Minimax Linear-quadratic Control Problems for Descriptor Systems with Properly Stated Leading Term*

Galina A. Kurina

Voronezh State Forestry Academy, ul. Timirjazeva, 8, Voronezh, 394613, Russia
Tel: 7 0732 737148; Fax 7 0732 538610; E-mail: kurina@kma.vsu.ru

Abstract-The controls in the feedback form and the optimal cost are obtained for the different minimax linear-quadratic control problems by descriptor systems with properly stated leading term in a Hilbert space. For that purpose the operators, which are solutions of the operator Riccati equation, are used, these operators act in all state space, satisfy the special symmetry condition, depending on the operators from the state equation, and determine optimal controls with the help of the state variable.

Index Terms-Singular systems, feedback, linear-quadratic control, minimax control.

I. Introduction

For the last twenty years many works devoted to the study of optimal control problems by systems, the state equations of which are not resolved with respect to the derivative have been published (see, for example, the reviews [1], [2], the monograph [3] and the paper [4]). In the scientific literature such systems are frequently named as descriptor, implicit, singular, differential-algebraic, etc.

Linear differential-algebraic equations, when noninvertible operators can be before the derivative and before the unknown state variable which is after the derivative sign, were studied in [5]. These equations are named as equations with properly stated leading term. The equations of the last form arise in practice (see[6]).

In the present paper, the controls in the feedback form and the optimal cost are obtained for the following minimax linear-quadratic control problems by time-varying descriptor systems with properly stated leading term in a Hilbert space: the problem with the fixed left point and a free right point, the problem with the fixed points, the periodic problem and for the regulation problem with constant coefficients on an infinite interval.

For that purpose it is not necessary to select from the state equation an equation resolved with respect to the derivative as it was made in numerous works devoted to linear-quadratic control problems for time-invariant descriptor systems. It should be noted that the forms of relations, defining the controls in the feedback form, are identical both for singular operators, standing on the left-hand side of the state equation, and for nonsingular operators, that is very convenient in a research of singularly perturbed control problems. For the entry of the controls in the feedback form the operators are used which are solutions of the special operator Riccati equations, they act in all state space, satisfy the peculiar symmetry condition, depending on the coefficients from the

*The work was supported by Russian Fundamental Research Foundation under grant 02-01-00351.

state equation, and optimal controls are determined by these operators and the value of the state variable.

II. Problem with fixed left point and free right point

Let us consider the linear-quadratic cost functional

$$J(u,v) =< x(T), d > +\frac{1}{2} < x(T), Fx(T) > +$$

$$\frac{1}{2}\int_0^T (< x(t), W(t)x(t) > + < u(t), R(t)u(t) > -$$

$$< v(t), S(t)v(t) >)dt \qquad (1)$$

on trajectories of the linear system

$$A(t)(B(t)x(t))' = C(t)x(t) + D(t)u(t)+$$

$$G(t)v(t) + h(t), \qquad (2)$$

$$A(0)B(0)x(0) = z_0. \qquad (3)$$

Here $t \in [0, T], T > 0$ is fixed, $d, x(t) \in X, u(t) \in U, v(t) \in V; z_0, h(t) \in Z; X, Y, Z, U, V$ are real Hilbert spaces, $< \cdot, \cdot >$ means a scalar product in appropriate spaces, $F, W(t) \in L(X), R(t) \in L(U), S(t) \in L(V), A(t), \in L(Y, Z), B(t) \in L(X, Y), C(t) \in L(X, Z), D(t) \in L(U, Z), G(t) \in L(V, Z); W(t) = W^*(t), R(t) = R^*(t), S(t) = S^*(t)$ for all $t \in [0, T], V = V^*$, the exponent star with the notation of an operator denotes the adjoint operator, the operators $W(t), R(t), S(t), A(t), C(t), D(t), G(t)$ and the function $h(t)$ are continuous with respect to t, the operator $B(t)$ is supposed to depend smoothly on t, the operators $R(t)$ and $S(t)$ are positive and invertible for all $t \in [0, T]$.

Admissible controls $u(t)$ and $v(t)$ are continuous functions with values in U, V respectively, for which there is a solution of the problem (2), (3) (a solution of (2) is a continuous function x:*[0, T]*$\rightarrow X$ that has a continuously differentiable product *B(t)x(t)* and which satisfies (2) pointwise).

Remark 1. From the relation (3) it follows that $z_0 \in Im(A(0)B(0))$, that is for some $x_0 \in X$ the equality $z_0 = A(0)B(0)x_0$ should take place.

We will seek controls $u_*(t), v_*(t)$ in a feedback form, for which

$$J(u_*, v) = \min_u J(u, v), J(u, v_*) = \max_v J(u, v).$$

We name such pair of controls (u_*, v_*) as the optimal controls pair and the value of the functional (1) when $u(t) = u_*(t), v(t) = v_*(t)$ we denote by J_*, i.e.

$$J_* = J(u_*, v_*).$$

Theorem 1. If the operator $K(t) \in L(X, Z)$ is a solution of the differential operator equation

$$B^*(t)(A^*(t)K(t))' = -C^*(t)K(t) - K^*(t)C(t)+$$

$$K^*(t)A(t)(B(t))' + K^*(t)(D(t)R^{-1}(t)D^*(t)-$$

$$G(t)S^{-1}(t)G^*(t))K(t) - W(t) \qquad (4)$$

with the condition

$$B^*(T)A^*(T)K(T) = F, \qquad (5)$$

$\varphi(t) \in Z$ is a solution of the problem

$$B^*(t)(A^*(t)\varphi(t))' = -(C^*(t)-$$

$$K^*(t)(D(t)R^{-1}(t)D^*(t)-$$

$$G(t)S^{-1}(t)G^*(t))\varphi(t) - K^*(t)h(t), \qquad (6)$$

$$B^*(T)A^*(t)\varphi(T) = d, \qquad (7)$$

and $x_*(t)$ is a solution of the problem (2), (3) with the controls, defined by the formulas

$$u_*(t) = -R^{-1}(t)D^*(t)(K(t)x_*(t) + \varphi(t)),$$

$$v_*(t) = S^{-1}(t)G^*(t)(K(t)x_*(t) + \varphi(t)), \qquad (8)$$

then the equalities (8) define the optimal controls pair for the problem (1)-(3) in the feedback form and

$$J_* =< A(0)B(0)x(0), \varphi(0)+ \frac{1}{2}K(0)x(0) > +\frac{1}{2}\int_0^T < \varphi(t), 2h(t)- (D(t)R^{-1}(t)D^*(t) - G(t)S^{-1}(t)G^*(t))\varphi(t) > dt. \qquad (9)$$

Remark 2. From (4), (5) and the symmetry of the operators F,W(t),R(t), S(t) it follows that the operator B*(t)A* (t)K (t) is symmetric for every t∈[0, T], that is

$$B^*(t)A^*(t)K(t) = K^*(t)A(t)B(t). \qquad (10)$$

Proof of theorem 1. Taking into account the equality (10), it is not difficult to verify the validity of the relation

$$(< x(t), B^*(t)A^*(t)(\varphi(t) + \frac{1}{2}K(t)x(t)) >)' = < A(t)(B(t)x(t))', \varphi(t) + \frac{1}{2}K(t)x(t) > + < x(t), B^*(t)(A^*(t)\varphi(t))'+ \frac{1}{2}(B^*(t)(A^*(t)K(t))'x(t)+ K^*(t)A(t)(B(t)x(t))') > - \frac{1}{2} < x(t), (B^*(t))'A^*(t)K(t)x(t) > .$$

Using the last relation let us find the derivative of the function $< x(t), B^*(t)A^*(t)(\varphi(t) + 1/2K(t)x(t)) >$ in view of the expressions (2), (4), (6), (10). Fulfilling the simple transformations we have

$$(< x(t), B^*(t)A^*(t)(\varphi(t) + \frac{1}{2}K(t)x(t)) >)' = -\frac{1}{2}(< x(t), W(t)x(t) > + < u(t), R(t)u(t) > - < v(t), S(t)v(t) >) + \frac{1}{2} < \varphi(t), 2h(t)- (D(t)R^{-1}(t)D^*(t) - G(t)S^{-1}(t)G^*(t))\varphi(t) > + \frac{1}{2} < u(t) + R^{-1}(t)D^*(t)(K(t)x(t)+ \varphi(t)), R(t)(u(t)+ R^{-1}(t)D^*(t)(K(t)x(t) + \varphi(t))) > -\frac{1}{2} < v(t)- S^{-1}(t)G^*(t)(K(t)x(t) + \varphi(t)), S(t)(v(t)- S^{-1}(t)G^*(t)(K(t)x(t) + \varphi(t))) > . \qquad (11)$$

Integrating this equality on the segment $[0, T]$, by virtue of the relations (1), (5), (7) we obtain

$$J(u, v) =< A(0)B(0)x(0), \varphi(0)+\frac{1}{2}K(0)x(0) > + \frac{1}{2}\int_0^T < \varphi(t), 2h(t) - (D(t)R^{-1}(t)D^*(t)- G(t)S^{-1}(t)G^*(t))\varphi(t) > dt + \frac{1}{2}\int_0^T (< u(t)+ R^{-1}(t)D^*(t)(K(t)x(t) + \varphi(t)), R(t)(u(t)+ R^{-1}(t)D^*(t)(K(t)x(t) + \varphi(t))) > - < v(t)- S^{-1}(t)G^*(t)(K(t)x(t) + \varphi(t)), S(t)(v(t)- S^{-1}(t)G^*(t)(K(t)x(t) + \varphi(t))) >)dt. \qquad (12)$$

Let us show, that the magnitude $< A(0)B(0)x(0), \varphi(0) + 1/2K(0)x(0) >$ does not depend on a controls $u(t), v(t)$. Indeed by virtue of the remark 1 and the relations (3), (10) we have

$$< A(0)B(0)x(0), \varphi(0) + \frac{1}{2}K(0)x(0) >= < z_0, \varphi(0) > +\frac{1}{2} < x_0, K^*(0)z_0 > .$$

The choice of an optimal controls pair is obvious, if we use the obtained expression for *J(u,v)*. That is as the operators R(t), S(t) are positive, from (12) it follows that the optimal controls pair is determined by the formulas (8) and the value J_* is calculated by the formula (9).

Remark 3. Taking into account the remarks 1 and 2 we can write down the formula (9) for J_* in the form

$$J_* =< z_0, \varphi(0) > +\frac{1}{2} < x_0, K^*(0)z_0 > +$$

$$\frac{1}{2}\int_0^T < \varphi(t), 2h(t)-$$

$$(D(t)R^{-1}(t)D^*(t) - G(t)S^{-1}(t)G^*(t))\varphi(t) > dt.$$

III. Problem with fixed points

Now we deal with the minimax control problem when the functional is determined by the formula

$$J(u,v) = \frac{1}{2}\int_0^T (< x(t), W(t)x(t) > +$$

$$< u(t), R(t)u(t) > - < v(t), S(t)v(t) >)dt \quad (13)$$

and trajectories of the system (2) satisfy the boundary values

$$A(0)B(0)x(0) = z_0, A(T)B(T)x(T) = z_T. \quad (14)$$

Here admissible controls are continuous functions $u(t), v(t)$ with values in U, V respectively, for which there is a solution of the problem (2), (14).

Theorem 2. If the operator $K(t) \in L(X,Z)$ is a solution of the differential operator equation (4) under the symmetry condition (10), $\varphi(t) \in Z$ is a solution of the equation (6) and $x_*(t)$ is a solution of the problem (2), (14) with the controls, defined by the formulas (8), then the equalities (8) determine the optimal controls pair for the problem (13), (2), (14) in the feedback form and

$$J_* =< A(0)B(0)x(0), \varphi(0) + \frac{1}{2}K(0)x(0) > -$$

$$< A(T)B(T)x(T), \varphi(T) + \frac{1}{2}K(T)x(T) > +$$

$$\frac{1}{2}\int_0^T < \varphi(t), 2h(t) - (D(t)R^{-1}(t)D^*(t)-$$

$$G(t)S^{-1}(t)G^*(t))\varphi(t) > dt. \quad (15)$$

Proof. Integrating the equality (11) on the segment *[0,T]*, by virtue of the relation (13) we obtain

$$J(u,v) =< A(0)B(0)x(0), \varphi(0)+\frac{1}{2}K(0)x(0) > -$$

$$< A(T)B(T)x(T), \varphi(T) + \frac{1}{2}K(T)x(T) > +$$

$$\frac{1}{2}\int_0^T < \varphi(t), 2h(t) - (D(t)R^{-1}(t)D^*(t)-$$

$$G(t)S^{-1}(t)G^*(t))\varphi(t) > dt+$$

$$\frac{1}{2}\int_0^T (< u(t) + R^{-1}(t)D^*(t)(K(t)x(t)+$$

$$\varphi(t)), R(t)(u(t)+R^{-1}(t)D^*(t)(K(t)x(t)+\varphi(t)) > -$$

$$< v(t) - S^{-1}(t)G^*(t)(K(t)x(t)+\varphi(t)), S(t)(v(t)-$$

$$S^{-1}(t)G^*(t)(K(t)x(t) + \varphi(t))) >)dt. \quad (16)$$

We further establish that the term outside the integrals on the right-hand side of (16) is equal to

$$< z_0, \varphi(0) > +\frac{1}{2} < x_0, K^*(0)z_0 > -$$

$$< z_T, \varphi(T) > -\frac{1}{2} < x_T, K^*(T)z_T >,$$

where $z_T = A(T)B(T)x_T$, that is it does not depend on a controls $u(t), v(t)$, it depends on the boundary values (14) only.

As the operators $R(t)$ and $S(t)$ are positive, from (16) it follows that the optimal controls pair for the problem (13),(2),(14) is defined by the formulas (8) and J_* is calculated by the formula (15).

IV. Periodic problem

Let us consider the periodic minimax control problem with the functional (13) when trajectories of the system (2) satisfy the condition

$$x(0) = x(T). \tag{17}$$

Here in addition to the previous conditions we assume that all operators and $h(t)$ are T-periodic in t functions.

Admissible controls are continuous T-periodic in t functions with values in U, V respectively, for which there is a solution of the problem (2),(17).

Theorem 3. If the operator $K(t) \in L(X, Z)$ is a solution of the differential operator equation (4) under the conditions (10) and

$$K(0) = K(T), \tag{18}$$

$\varphi(t) \in Z$ is a solution of the equation (6) under the condition

$$\varphi(0) = \varphi(T), \tag{19}$$

and $x_*(t)$ is a solution of the problem (2), (17) with the controls, defined by the formulas (8), then the equalities (8) determine the optimal controls pair for the problem (13), (2), (17) and

$$J_* = \frac{1}{2}\int\limits_0^T < \varphi(t), 2h(t) - (D(t)R^{-1}(t)D^*(t) -$$

$$G(t)S^{-1}(t)G^*(t))\varphi(t) > dt.$$

For the proof of the theorem 3 it is necessary to integrate the equality (11) on the segment $[0, T]$ using the relations (13), (17), (18), (19).

V. Regulation problem on infinite interval

Now we consider the minimax control problem with constant coefficients and the functional

$$J(u, v) = \frac{1}{2}\int\limits_0^{+\infty} (< x(t), Wx(t) > +$$

$$< u(t), Ru(t) > - < v(t), Sv(t) >)dt \tag{20}$$

on trajectories of the system

$$(Bx(t))' = Cx(t) + Du(t) + Gv(t), \tag{21}$$

$$Bx(0) = y_0. \tag{22}$$

(Here it is assumed that $Z = Y$.)

Admissible controls are continuous functions $u(t), v(t)$ with values in U, V respectively, ensuring a finite value of the functional (20) and the relation

$$x(+\infty) = 0,$$

where $x(t)$ is a solution of the problem (21), (22) corresponding to the controls $u(t)$ and $v(t)$.

Theorem 4. If the invertible operators R and S are symmetric positive, the operator $K \in L(X, Y)$ is a solution of the algebraic operator Riccati equation

$$C^*K + K^*C - K^*(DR^{-1}D^* - GS^{-1}G^*)K + W = 0$$

under the condition

$$B^*K = K^*B, \tag{23}$$

$x_*(t)$ is a solution of the problem (21), (22) with the controls, defined by the formulas

$$u_*(t) = -R^{-1}D^*Kx_*(t), v_*(t) = S^{-1}G^*Kx_*(t), \tag{24}$$

and these controls are admissible, then the equalities (24) determine the optimal controls pair for the problem (20)-(22) and

$$J_* = \frac{1}{2} < x(0), B^* K x(0) > .$$

The proof of this theorem is analogous to the proof of the theorem 1.

Remark 4. The solvability of an algebraic operator Riccati equation under the symmetry condition of the form (23) was considered in [7]. The solvability of an operator differential Riccati equation unresolved with respect to the derivative was discussed in [8].

Remark 5. The statements, which are similar to the heorems 1-4, were obtained for the linear-quadratic optimal control problems by descriptor systems in a Hilbert space in [9].

References

[1] F.L. Lewis, A survey of linear singular systems. Circuits, Systems, and Signal Processing, vol.5, no.1, 1986, pp.3-36.

[2] G.A. Kurina, Singular perturbations of control problems with state equation unresolved with respect to derivative. Survey, Izvestija RAN. Tehnicheskaja kibernetika, no.4, 1992, pp.20-48, (in Russian).

[3] V.L. Mehrmann, The Autonomous Linear Quadratic Control Problem, Lecture Notes in Control and Information Sciences, vol.163, 1991.

[4] D.J. Bender and A.J. Laub, The linear-quadratic optimal regulator for descriptor systems, IEEE Trans. Autom. Control, AC-32, 1987, pp.672-688.

[5] R. März, Differential algebraic systems anew, Applied Numerical Mathematics, vol.42, 2002, pp.315-335.

[6] D. Estévez Schwarz and C. Tischendorf. Structural analysis of electric circuits and consequences for MNA, Int. J. Circ. Theor.Appl., vol.28, 2000, pp.131-162.

[7] G.A. Kurina, On regulating by descriptor system on infinite interval, Izvestija RAN. Tehnicheskaja kibernetika, no.6, 1993, pp.33-38, (in Russian).

[8] G.A. Kurina, On operator Riccati equation unresolved with respect to derivative, Differencial'nye uravnenija,vol. XXII, no.10, 1986, pp.1826-1829 (in Russian).

[9] G.A. Kurina, Feed-back control for time-varying descriptor systems, Systems Science, vol.26, no.3, 2000, pp.47-59.