On Distributed Coordination of Mobile Agents with Changing Nearest Neighbors

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Abstract— **In a recent paper [10], we provided a formal analysis for a distributed coordination strategy for coordination of a set of agents moving in the plane with the same speed but variable heading direction. Each agents heading is updated as the average of its heading and a set of its nearest neighbors. As the agents move, the graph induced by the nearest neighbor relationship changes, resulting in switching. We recently demonstrated that by modelling the system as a discrete linear inclusion (in a discrete time setting) and a switched linear system (in continuous time setting), conditions for convergence of all headings to the same value can be provided. In this paper, we extend these results and demonstrate that switching among graphs has to stop in a finite time in order to get convergence. Moreover, we will show that a necessary and sufficient condition for convergence is that the switching stops on a connected graph. We also provide connections between this problem and Left convergent product (LCP) sets of matrices.**

I. Introduction

In a recent paper [17], Vicsek *et al.* propose a simple but compelling discrete-time model of n autonomous agents {i.e., points or particles} all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its "neighbors." Agent i's *neighbors* at time t, are those agents which are either in or on a circle of pre-specified radius r centered at agent i's current position. The Vicsek model turns out to be a special version of a model introduced previously by Reynolds [13] for simulating visually satisfying flocking and schooling behaviors for the animation industry. In their paper, Vicsek *et al.* provide a variety of interesting simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves.

More recently, a theoretical explanation for this observed behavior was provided [10]. It was shown that studying the behavior of headings of all agents amounts to studying left-infinite products of certain matrices, chosen from a finite set.

We now describe this model in detail: The distributed coordination model under consideration consists of n autonomous agents {e.g., points or particles}, labelled 1 through n, all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a simple local rule based on the average of its own heading plus the headings of its "neighbors." Agent i's *neighbors* at time t, are those agents which are inside a circle of pre-specified radius r centered at agent i's current position. In the sequel $\mathcal{N}_i(t)$ denotes the set of labels of those agents which are neighbors of agent i at time t. Agent i's heading, written θ_i, evolves in discrete-time in accordance with a model of the form

$$\theta_i(t+1) = < \theta_i(t) >_r \tag{1}$$

where t is a discrete-time index taking values in the non-negative integers $\{0, 1, 2, \ldots\}$, and $< \theta_i(t) >_r$ is the average of the headings of agent i and agent i's neighbors at time t; that is

$$< \theta_i(t) >_r = \frac{1}{1 + n_i(t)} \left(\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) \right) \tag{2}$$

where $n_i(t)$ is the number of neighbors of agent i at time t.

The explicit form of the update equations determined by 1 and 2 depends on the relationships between neighbors which exist at time t. These relationships can be conveniently described by a simple, undirected graph with vertex set $\{1, 2, \ldots, n\}$ which is defined so that (i, j) is one of the graph's edges if and only if agents i and j are neighbors. Since the relationships between neighbors can change over time, so can the graph which describes them. To account for this all possible such graphs need to be considered. In the sequel, the symbol $\mathcal{P}$ is used to denote a suitably defined set, indexing the class of all simple graphs $\mathbb{G}_\mathsf{l}$ defined on n vertices. Before going further however, we need to introduce some concepts from graph theory.

For any given $p \in \mathcal{P}$, A_p is the *adjacency matrix* of the graph, that is, a Boolean matrix with the element $a_{ij} = 1$ if and only if there is an edge between vertex i and vertex j, and $a_{ij} = 0$ otherwise. Note that since all agents use circles with the same radius, the graph is *undirected* and as a result the adjacency matrix is symmetric. For each vertex i, the number d_i denotes the *degree* or *valence* of node i, i.e., the number of vertices to which vertex i is connected, or in other words, the number of neighbors of agent i. A diagonal matrix D_p whose *ith* diagonal element is the number d_i is called the matrix of valences. Using the above definitions, for each $p \in \mathcal{P}$, define

$$F_p = (I + D_p)^{-1}(A_p + I). \tag{3}$$

We denote such matrices as *formation* matrices. The heading update equation can be now written as follows

$$\theta(t+1) = F_{\sigma(t)}\theta(t), \quad t \in \{0, 1, 2, \ldots\} \tag{4}$$

where θ is the heading vector $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \ldots & \theta_n \end{bmatrix}'$ and $\sigma : \{0, 1, \ldots\} \to \mathcal{P}$ is a switching signal whose value at time t, is the index of the graph representing the neighboring relationships at time t, and denoted by $\mathbb{G}_{\sigma(t)}$. A complete description of this system would have to include a model which explains how σ changes over time as a function of the positions of the n agents in the plane. In [10], it was shown that for a large class of switching signals and for any initial set of headings that the headings of all n agents will converge to the same steady state value θ_{ss}. Convergence of the θ_i to θ_{ss} is equivalent to the state vector θ converging to a vector of the form $\theta_{ss}\mathbf{1}$ where $\mathbf{1} := \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}'_{\mathbf{n}\times\mathbf{1}}$. This will result in all agents moving in the same direction (with the same speed). We will show that alignment of all agents results in agents moving in a formation, and converging to a constant relative distance. Alternatively, the above problem can be thought of as a *random walk* on an *edge-colored graph* [8].

II. Mathematical Preliminaries

Below we will review some known results about nonnegative matrices which will be the main tool in analyzing the problem. The notation used is fairly standard. $\mathbb{R}$ is the set of reals, $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is the vector space of $n \times n$ matrices with real elements.

A matrix $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is called *positive*, denoted by $A > 0$ if each of its entries are positive. Similarly, A is called *nonnegative*, $A \geq 0$ if it is nonnegative entry-wise. Similar definition holds for vectors, i.e., a vector is positive if all its elements are positive, and it is non-negative if all elements are greater than or equal to zero. It is a well known result from Perron-Frobenius Theory that if a nonnegative matrix has a positive eigenvector, the corresponding eigenvalue is the spectral radius $\rho(\cdot) > 0$, i.e., the eigenvalue with the maximum modulus.

A nonnegative matrix $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is called *irreducible* if and only if $(A + I)^{n-1}$ is positive. For irreducible matrices, $\rho(A) > 0$ is a simple eigenvalue corresponding to a positive eigenvector, however, there might be other eigenvalues with maximum modulus. A graph is connected if and only if its adjacency matrix is irreducible. It turns out that irreducibility is not sufficient to analyze asymptotic properties of nonnegative matrices, hence we need the following definition: A nonnegative, irreducible matrix is called *primitive* (also known as irreducible and aperiodic) if $\rho(\cdot)$ is the only eigenvalue of maximum modulus. A matrix A is primitive if and only if $A^N > 0$ for some $N > 0$. If $A^N > 0$ for some N, then $A^m > 0$ for any $m \geq N$. Moreover, the primitivity of a matrix does not depend on the value of nonzero elements and it is solely determined by the location of the zero elements. Therefore, in determining whether a matrix is primitive or not, it can be assumed without loss of generality that the matrix is binary. The smallest integer N_1 such that $A^{N_1} > 0$ is called the *index of primitivity*, or index for short, of a primitive matrix. It is a theorem of Weilandt which states that for a general non-negative, primitive matrix, $N_1 \leq (n-1)^2 + 1$ [9]. More importantly, this is a tight bound, i.e., there is a non-negative matrix such that $A^{(n-1)^2} \geq 0$ and $A^{(n-1)^2+1} > 0$. However, if all diagonal elements are positive, then the matrix is primitive if and only if $A^{n-1} > 0$, i.e., the index is reduced. (cf. [9] chapter 8 for further details and proofs). It is possible to show that if $A \geq 0$ is primitive, $\lim_{m\to\infty} \frac{A^m}{\rho(A)^m}$ exists and is equal to a rank-one matrix $\mathbf{ed}$, where $\mathbf{e} > \mathbf{0}$, $\mathbf{d} > \mathbf{0}$ are the right and left eigenvectors (known as the Perron vectors) associated with the eigenvalue $\rho(A)$ and are both positive.

An important class of non-negative matrices are *(row) stochastic* matrices. A non-negative matrix is called row stochastic, or stochastic for short, if all row sums are equal to one. A necessary and sufficient condition for a non-negative matrix to be stochastic is that $\mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}'$ be a fixed point, i.e., $\mathbf{1}$ be a right eigenvector vector corresponding to the spectral radius $\rho(\cdot) = 1$. In other words, $A \geq 0$ is stochastic if and only if $A\mathbf{1} = \mathbf{1}$. A stochastic matrix whose powers converge to a rank-one matrix $\mathbf{1c}$ for some row vector $\mathbf{c}$, is called *ergodic*. All primitive stochastic matrices are ergodic, but not all ergodic matrices are primitive. Primitive matrices canbe thought of as ergodic matrices whose powers converge to a positive rank-one matrix, i.e., $\mathbf{c} > 0$.

A set of matrices is called a Left Convergent Product (LCP) set [4], [5] if any left infinite product of matrices from the set converge to a limit.

III. Conditions for convergence

The solution to the equation 4 (in discrete setting) can be given as the product of matrices F_p for different values of p. In other words, $\theta(t) = F_{\sigma(t)} F_{\sigma(t-1)} \cdots F_{\sigma(1)} \theta_0$, where θ_0 is the initial vector of headings. It is clear that in order to study the asymptotic convergence of all headings to the common steady state value, one needs to study the properties of left-infinite products of all F_ps. The above system is an example of a discrete linear inclusion in which the switching function $\sigma(\cdot)$ determines which matrix (chosen from a finite collection of matrices) should be chosen at each sampling instant.

A close examination of the formation matrices, reveals that all formation matrices are stochastic matrices with positive diagonal elements. The headings of all agents converge to the same value if and only if the infinite products of F_ps converge to a rank-one matrix $\mathbf{1c}$ for some row vector $\mathbf{c}$. This would guarantee that

$$\lim_{t \to \infty} \theta(t) = \mathbf{1c}\theta_0 = \mathbf{1}\theta_{ss}.$$

The study of sets of stochastic matrices infinite product of which converge, arise in many problems such as in constructing self-similar objects, in various interpolation schemes, in constructing wavelets with compact support, and in studying nonhomogeneous Markov chains [7], [15], [12], [19], [14]. As mentioned earlier, such a collection of matrices is called a Left Convergent Product (LCP) set [4]. Unfortunately, an effective algorithm which could decide whether a finite set of matrices is LCP or not is not known [5]. Several equivalent conditions for a set to be LCP has been developed in the literature and its connections to other forms of stability has been explored [2], [3], [18]. In what follows, we review some known necessary and sufficient conditions for a set of stochastic matrices to be LCP, however, as mentioned earlier, these conditions to not result in an effective computational procedure.

Theorem 1: For a finite set $\mathcal{S} = \{F_1, F_2, \cdots, F_m\}$ of $n \times n$ row stochastic matrices, the following conditions are equivalent.

- $\mathcal{S}$ is an LCP set in which each 1-right eigenspace is one dimensional. (Each element of $\mathcal{S}$ has precisely one eigenvalue at 1). Moreover, all left infinite products of members of $\mathcal{S}$ converge to a limit rank-one matrix $\mathbf{1c}$ for some positive row vector $\mathbf{c}$, where $\mathbf{1}$ is the common right eigenvector, $\mathbf{1} = [1 \cdots 1]'$.
- All finite products $F_{p_1} \cdots F_{p_k}$ are ergodic matrices for all $k > 0$ [19].
- There exist a vector norm $||\cdot||$ with respect to which the set $\mathcal{S}$ is paracontractive [6], i.e., $\forall F_p \in \mathcal{S}$, and $\forall x \in \mathrm{I\!R}^n, F_p x \neq x$, $||F_p x|| < ||x||$ [18].
- Let V be an $n-1 \times n$ matrix whose rows form an orthonormal basis for the $n-1$-dimensional subspace perpendicular to $\mathbf{1}$, where $\mathbf{1}$ is the vector of all ones. Then $\rho(P\Sigma P') < 1$ where $P\Sigma P' = \{PF_pP, \quad p = 1, \cdots, m\}$, and $\rho(\cdot)$ is the Rota-Strang joint spectral radius [1] [5].

It can be shown that a necessary and sufficient condition for convergence to zero of infinite products of matrices chosen from Σ is that $\rho(\Sigma) < 1$.

The first two conditions of the theorem basically states that a finite set of stochastic matrices is LCP if and only if all finite products formed from the finite set of matrices are ergodic matrices themselves. This is a classical result due to Wolfowitz [19]. Note that ergodicity of each matrix is *not* sufficient for convergence of the infinite products, but rather, ergodicity of

[1]The Rota-Strang joint spectral radius, is the generalization of the concept of spectral radius of a single matrix to a set of matrices. Let $\mathcal{M} := \{\hat{F}_p, p = 1, \cdots, m\}$ be a finite collection of matrices. Denote by $\mathcal{M}^k$ the set of all words of length k, with the alphabet being elements of $\mathcal{M}$. Then,

$$\rho(\mathcal{M}) := \limsup_{k \to \infty} (\max_{M \in \mathcal{M}^k} ||M||)^{\frac{1}{k}}. \tag{5}$$

Note that convergence of matrix products to a rank-one matrix is equivalent to convergence of the infinite products of the $n-1$ dimensional projections $\hat{F}_{\sigma(t)} \cdots \hat{F}_{\sigma(1)}$ to zero, as t approaches infinity.

all finite products is needed. Since all finite products of all lengths need to be checked, the above condition is not effective from a computational point of view. Thomasian [16] showed that only a large but finite number of products need to be considered and later Paz [12] showed that one actually needs to check all products of length at most $\frac{1}{2}(3^n - 2^{n+1} - 1)$, still a very large number. In fact, The problem of deciding whether infinite products of stochastic matrices chosen from a finite set converge or not, has been shown to be PSPACE-complete [8], which roughly speaking, amounts to the hardest problem in the polynomial in space in the hierarchy. Such problems are believed to be generally harder to solve, than NP-complete problems.

Nevertheless, it can be shown that when the infinite products converge, the convergence is indeed exponential [1], [11].

Let $\mathcal{Q}$ denote the index set corresponding to formation matrices which correspond to a connected graphs. It can be shown that the set $\{F_p, p \in \mathcal{Q}\}$ of connected formation matrices is an LCP set [10], and all products of all lengths are indeed ergodic. This means that if switching is occurred *only* among formation matrices corresponding to connected graphs, The heading of all agents would exponentially converge to the same value θ_{ss}. As was shown in [10], this is a direct consequence of inclusion of each agent itself in the averaging, which results in self-loops in the graph.

It is possible to substantially relax the constraints on σ beyond this and still prove convergence. Specifically, what is really required is the connectedness of the graph "in time" rather than at each particular instant. By connected "in time" it is meant that there exists a number $T > 0$, such that over each interval of length less than or equal to T, the graph resulting from taking the union of all edges of all graphs occurring over that interval is connected. This will ensure ergodicity of all products of length T, which in turn implies convergence of all infinite products and the LCP property for the products of products of length T, primarily as a result of the positivity of diagonal elements of each matrix in the product. The uniformity is required to ensure the finiteness of the set of products, which is required if case 1 of the above theorem is to be used.

The above argument shows that the headings of all agents converge provided that there is a uniform $T > 0$ such that over any interval of length at most T, there is a "path" between any two vertices. Note that translation of ergodicity of products of length T into connectivity of the union graph is a direct consequence of existence of self-loops in the graph, which is a direct consequence of inclusion of the agent itself in the averaging of the heading angles. This simple inclusion dramatically reduced the complexity of the problem.

Similar results can be proven in a continuous-time setting, where the dynamics of the problem is described by a linear differential inclusion, or a switched linear system of the form

$$\dot{\theta} = -(I - F_{\sigma(t)})\theta(t).$$

It can be shown that the solution of the LDI is a product of matrix exponentials $\exp(-I + F_{\sigma(t)})$ corresponding to the continuous dynamics in each interval. Interestingly, the matrix exponentials can be shown to be non-negative, stochastic matrices, which are paracontractive with respect to the infinity norm if the graph at that instant is connected. Under the assumption that there is a minimum time such that each continuous dynamics is present, i.e., there are finite number of switchings in any finite interval, and the switching times are rational with respect to each other, the set of matrix exponentials is finite and the same arguments hold as in the discrete case. Alternatively, one can show that the exponential of a connected formation matrix is ergodic with positive diagonal elements, and it satisfies the necessary and sufficient conditions given in Theorem 1. In any case, similar result holds in a continuous time setting.

IV. Main Result

The previous section provided a review of recent results for proving convergence of all headings, under some assumptions on the switching signal $\sigma(\cdot)$. However, the question of whether the underlying graph stops changing was not considered. In this section, we show that a necessary condition for convergence is that the graph stops changing after a finite time:

Theorem 2: Consider the distributed coordination strategy described above. If all headings converge to the same value θ_{ss}, then the graph induced by the nearest neighbor relationship stops changing after a large enough but finite number of steps N.

Proof: Since all agents are moving with the same speed, the update equations for the components of the position vector of each agent can be written as follows:

$$\begin{aligned} x_i(t+1) &= x_i(t) + VT\cos(\theta_i(t)) \\ y_i(t+1) &= y_i(t) + VT\sin(\theta_i(t)). \end{aligned} \tag{6}$$

Where $x_i(t)$ and $y_i(t)$ are the components of the position vector of the ith agent. We can easily solve for $x_i(t)$ and $y_i(t)$ in terms of the initial position vector $(x_i(0),\ y_i(0))$ as:

$$\begin{aligned} x_i(t) &= x_i(0) + VT\sum_{l=0}^{t}\cos(\theta_i(l)) \\ y_i(t) &= y_i(0) + VT\sum_{l=0}^{t}\sin(\theta_i(l)). \end{aligned} \quad (7)$$

We know look at the relative positions $x_{ij} := |x_i - x_j|$, and $y_{ij} := |y_i - y_j|$. Then,

$$x_{ij}(t) = x_{ij}(0) + VT\sum_{l=0}^{t}\cos(\theta_i(l)) - \cos(\theta_j(l)).$$

Similarly,

$$y_{ij}(t) = y_{ij}(0) + VT\sum_{l=0}^{t}\sin(\theta_i(l)) - \sin(\theta_j(l)).$$

We now have to show that when θ_i, and θ_j converge to θ_{ss}, x_{ij} and y_{ij} both converge to their limits $\bar{x}_{ij}$ and $\bar{y}_{ij}$ respectively. In other words, we need to show that the two infinite sums

$$\sum_{l=0}^{\infty}\cos(\theta_i(l)) - \cos(\theta_j(l))$$

and

$$\sum_{l=0}^{\infty}\sin(\theta_i(l)) - \sin(\theta_j(l))$$

converge when θ_i and θ_j converge to θ_{ss} exponentially. We demonstrate this for the first infinite sum, and the second one follows in the same manner.

To prove that $\sum_{l=0}^{\infty}\cos(\theta_i(l)) - \cos(\theta_j(l))$ converges, we first note that $\theta_i(t)$ and $\theta_j(t)$ converge exponentially to θ_{ss} [10]. We also note that $\cos(a) - \cos(b)$ can be written as $-2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$. To establish convergence, we use a partial sum argument to show that $\{s_t\} := \sum_{l=0}^{t}\cos(\theta_i(l)) - \cos(\theta_j(l))$, is a Cauchy sequence. For this, we will show that $|s_m - s_n| \to 0$ as $m,\ n \to \infty$ $(n > m)$.

$$\begin{aligned} |s_m - s_n| &\leq \sum_{l=m}^{n}|\cos(\theta_i(l)) - \cos(\theta_j(l))| \\ &= \sum_{l=m}^{n}|2\sin(\frac{\theta_i(l)-\theta_j(l)}{2})||\sin(\frac{\theta_i(l)+\theta_j(l)}{2})| \\ &\leq \sum_{l=m}^{n}|2\sin(\frac{\theta_i(l)-\theta_j(l)}{2})|. \end{aligned}$$

Noting that $|\sin(a)| \leq |a|$, we have

$$\begin{aligned} |s_m - s_n| &\leq \sum_{l=m}^{n}|\theta_i(l) - \theta_j(l)| \\ &\leq \sum_{l=m}^{n}|\theta_i(l) - \theta_{ss}| + |\theta_j(l) - \theta_{ss}|. \end{aligned}$$

From exponential convergence of θ_i and θ_j it follows that $|\theta_i(l) - \theta_{ss}| \leq \lambda^l$ for some $0 < \lambda < 1$ and all $i = \{1, \cdots, n\}$. We immediately conclude that $|s_m - s_n|$ goes to zero as m and n go to infinity, i.e., $\{s_t\}$ is a Cauchy sequence and therefore convergent. It follows that x_{ij} converges to some $\bar{x}_{ij}$. Similarly, one can show y_{ij} converges to $\bar{y}_{ij}$, and in fact the convergence is exponential. This implies that $d_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2}$ converges to a constant $\bar{d}_{ij}$. ■

The exponential convergence of pair-wise distances $d_{ij}(t)$ to $\bar{d}_{ij}$, implies that for any pair (i, j), and any given ϵ, there is a constant $N_{ij}(\epsilon)$ such that for any $t \geq N_{ij}(\epsilon)$, we have $|d_{ij}(k) - \bar{d}_{ij}| < \epsilon$. This means that after N_{ij} steps, the pair-wise distances are ϵ apart from the limit. As a result, one can always choose ϵ small enough and find an N such that if $d_{ij}(N) - \bar{d}_{ij} < \epsilon$, then those distances which are less then r (the neighborhood radius) stay that way for ever, and those which are greater than are stay greater than or equal to r for ever, and as a result, The graph induced by a neighboring relation should stay the same after step N. We therefore have the following corollary:

Corollary 1: Given any initial configuration, there always exist a large enough, but finite $N > 0$, after which the graph does not change, i.e.,

$$\theta(t) = F_{\sigma(1)} \cdots F_{\sigma(N)}^{(t-N+1)} \quad \forall \ t \geq N-1.$$

The above corollary has another very important consequence, namely, that if the headings converge, the graph after N steps should be connected. This is the immediate consequence of the fact that if the graph stops changing after N steps and is disconnected, then the powers of $F_{\sigma(N)}^{t+N-1}$ will not converge, as t approaches infinity. In this scenario, the headings in each connected component will converge. Therefore, for convergence, it is necessary that the graph remain unchanged after a finite number of steps.

On the other hand, from Perron-Frobenius theory of non-negative matrices, it is clear that if the graph stops changing after N steps and is connected, then the powers of $F_{\sigma(N)}$ converge to a rank-one matrix $\mathbf{1d_N}$, where $\mathbf{d_N}$ is the Left Perron vector associated with $F_{\sigma(N)}$, i.e., the left eigenvector corresponding to the eigenvalue at 1. We therefore have the following:

Theorem 3: A necessary and sufficient condition for all headings to converge to the same value (and as a result have all agents aligned) is that the graph induced by the nearest neighbor relationship does not change after N steps and is connected, for some large enough but finite integer $N > 0$.

V. Concluding Remarks

We provided a connection between the problem of distributed coordination of a set of mobile autonomous agents with nearest neighbor interaction, with convergence of infinite products of stochastic matrices. In a continuous time setting, the problem amounts to analyzing infinite products of matrix exponentials, which turn out to be stochastic matrices as well.

By building upon recent results in [10], we showed that a necessary and sufficient condition for convergence of all headings is that the switching stops in a finite time, and the resulting graph is connected. The problem would be more complicated if the neighborhood region is chosen to be closed as opposed to open, i.e., if the boundary of the circle with radius r is also included in the sensing region. It is possible to extend these results to the case of agents with second order dynamics, with attraction and repulsion forces present.

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