the square plate seen in figure 2 to be at a temperature value of 200 degrees and falling linearly to zero as approaching the midpoint**.** In all simulations conducted a zero initial temperature is assumed. Due to the symmetry of the thermal loading on figure 2, the quadrant of figure 3 is employed.

In the first simulation conducted with the Green-Galerkin method, the increment of time, or step Δt, is chosen to be two seconds. The instant of time to achieve the desired distribution is found to be t = 18 seconds. Five input functions along the horizontal length of figure 3 and five equal and symmetrical input functions on the vertical length are used. Each of the five input functions uses 9 input values of variable intensity and duration of two seconds each. Snapshots of the resulting temperature profile, obtained from the application of the algorithm generated heat input functions, are seen in figure 4.

To verify the analytical temperature profile of figure 4.c an FEA simulation is conducted where the numerical result of the heat input functions generated by the algorithm is studied.

In the next simulation the effect of the time step employed in the iterative process on the convergence of the solution generated by the Green-Galerkin method is investigated. The same temperature profile is once again studied, this time however, $\Delta t = 1$ second while the duration of thermal processing remains at 18 seconds. Once again, five input functions along the horizontal length of figure 3 and five equal and symmetrical input functions on the vertical length are used. Each of the five input functions uses 18 input values of variable intensity and duration of one second each. The thermal field that results from the application of the algorithm generated heat input functions is seen in figure 6. In addition, figure 7 shows the result of the FEA simulation.

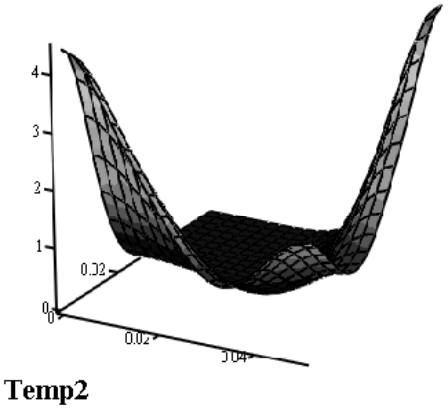

Fig. 4.a. Time t = 2 sec.

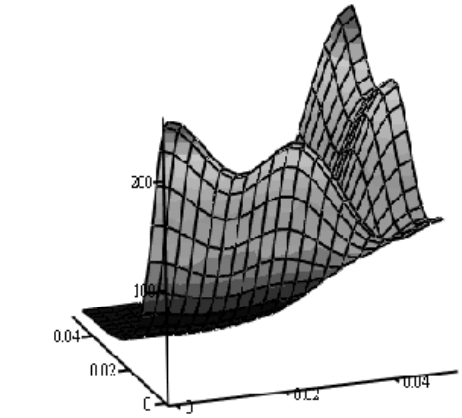

Fig. 4.b. Time t = 12 sec.

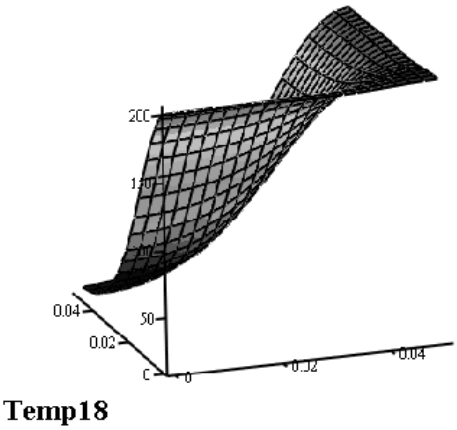

Fig. 4.c. Time t = 18 sec.

Fig. 4. Simulated temperature profiles.

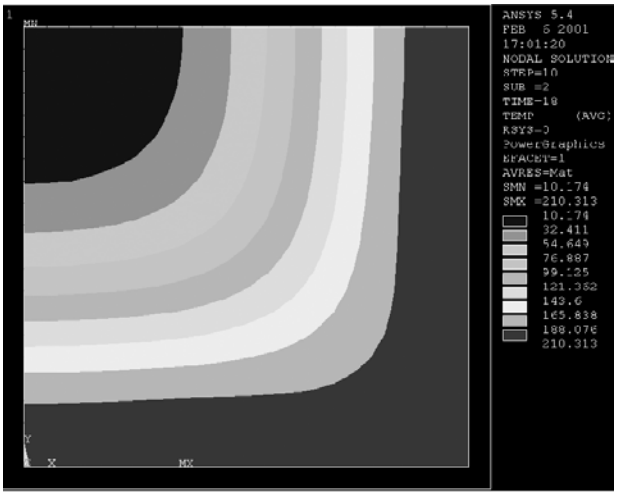

Fig. 5. FEA temperature profile.

It is worth noting that the colors seen in figures 5 and 7 represent approximately the same temperature ranges, thus allowing for a direct comparison of the two temperature profiles achieved by a different value of Δt in the iterative technique.

Comparing the results of the two simulations conducted, it appears that the algorithm is able to generate heat input functions that will achieve the desired temperature profile despite the variations in the iterative time step used in the numerical processing.

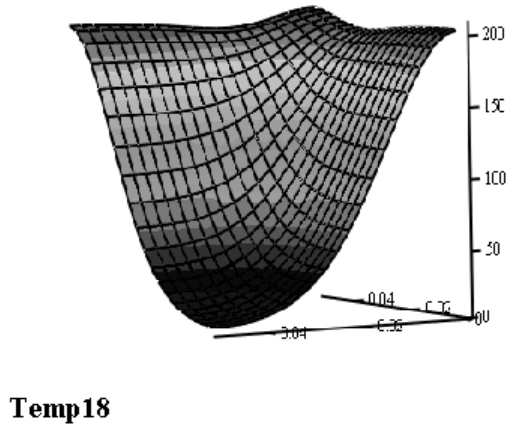

Fig. 6. Simulated temperature profile.

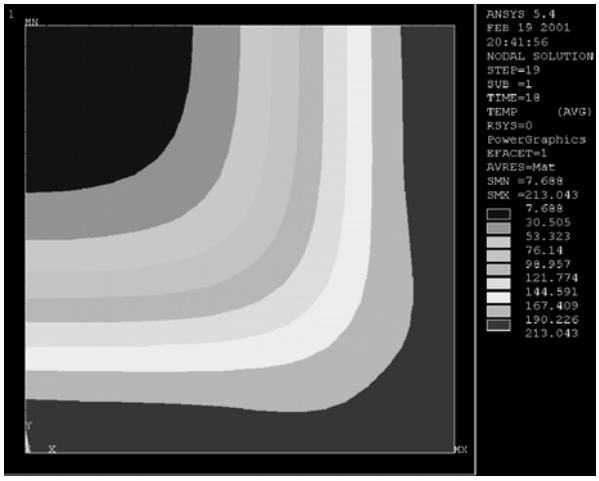

Fig. 7. FEA temperature profile.

Finally, the ability of the method to generate heat input functions for as many locations chosen on the boundary of the two-dimensional plate is investigated. A simulation is conducted for the same temperature profile studied above, using nine heat input functions along the horizontal boundary of figure 3 and nine equal and symmetric heat input functions on the vertical boundary. Each input function uses 9 input values of variable intensity and duration of two seconds (step Δt used in the iterative process) each. The thermal field that results from the application of the algorithm generated heat input functions is seen in figure 8. In addition, figure 9 shows the result of the FEA simulation. It is again noted that the colors seen in this FEA figure represent approximately the same temperature ranges as the above two, thus allowing for a direct comparison of the temperature profiles achieved in each case studied.

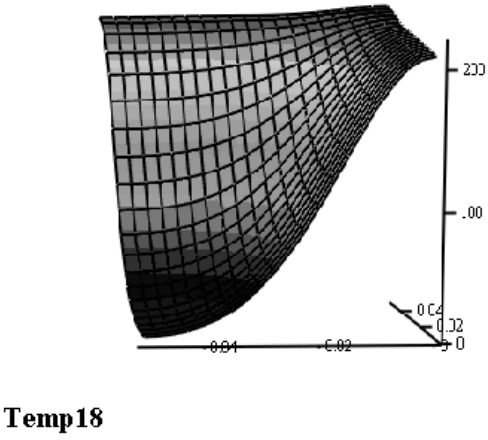

Fig. 8. Simulated temperature profile.

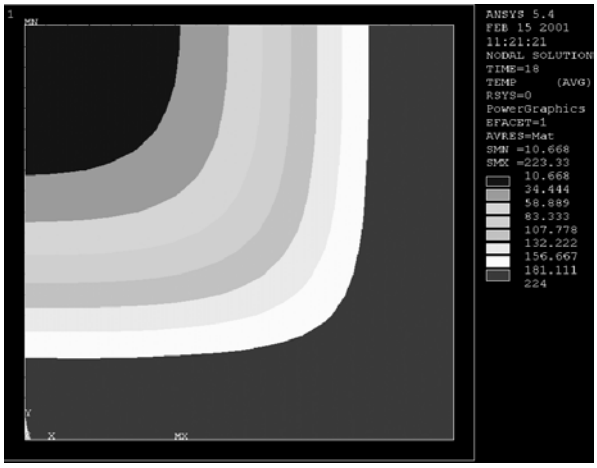

Fig. 9. FEA temperature profile.

Comparing the simulation results presented in the figures above, shows that the Green-Galerkin method is able to generate heat input functions for the number of locations chosen to exist on the boundary so as to achieve the desired temperature profile.

V. Conclusion

In summary, this article investigated the convergence of the multi-dimensional solution of the Green-Galerkin method when a different time step in the iterative process of the numerical technique is selected. In addition, the method's ability to generate heat input functions for any number of heating locations chosen to exist on the surface of a solid body was addressed. The results indicate that the proposed method is able to generate the surface heat input functions required to achieve the desired temperature profile, irrespective of the time step and number of heating locations chosen.

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