On Transformation of Nonholonomic Systems To Time-State Control Form

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***Abstract*— Time-state control form (TSCF) provides and alternative way to stabilization of symmetrical affine systems. These systems usually arise as nonholonomic driftless systems. It is, however, not precisely determined which systems can be transformed into this form. Conversion of a general driftless system to the TSCF is a subject of this paper. For control purposes, it is desired that the state variable part of TSCF has a controllable approximate linearization. We give a procedure for conversion once a special vector field called the time axis vector field is known. Since this vector field is hard to find, we turn for help to the machinery of exterior differential systems. It turns out that any flat system can be transformed to TSCF with controllable linearization of state variable part. We show an example of a system of dimension 5. As a side product, we obtain a result on linearization by time scaling.**

I. Introduction

An interesting and important class of nonlinear systems is formed by systems with uncontrollable linearization. They are challenging as they can not be stabilized by a smooth (not even continuous) feedback and they are important as they contain systems subject to nonintegrable velocity constraints referred to as nonholonomic systems. These are often encountered among mechanical systems and in some sense are the most generically nonlinear systems. For symmetrical affine systems, i.e., nonholonomic systems without drift, transformation into chained form [1] greatly simplifies the control problem and leads to a number of control designs based on discontinuous, time varying or hybrid feedback, see, e.g., [2] and the references within.

A different approach to tackle the problem was taken in [3]. The proposed time-state control form (TSCF) is based on the fact that driftless systems are time scalable and decouples the original system into a state part and a time part. If the linear approximation of the state part is controllable, one can stabilize the system using traditional linear control methods. Time flow of the state part is directly controlled by the time part. This approach is appealing for its simplicity and also for the fact that the class of systems that can be transformed into the TSCF is obviously larger than the class of systems that can be transformed into the one generator chained form. To which extent is the class larger remains, however, an open problem.

We are interested in converting a driftless controllable nonholonomic system to the TSCF. In this paper, we first review the structure of the TSCF and then show the possibilities of conversion in a vector field setting. We give a methodology of transformation for the case a special vector field called the time axis vector field is known and also propose a complete transformation algorithm for systems that admit the time axis vector field as a constant vector field. After introducing a dual approach of treating nonholonomic systems based on Pfaffian systems, we briefly review the basic facts on dynamic feedback linearization. Then, search for systems that can be transformed into TSCF with controllable linearization of the state part. Connection to flat systems is investigated. We also obtain a new result on linearization by time scaling which directly generalizes the known facts on exact linearization of single input system on plane. An example of a flat system of dimension 5 with 3 inputs is given.

Notation. The following notation will be used through the paper: $\mathbb{R}$ denotes the field over the reals, δ_{ij} the *Kronecker product*. The *exterior derivative* of a k-form α is denoted by $\mathbf{d}\alpha$. A *differential* of a smooth function h is denoted by dh and for a one-form α and vector field g, the *interior product* is written as $\alpha(g)$. Wedge product is represented by $\wedge$. $\mathcal{L}_g h$ is the *Lie derivative* of a function h along the vector field g and $\mathcal{L}_g \alpha$ denotes the Lie derivative of a covector field α along g defined by [4] $\mathcal{L}_g \alpha = g^T \left(\frac{\partial \alpha^T}{\partial x}\right) + \alpha \frac{\partial g}{\partial x}$. For a map $\varphi : M \to N$, φ_* denotes the *push forward* $\varphi_* : T_p M \to T_{\varphi(p)} N$.

II. Time-state control form

Consider a driftless system given by

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m, \qquad x \in \mathbb{R}^n, u_i \in \mathbb{R}, \quad (1)$$

where $m < n$, $g_i(x)$ are smooth, mutually independent vector fields on $\mathbb{R}^n$. The control problem is to stabilize (1) to $x_0 \in \mathbb{R}^n$ using a feedback controller. We also assume the controllability Lie algebra $\mathfrak{C}(x)$ of (1) [4] spans $\mathbb{R}^n$ around x_0 which implies local controllability at x_0.

We are looking for a (static) feedback transformation

$$u = \sigma_1(x)v, \quad (2)$$

where $u = (u_1, \ldots, u_m), v \in \mathbb{R}^m$, and a diffeomorphic change of coordinates

$$z = \varphi(x), \quad \varphi(x) = (h_1(x), \ldots, h_n(x)) \quad (3)$$

such that (1) is transformed into the *first control form* (1st CF)

$$\dot{z} = \begin{bmatrix} \dot{\zeta} \\ \hdashline \dot{z}_n \end{bmatrix} = \hat{f}_1(\zeta)v_1 + \hat{f}_2(\zeta)v_2 + \cdots + \hat{f}_m(\zeta)v_m, \quad (4)$$

where $\zeta = (z_1, \ldots, z_{n-1})$ and it holds that all $\hat{f}_i$ are free of z_n. Moreover, we require that

$$\hat{f}_1(z_0) = \hat{f}_1(\zeta_0) = (0, \ldots, 0, 1)^T. \quad (5)$$

Once the system (1) is in the 1st CF, there always exists another input transformation $v = \sigma_2(\zeta)\mu$, $\mu \in \mathbb{R}^m$ such that after scaling by μ_1 we obtain the *time-state control form (TSCF*

$$\begin{aligned}\frac{d\zeta}{dz_n} &= \beta_1(\zeta) + \beta_2(\zeta)\mu_2/\mu_1 + \cdots + \beta_m(\zeta)\mu_m/\mu_1 \\ \dot{z}_n &= \mu_1\end{aligned} \tag{6}$$

and (5) renders $\beta_1(\zeta_0) = 0$. The z_n part of (6) is referred to as the *time state* (TS) and the subsystem having ζ as the state variable is called the *state part* (SP) and has an equilibrium at ζ_0.

If the linear approximation of SP is controllable, the system (1) can be stabilized by means of standard tools from linear control theory. The TS is increasing/decreasing at a constant rate and linear state feedback stabilizing SP is easily designed for both modes.

III. Transformation using vector fields

The requirement that the vector fields $\hat{f}_i(\zeta)$ do not contain the last coordinate z_n suggests existence of a vector field $e(x)$, such that the Lie bracket $[e(x), f_i(x)] = \varphi_*^{-1}[\hat{e}(\zeta), \hat{f}_i(\zeta)]|_{\zeta=\varphi(x)} = 0$, $i = 1, \dots, m$. Indeed, the existence of a time axis vector field satisfying one more property to ensure the TSCF will have an equilibrium at $(\zeta_0, z_{n0}) = \varphi(x_0)$ is the key idea of a necessary and sufficient condition stated below.

Theorem 1 (Sampei et al. [3]): The following conditions are equivalent:

A. (1) can be transformed to (6).

B. There exists a vector field $e(x)$ such that

$$\begin{aligned}[e(x), g_i(x)] &\in \Delta_g(x), \quad i = 1, \dots, m \\ e(x_0) &\in \Delta_g(x_0),\end{aligned}$$

where $\Delta_g(x) = \text{span}(g_1(x), \dots, g_m(x))$ is a distribution and the span is over all smooth real functions.

C. There exists a vector field $e(x)$ such that for

$$(f_1(x), \dots, f_m(x)) = (g_1(x), \dots, g_m(x))\sigma_1(x),$$

where $\sigma_1(x)$ of dimension $m \times m$ is nonsingular for all x, holds that

$$\begin{aligned}[e(x), f_i(x)] &= 0, \qquad i = 1, \dots, m \\ e(x_0) &= f_1(x_0).\end{aligned}$$

Proof: See [3]. ■

These conditions, although necessary and sufficient, are based on existence of another object, the vector field $e(x)$. We will call this vector field the *time axis vector field*. It not clear whether for a given system such vector field exists, neither is it clear how to find it besides solving a set of partial differential equations (PDE) given in item *B.* or *C.* of Theorem 1. Moreover, even assuming we have the vector field $e(x)$ available, there is no method to find the feedback transformation (2). Note that the coordinate transformation (3) can be found from Frobenius' theorem.

In order to remedy these shortages, we find the feedback (2) in case $e(x)$ is known and then note the problem is greatly simplified for systems for which $e(x)$ is *constant*.

We will need covector fields of certain properties. Let α_j, $j = 1, \dots, n-m$, and η_k, $k = 1, \dots, m$, denote linearly independent covector fields (one-forms) such that for $i = 1, \dots, m$ holds that $\alpha_j(g_i) = 0$ and $\eta_k(g_i) = \delta_{ki}$. Now, we can proceed to the following lemma.

Lemma 1: Let $f(x)$ be a smooth vector field on $\mathbb{R}^n$ that can be written as a linear combination of smooth vector fields $g_1(x), \dots, g_m(x)$:

$$f = (g_1, \dots, g_m)y, \qquad y = (y_1, \dots, y_m)^T. \tag{7}$$

Then, for any given smooth vector field $e(x)$, there always exists a solution $y(x)$ which satisfies

$$\mathcal{L}_e(\eta_i(f)) = (\mathcal{L}_e\eta_i)(f), \qquad i = 1, \dots, m. \tag{8}$$

The coefficient vector $y(x)$ is given by one of the m linearly independent fundamental solutions of the partial differential equation

$$\frac{\partial y(x)}{\partial x}e(x) = A(x)y(x), \tag{9}$$

where

$$A(x) = \begin{bmatrix} (\mathcal{L}_e\eta_1)(g_1) & \cdots & (\mathcal{L}_e\eta_1)(g_m) \\ (\mathcal{L}_e\eta_2)(g_1) & \cdots & (\mathcal{L}_e\eta_2)(g_m) \\ \vdots & & \vdots \\ (\mathcal{L}_e\eta_m)(g_1) & \cdots & (\mathcal{L}_e\eta_m)(g_m) \end{bmatrix}.$$

Proof: By definition, $\eta_i(g_j) = \delta_{ij}$ and thus

$$\eta_i(f)(x) = y_i(x), \quad (\mathcal{L}_e\eta_i)(f) = \sum_{k=1}^m y_k(\mathcal{L}_e\eta_i)(g_k).$$

Therefore, (8) becomes

$$\frac{\partial y_i}{\partial x}e = \sum_{k=1}^m y_k(\mathcal{L}_e\eta_i)(g_k),$$

and putting the equations together for all $i = 1, \dots, m$ yields (9).

Now, we show that the PDE (9) can be solved in time domain. First, note that the solutions $y(x(t))$ to a set of differential equations

$$\dot{x} = e(x), \qquad \dot{y} = A(x)y \tag{10}$$

provide the solution $y(x)$ to (9), since

$$\dot{y} = \frac{\partial y}{\partial x}\dot{x} = \frac{\partial y}{\partial x}e = Ay.$$

We will solve (10) using a coordinate change $z := \varphi(x)$ such that

$$\dot{z} = (\dot{\zeta}, \dot{z}_n) = \varphi_* e(x) = (0, \dots, 1).$$

Then, we have $\dot{z}_n = 1$ and $\zeta = c = \mathit{const}$, and from $x = \varphi^{-1}(z)$, the second equation in (10) becomes

$$\dot{y} = A(c, z_n)y. \tag{11}$$

Scaling(11) by $\dot{z}_n$ yields

$$\frac{dy}{dz_n} = A(c, z_n)y,$$

and y can be solved as a function of z_n and there are m fundamental solutions $y_1, \ldots, y_m$ for linearly independent initial conditions $y_1(0), \ldots, y_m(0)$. After deriving $y(z_n)$, z_n is replaced by a function of x which can be calculated from $z = \varphi(x)$. ■

Remark 1: If $e(x)$ is such that the distribution generated by the vector fields $g_i(x)$ is invariant under $e(x)$, then Lemma 1 is equivalent to the case when $y(x)$, defined in (7), is to be found such that $[e, f] = 0$ holds.

Hence we have obtained an algorithm for calculating the feedback transformation (2), provided the time axis vector field is known. Although there is another PDE to be solved, it was shown that the solution can be obtained in time domain setting.

Investigation of existence of $e(x)$ starts with a review of a well-known lemma:

Lemma 2: If a smooth distribution Δ is invariant under the vector field f, then the codistribution $\Omega = \Delta^\perp$ is also invariant under f. If a smooth codistribution Ω is invariant under the vector field f, then the distribution $\Delta = \Omega^\perp$ is also invariant under f.

Proof: See [4]. ■

From Theorem 1 and Lemma 2 directly follows the next observation.

Corollary 1: When e is a constant vector, checking the invariance of Δ_g under e reduces to checking either of the following two equivalent conditions

$$\frac{\partial g_i}{\partial x} e \in \Delta_g \quad \text{or} \quad \left(\frac{\partial \alpha_j^T}{\partial x} e\right)^T \in \Delta_g^\perp, \tag{12}$$

where $i = 1, \ldots, m$, $j = 1, \ldots, n-m$ and the covector fields α_j were defined above.

Remark 2: If, moreover, e can be found such that $e \in \Delta_g(0)$, we have the time axis vector field satisfying Theorem 1. For constant e, the matrix $A(x)$ in (9) becomes a zero matrix and hence the input transformation (2) becomes a linear combination with real *constant* coefficients.

The resulting algorithm for transformation of (1) to the 1st CF (4) is then given as follows:

Step 1. Check whether there exists a constant vector field e satisfying any of the conditions in (12). Then check if the candidate satisfies $e \in \Delta_g(x_0)$. If so, proceed to Step 2.

Step 2. Solve $f_1(x_0) = (g_1(x_0), \ldots, g_m(x_0))d_1(x_0)$ for $d_1(x_0)$. Then, complete $d_2(x_0), \ldots, d_m(x_0)$ to have a set of m independent vectors. It follows from Lemma 1 and Remark 2 that the feedback transformation (2) is given as $u = \delta_1(x)v = [d_1(x), \ldots, d_m(x)]v$. Use Frobenius' theorem to the find coordinate transformation (3) and transform (1) to (4). ◃

The algorithm above is based on the assumption that the system (1) admits a constant time axis vector field e. While this holds for some simple systems, the condition is not satisfied for all of practically interesting systems. For this reason, we turn to a different setup to find more about systems for which a general $e(x)$ exists.

IV. Differential form setting

Different point of view can bring more insight into the problem. We note that control systems can also be modeled by the use of one-forms instead of vector fields. This framework proved itself useful in nonholonomic motion planning and control or in (dynamic) exact linearization. We refer to [5] for monograph on exterior systems and to [6] for their use in control theory.

Let M be a smooth manifold of dimension n and let $\Omega^P(M)$ denote the set of smooth exterior p-forms on M. Then, we define $\Omega(M) = \oplus\Omega^P(M)$ to be the set of smooth exterior forms of all orders on M. An *exterior differential system* is given by an ideal $I \subset \Omega(M)$ that is closed under differentiation.

An exterior differential system of the form

$$\alpha^1 = \alpha^2 = \cdots = \alpha^s = 0, \tag{13}$$

where α^i are independent one-forms on manifold M is called a *Pfaffian system of codimension* $n - s$.

If $\{\alpha^i, \ldots, \alpha^n\}$ is a basis for $\Omega^1(M)$, then the set $\{\alpha^{s+1}, \ldots, \alpha^n\}$ is called the complement to the Pfaffian system (13). The one-forms $\alpha^{s+1}, \ldots, \alpha^n$ generate the algebraic ideal

$$I = \{\sigma \in \Omega(M) : \sigma \wedge \alpha^1 \wedge \cdots \wedge \alpha^s = 0\}.$$

An important role is played by the *independence condition* for the Pfaffian system, which is a one form τ which does not vanish on integral curves $c(t)$ of the system, that is $\tau(c(t))(c'(t)) \neq 0$. If τ is integrable, the Pfaffian system corresponds to a system of first order ordinary differential equations.

The first derived system of I is defined as

$$I^{(1)} = \{\omega \in I : \mathbf{d}\omega \equiv \ mod \ I\}.$$

When proceeding iteratively, one can construct the derived flag of I as a filtration

$$I = I^{(0)} \supset I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(N)},$$

which stops decreasing for some finite N, as dimension of I is finite. If $I^{(N)} = 0$, the system is called *completely nonholonomic*.

Control system $\dot{x} = f(x, u)$ on $\mathbb{R}^n \times \mathbb{R}^m$ can be regarded as a Pfaffian system on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$

$$I_f = \{dx^i - f^i(x, u)dt, \ i = 1, \ldots, n\}. \tag{14}$$

Clearly a completely nonholonomic system corresponds to a strongly accessible control system.

To a driftless control system (1) is naturally associated a Pfaffian system

$$I := \{g_q, \ldots, g_m\}^\perp \tag{15}$$

of dimensional $s = n - m$ and it holds that $I_f^{(1)} = I$. Recall that Chow's theorem asserts that a completely nonholonomic system is equivalent to a controllable driftless system.

On the other hand, for a Pfaffian system to correspond to a control system $\dot{x} = f(x, u)$, the ideal $\{I_f, \tau\}$ must be integrable. As expected, this is usually satisfied for $\tau = dt$.

By replacing a system of differential equations by a Pfaffian systems, one gains the tools and methods of coordinate independent differential geometry. But something is also lost: the sense of an independent variable. We are actually going to use this fact to our advantage. Indeed, when dealing with driftless systems in Pfaffian setting, one does not have to carry around neither input variables, nor the time, which simplifies the analysis.

a) Canonical forms.: First, we introduce a special structure of Pfaffian systems. If for a Pfaffian system I on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$ of codimension $m+1$ there exist coordinates z such that

$$I = \{dz_i^j - z_{i+1}^j dz^0,\ i = 1, \dots, s_j, j = 1, \dots, m\} \qquad (16)$$

we say that I is in *extended Goursat normal form*. This form is closely related to Brunovsky canonical form [7] known from linear systems theory, which is formed by m chains of s_j integrators

$$\omega_i^j := dy_i^j - y_{i+1}^j dt,\ i = 1, \dots, s_j, j = 1, \dots, m. \qquad (17)$$

Note that the two forms differ only by the independence condition.

b) Associated system.: Having two vector fields f and g on M that describe an affine control system $\dot{x} = f(x) + g(x)u$, one may construct an associated system on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ as

$$g_1 = \begin{bmatrix} 1 \\ f \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 \\ g \end{bmatrix}. \qquad (18)$$

Obviously, if the original affine system is feedback linearizable, i.e., it can be transformed to the Brunovsky canonical form (17) by a static feedback and coordinate change, then (18) can be transformed to the Goursat form. It can be shown that the converse is not true. For, by creating the associated system we gain the possibility to choose different independence condition τ.

c) Generator.: Having a collection of vector fields $g_1, \dots, g_m$ on M such that its controllability Lie algebra $\mathfrak{C}(q) = ad_{g_j}^k g_i,\ k = 0, 1, \dots$ spans the tangent space T_qM around the equilibrium for some j, we call the vector field g_j a *generator* for a driftless system (1). Here $ad_f^{k+1} g = [f, ad_f^k g]$ and $ad_f^0 g = g$. The generator may be a pointwise combination of the original vector fields. One proceeds similarly for multiple generators.

V. TSCF in Pfaffian systems

After introducing a new framework, we need a definition of the TSCF dual to (6). Since (6) can always be obtained from the 1st CF (4), we will give an alternative definition of the 1st CF:

Lemma 3 (1st CF in codistribution): The Pfaffian system $I = \{\alpha^1, \dots, \alpha^s\}$ on M, $\dim(M) = n$ corresponds to a driftless control system in the 1st CF (4) defined on $M\times\mathbb{R}^{n-s}$, if (and only if)

1) I maximally nonholonomic, and
2) the one-forms $\alpha^i = \sum_{k=1}^n a_k^i(x) dx^k$ are such that $a_k^i(x)$ are smooth functions and there exists j, $1 \le j \le n$ such that $\partial a_k^i / \partial x^j = 0$, $i = 1, \dots, n-m$, $k = 1, \dots, n$. Moreover,
3) $a_j^i(x_0)$, $i = 1, \dots, n-m$ and $a_j^i(x) \neq 0$, $\forall x \in M\setminus\{x_0\}$ for at least one $i = 1, \dots, n-m$.

Proof: The first condition is equivalent to controllability of the system (1). The next conditions directly follow from construction of an associated Pfaffian system as shown in (14) and (15). The second one requires that there exists a state x^j such that all vector fields dual to the Pfaffian system are independent of this state. The last condition is to ensure that the drift vector obtained by selecting x_j as the time state has an equilibrium at x_0 and is not identically zero. ∎

It is clear that any system in (extended) Goursat form satisfies the conditions of Lemma 3. Moreover, the TSCF then has an exactly linearized state part (hence it is controllable). Also note that the dx^j is the independence condition τ and thus belongs to the complement of I.

We also note that as well as the 1st CF (4) does not ensure that the linear approximation of the state part is controllable, the same statement holds for Lemma 3. This is clear from the following example.

Example 1: Consider a symmetrical affine system in the form

$$\begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3 \end{bmatrix} = E_3 u, \quad \begin{bmatrix} \dot{q}^4 \\ \dot{q}^5 \\ \dot{q}^6 \end{bmatrix} = \begin{bmatrix} q^2 & -q^1 & 0 \\ q^3 & 0 & -q^1 \\ 0 & q^3 & -q^2 \end{bmatrix} u,$$

where E is an identical matrix. This is an example of a system in two-generator chained form and it is easy to show that its Pfaffian system I given by

$$I = \{dq^4 - q^2 dq^1 + q^1 dq^2, dq^5 - q^3 dq^1 + q^1 dq^3, \\ dq^6 - q^3 dq^2 + q^2 dq^3\}$$

is completely nonholonomic. Applying the coordinate change $z := (q^1, q^2, q^3, q^1q^2 - q^4, q^1q^3 - q^5, q^2q^3 - q^6)$ we obtain $I = \{dz^4 - z^1 dz^2, dz^5 - z^1 dz^3, dz^6 - z^2 dz^3\}$ which satisfies conditions of Lemma 3 for $\tau = dx^j = dz^3$. The linear approximation of the state part is, however, uncontrollable. ◁

The system in the above example is not controllable using the usual control method developed for TSCF. It can be stabilized by switching between the generators, but that is out of scope of this paper. Since the simplicity of the original stabilization methods [3] is the main merit of the TSCF, we focus on systems that yield TSCF with controllable approximation of the state part. Then, we can look at relations with the structure of Goursat forms, which is of one generator, that is equal to a vector field associated to the independence condition. We conclude that the existence of a vector field g_j whose iterated Lie brackets generate the controllability distribution becomes the *necessary condition* for such TSCF.

When looking for a *sufficient condition*, we may start by analysis of systems according to dimension of their Pfaffian systems. The simplest case is covered in the following proposition.

Proposition 1: Any controllable driftless system underactuated by one control can be put into TSCF with controllable linear approximation of the state part.

Proof: Pfaffian system associated to a system underactuated by one control is given by $I = \alpha$, $\alpha \in \Omega^1(M)$. The rank r of α is defined by

$$(\mathbf{d}\alpha)^r \wedge \alpha \neq 0, \text{ and } (\mathbf{d}\alpha)^{r+1} \wedge \alpha = 0.$$

By Pfaff's theorem [6], there exists a coordinate chart on M such that in these coordinates

$$\alpha = dz^1 + z^2 dz^3 + \cdots + z^{2r} dz^{2r+1},$$

where dz^{2r+1} is such that $(\mathbf{d}\alpha)^r \wedge \alpha \wedge dz^{2r+1} = 0$. This ideal satisfies the conditions of Lemma 3 for any of dz^{2i+1}, $i = 1, \ldots, r$ as a generator and hence corresponds to the 1st CF. If we interpret that ideal I back in terms of vector fields, we get

$$\begin{aligned} \dot{z}^i &= u_i, \quad i = 2r+1, 2r, \ldots, 2 \\ \dot{z}^1 &= z^2 u_3 + z^4 u_5 + \cdots + z^{2r} u_{2r+1}. \end{aligned}$$

Using any of the coordinates z^{2i+1}, $i = 1, \ldots, r$ for the time state (TS), the system is in the TSCF. It is easy to check that the linear approximation of state part is controllable for any such choice. Moreover, as $G_0 = \mathrm{span}(\beta_1, \ldots, \beta_{2r-1})$ is not involutive for $r > 1$, the state part is not exactly linearizable [4] and the system can not be transformed into the chained form for $n > 3$. ■

This transformation requires solving one PDE at each step of generating new coordinate z^i. For $r > 1$ the system can not be transformed into extended Goursat form (i.e. the chained form) by static feedback and coordinate change. This is, however, not true if one considers a wider class of transformations. Indeed, it was shown by Charlet et. al. [8] that a system with n states and $n-1$ inputs is dynamic feedback linearizable.

d) Dynamic feedback.: Linearization of a smooth control system

$$\dot{x} = f(x, u) \tag{19}$$

defined on $X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ consists of finding a feedback

$$\begin{aligned} \dot{z} &= a(x, z, v) \\ u &= \sigma(x, z, v) \end{aligned} \tag{20}$$

defined on a subset $\tilde{X} \times Z \times V \subset X \times \mathbb{R}^r \times \mathbb{R}^q$ such that the closed loop system

$$\begin{aligned} \dot{x} &= f(x, \sigma(x, z, v)) \\ \dot{z} &= a(x, z, v) \end{aligned} \tag{21}$$

is diffeomorphic to a controllable linear system on $\tilde{X} \times Z$.

Note that in Pfaffian systems, *static* linearization can be written as a simple diffeomorphism φ (recall that we have lost the notion of independent variables, inputs, etc.), which is a map $X \times U \to \tilde{X} \times \tilde{U}$. This is no longer true for dynamic feedback, where $\dim(\tilde{X}) > \dim(X)$ and one has to use the term of dynamic immersion. The Pfaffian system I is then prolonged into a system J and the systems are equivalent if their trajectories are equal [9].

e) Differential flatness.: A control system (19) is called *differentially flat* if there exists a (dynamic) feedback (20), such that the system is diffeomorphic to Brunovsky canonical form (17). For driftless system, this condition is relaxed to diffeomorphism to extended Goursat form (16). The linearizing outputs are then referred to as flat outputs.

Proposition 2: Any symmetrical affine system (1) that is differentially flat can be transformed into TSCF with controllable state part.

Proof: For space reason, we only give a sketch here. By assumption, the system (1) can be transformed to an extended Goursat form by dynamic feedback (prolongation in Pfaffian systems). That implies existence of a single generator associated to the independence condition $\tau = dz^n$. This implies that the coordinate z^n is never prolonged and also that z^n does not appear in the extended Goursat form (only in a form of dz^n).

Now, we need to show that the coordinate z^n does not appear in the system after application of a coordinate transformation and static feedback transforming it to the 1st CF, so that all conditions of Lemma 3 are satisfied. This is shown by contradiction. Construct the state part of TSCF by choosing the time state as state corresponding to generator of the extended Goursat form and applying a construction inverse to that we showed when designing an associated system (18). Appearance of z^n in any of the vector fields f and g^i implies that we have obtained a time-variant system. Removing the time variance by a dynamic feedback, however, yields an uncontrollable system. This contradicts the assumption that the state part can be converted to Brunovsky canonical form (controllable linear system) in a time scale z^n.

To prove the controllability of the linear approximation of the state part, assume an affine system (state part of TSCF). By assumption, this system can be linearized in the z^n time scale by (possibly dynamic) feedback. Recall that the necessary condition for static feedback linearizability is controllability of the linear approximation around the origin. This was generalized in [8]: if a system $\dot{z} = f(z) + \sum_{i=1}^{m-1} g_i v_i$ is dynamic feedback linearizable, then its linear approximation at the origin $\dot{z} = \nabla_z f(0) z + \sum_{i=1}^{m-1} g_i(0) v_i$ is controllable. ■

Example 2: Consider a controllable system (1), where $x \in \mathbb{R}^5$ and $u \in \mathbb{R}^3$. It was shown in [10] that its Pfaffian system $I = \{\omega^1, \omega^2\}$ can always be written as

$$\omega^1 = dx^1 - a(x) dx^3 - x^5 dx^4, \quad \omega^2 = dx^2 - z^3 dz^4,$$

which can be converted to the Goursat form by prolongation $J := I + \{dx^3 - x^6 dx^4\}$ and coordinate change $z := (x^1, x^1, x^3, x^4, x^5 + a(x) x^6, x^6)$. Hence it is a flat system.

Obviously, the system satisfies Lemma 3 if $a_4 = \partial a / \partial x^4 = 0$. This may not be always the case and so we write the structure equations

$$\begin{aligned} \mathbf{d}\omega^1 &\equiv a_1 dx^3 \wedge \omega^1 \\ &\quad + a_2 dx^3 \wedge \omega^2 + (a_1 x^5 + a_2 x^3 + a_4) dx^3 \wedge dx^4 \\ &\quad + a_5 dx^3 \wedge dx^5 + dx^4 \wedge dx^5 \\ \mathbf{d}\omega^2 &\equiv dx^4 \wedge dx^3 \quad \mod \omega^1, \omega^2. \end{aligned}$$

For $a_5 = 0$ around the equilibrium, we have $\{I, dx^4\}$ integrable and, by Engel's theorem, I can be transformed to $I = \{dz^1 - z^2 dz^0, dz^3 - z^4 dz^0\}$, that is, a Goursat form with 2 towers.

Finally, assume for example $a(x) = x^4 x^5$ which means there exists no static feedback to transform the system to the Goursat form and the system is not in the 1st CF. The coordinate change $z := (x^1, x^2 - x^3x^4, -x^4, x^3, x^5)$ yields $I = \{dx^1 - x^5dx^3 - x^3x^5dx^4, dx^2 - x^3dx^4\}$ which is that of the 1st CF. ◁

From the above arguments follows an interesting corollary, which is a generalization of the well known fact that a locally accessible affine system on a plane with a single input is always linearizable.

Corollary 2: Any locally accessible affine system $\dot{x} = f(x) + g(x)u$ with single input and $x \in \mathbb{R}^3$ is exactly linearizable by time scaling.

Proof: First, construct the associated driftless system (18). It is easy to see that we can apply Engel's theorem to obtain Goursat form with one tower in four dimensions. Now, we only need to show that the independence condition of the Goursat form is not independent of time t. This can be shown by contradiction and it goes along the same lines as the proof of Proposition 2. ■

VI. Summary and Future Work

Time-state control form is an interesting approach to control of nonholonomic driftless systems. We were interested in conversion of given systems into the TSCF. This is not a simple task as in its full generality it requires solving a set of partial differential equations. This, even if some existence criteria were available is barely tractable for general systems.

By showing how to proceed if the time axis vector field is known we have given and algorithm to design a static feedback transformation for converting the system (1) to 1st control form. The time axis vector field is easy to find if there exists a constant one for system (1).

Using the machinery of Pfaffian systems we have shown that any differentially flat system can be transformed to TSCF with a controllable approximation of its state part, by static feedback and coordinate change. TSCF satisfying the controllability condition are of special interest as their stabilization is very intuitive and easy.

Last, we note that the condition in Proposition 2 is only sufficient. For, if it was a necessary one, it would imply that every system with controllable linear approximation can be converted to a linear systems by a (possibly dynamic)feedback and time scaling.

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