STABILITY RESULTS FOR DISCRETE SYSTEMS CONTROLLED IN THE PRESENCE OF INTERRUPTED OBSERVATIONS

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Abstract: This paper presents and proves two sufficient conditions dealing with the uniform exponential stability of an unstable discrete linear time invariant model, rendered stable through appropriate feedback control, when interrupted observations of the plant state occur. The state is assumed to be available and directly measured by sensors. The first sufficient condition establishes a linear relation between the total time when at least one of the sensors' measure is not available (T_u), and the total time when all sensors' measures are available (T_a), bounding T_u in order to say the closed loop system is uniformly exponentially stable. The second sufficient condition establishes another linear bound between each "unavailability" time interval T_{u_i} and the previous "availability" time interval $T_{a_{i-1}}$, in order to state that the Euclidean norm of the plant state, at the end of each T_{u_i} interval, is a monotonic descent series.

Keywords: Detection and estimation; Bayesian methods; Discrete linear system; Uniform exponential stability; Interrupted observations.

1. INTRODUCTION

In recent years the mass advent of digital communication networks and systems has boosted Teleoperation integration in feedback control systems. Applications like unmanned vehicles (Hallberg et al., 1999) or Internet based real time control (Overstreet et al., 1999) provide significant examples raising, in turn, new problems.

This paper deals with one of such problems: if the communication channel through which feedback information passes is not completely reliable, sensors' measurements may not be available to the controller during some intervals of time ("unavailability" time T_u). In such a situation, one has to couple the controller with a block, hereafter called supervisor, which is able to discriminate between situations of signal "availability" and "unavailability", and to generate an estimate of the plant's state during this "unavailability" interval. Methods for detection and estimation for abruptly changing systems (Tugnait, 1982) can be applied in the problem considered here. For that purpose an algorithm based on Bayesian decision could be implemented, for example.

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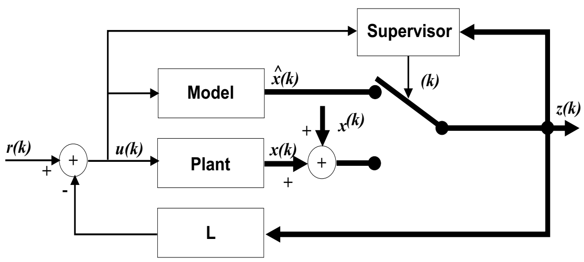

Fig. 1. Block diagram of a discrete control system with interrupted observations.

Fig.1 represents a block diagram of a discrete control system with interrupted observations. The supervisor decides whether the state $x(k)$ is being correctly measured by the sensors or not, and commands the switch signal $\sigma(k)$. During the time intervals in which the sensors do not provide a reliable measure of the actual plant state (it is admitted that the state coincides with the output) one possibility is to replace it by an estimate $\hat{x}(k)$ obtained from a plant model. This yields a loss of performance with respect to the ideal situation in which the sensors are always available, and may pose stability problems if the plant is open-loop unstable. Moreover if the plant model has modelling uncertainties δA, and if the state's measurements $x(k)$ are disturbed by $\delta_x(k)$. It is shown that, with the above described scheme, the controlled plant will be stable if the time interval, during which at least one of the sensors' measure is unavailable, is somehow "small", and that the Euclidean norm of the state $x(k)$, at the end of each T_{u_i} interval, is a monotonic descent series.

Somehow related with the problem of temporary sensor "unavailability" presented in this paper is the problem of data packet dropout, and the problem of network-induced delay, in Networked Control Systems (Zhang et al., 2001).

This paper is organized in five sections and two Appendices. After this introduction, section 2 refers a possibility for a supervisor algorithm, even though it is not studied in detail. Section 3 presents two theorems with sufficient conditions for uniform exponential stability of the controlled plant under the occurrence of interrupted observations. In section 4, simulation examples are presented in order to illustrate the two previous theorems. These simulation examples highlight the supervisor performance on the detection of the interrupted observations, and include an unstable discrete linear time invariant plant of second order. Section 5 draws conclusions and in Appendices A and B the two theorems are proved.

2. DETECTION AND ESTIMATION

One possibility for modelling sensors' measures interruptions is to consider that each observation (sensors' measures) $x(k)$ (it is assumed that $x_{min} \leq x(k) \leq x_{max}$), made at discrete time k, occurs under hypothesis H_0 with probability p_0, close to one, or under hypothesis H_1 with probability $1-p_0$, close to zero. Under hypothesis H_0 the observation is equal to the value of the state $x(k)$, added by zero mean white Gaussian noise of (constant) variance σ_e^2. Under hypothesis H_1 a measure interruption occurs. In this case the observation is no longer related to the state $x(k)$, but, instead, is given by a random variable $\eta(k)$ with a probability density function (p.d.f.) which is uniform in the range of measurement, from x_{min} to x_{max}.

According to a Bayesian approach, in order to detect that a given observation is actually noise, the probability of both hypothesis, given the observations, is computed. They are, then, compared, and if the probability associated with hypothesis H_1 is greater than the one associated with hypothesis H_0 it is decided that a measurement interruption has occurred. Under this decision, the observation is discarded and replaced by a forecast $\hat{x}(k)$ of the true value of $x(k)$, made from a plant model.

3. UNIFORM EXPONENTIAL STABILITY RESULTS

The plant of Fig.1 is described in the state-space form by

$$\begin{aligned} x(k+1) &= (A+\delta A)x(k) + bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$, accessible for direct measurement ($C = I_n$, where I_n is the identity matrix with dimension $n \times n$), $u \in \mathbb{R}$, A, δA and b of appropriate dimensions, and (A, b) controllable. Moreover, δA represents the modelling uncertainties. It is assumed the plant is time invariant and the controller is a state feedback of the signal $z(k)$, yielded by the sensor

$$u(k) = r(k) - Lz(k) \tag{2}$$

where L is a vector of feedback gains. $z(k) = x(k) + \delta_x(k)$ during the interval when all sensors are working properly, and $z(k) = \hat{x}(k)$ when measuring interruptions take place. The state disturbances being represented by $\delta_x(k)$.

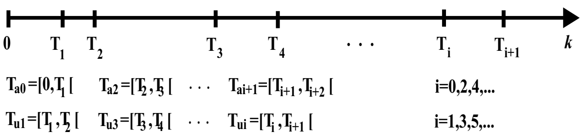

Fig. 2. Operation time line with "availability" intervals alternating with "unavailability" intervals.

Define the plant closed loop dynamics matrix as

$$A_{\delta L} := (A + \delta A) - bL = A_\delta - bL \tag{3}$$

and the model closed loop dynamics matrix as

$$A_L := A - bL \tag{4}$$

Furthermore, define the plant open and closed loop transition matrices

$$\Phi_\delta(k, k_0) := A_\delta^{k-k_0} \tag{5}$$

$$\Phi_{\delta L}(k, k_0) := A_{\delta L}^{k-k_0} \tag{6}$$

and the model open and closed loop transition matrices

$$\Phi(k, k_0) := A^{k-k_0} \tag{7}$$

$$\Phi_L(k, k_0) := A_L^{k-k_0} \tag{8}$$

Consider the time line of operation divided in alternate intervals where all sensors operate correctly (T_{a_i}, with $i = 0, 2, 4, ...$), and where, at least, one of them fails (T_{u_i}, with $i = 1, 3, 5, ...$) being replaced by the model estimate (see Fig.2). Note that the index i does not represent discrete instants of time, but is rather used to enumerate both the "availability" and the "unavailability" intervals. Let k_0 denote the beginning of one such intervals. It is assumed that the first interval corresponds to an "availability" interval, and that the intervals are open at their end. The sate $\hat{x}(k)$ of the model is made equal to the last available observation of the state $x(k)$ when an interrupted observation occurs.

Theorem 1 *Consider the closed loop system of Fig.1 with the unstable model in open loop (bounded by $\|\Phi(k, k_0)\| \leq \delta' \beta^{k-k_0}$, with $\delta' \geq 1$ a finite positive constant, and $\beta > 1$), rendered stable, through proper design of L, in closed loop ($\|\Phi_L(k, k_0)\| \leq \gamma \lambda^{k-k_0}$, with $\gamma \geq 1$ a finite positive constant, and $0 \leq \lambda < 1$). Consider, also, that $\xi := \|b\|$, $\varphi := \|L\|$, and that model uncertainties are bounded ($\|\delta A\| \leq \Delta$). The system with initial condition $x(0) = x_0$ is uniformly exponentially stable provided that the total "unavailability" time T_u, up to discrete time k inside the "unavailability" interval T_{u_i}, is*

$$T_u < -\frac{i+1}{2} \frac{\log\left(\delta'\left(1 + \frac{\xi\varphi\gamma}{\beta+\delta'\Delta-\lambda}\right)\gamma\right)}{\log(\beta + \delta'\Delta)} - T_a \frac{\log(\lambda + \gamma\Delta)}{\log(\beta + \delta'\Delta)}$$

and

$$\Delta < \frac{1-\lambda}{\gamma}$$

Theorem 2 *Under the premisses of Theorem 1, $\|x(T_{i+1} - 1)\|$, for $i = 1, 3, 5, ...$, is a monotonic descent series provided that the "unavailability" interval T_{u_i} is*

$$T_{u_i} < -\frac{\log\left(\delta'\left(1 + \frac{\xi\varphi\gamma}{\beta+\delta'\Delta-\lambda}\right)\gamma\right)}{\log(\beta + \delta'\Delta)} - T_{a_{i-1}} \frac{\log(\lambda + \gamma\Delta)}{\log(\beta + \delta'\Delta)}$$

and

$$\Delta < \frac{1-\lambda}{\gamma}$$

A proof of these two theorems is presented in the Appendices.

Note that for time-invariant systems the adjective "uniform" is superfluous, and that for linear time-invariant equations exponential stability is usually understood as asymptotic stability. Nevertheless, the exact terminology will be kept throughout this paper.

4. SIMULATION EXAMPLES

Simulation examples are presented in order to permit some critical comments on the two theorems, and are illustrated by an unstable discrete linear time invariant plant of second order. All the simulations are in discrete time with a sample time of 1*second*, and refer to the discrete control system of Fig.1.

Example 1 A second order system with open loop poles in $z_{o1} = 1.5$ and $z_{o2} = 0.95$, closed loop poles in $z_{c1} = 0.9$ and $z_{c2} = 0.9$, $\Delta = 1.4e-4$, $\gamma = 15$, $\lambda = 0.95$, $\delta' = 5$, $\beta = 1.5$, $\xi = 1$, and $\varphi = 0.89$ is considered, originating $\alpha = 5$, and $\mu = 1.5$. Clearly, the theorems' conditions are respected for $T_{u_1} = 5$ and $T_{a_0} = 200$. Fig.3 shows the time evolution of $log||x(k)+\delta_x(k)||$, with $\delta_x(k)$ a Gaussian random variable with variance $\sigma_x = 1e-6$, and $log(||z(k)||)$ (top), and the "unavailability" interval T_{u_1}, $\sigma = 1$ (bottom). Fig.4 highlights the differences between $log||x(k) + \delta_x(k)||$

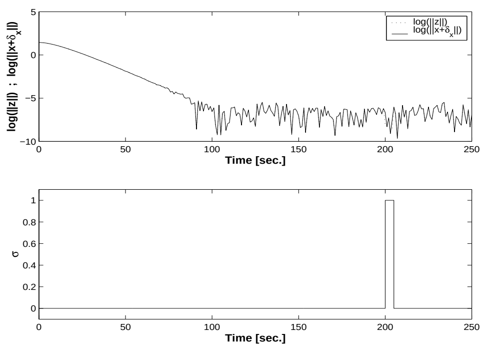

Fig. 3. $log||x(k)+\delta_x(k)||$ and $log||z(k)||$ for second order system verifying the theorems (top). T_{u_1} interval, $\sigma = 1$ (bottom).

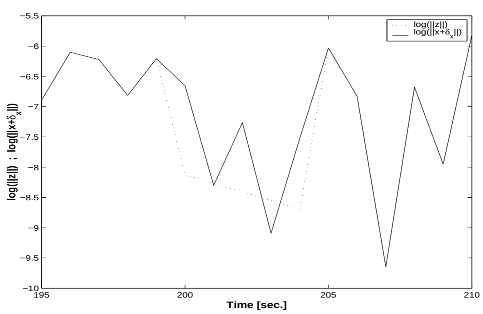

Fig. 4. Detail of Fig.3.

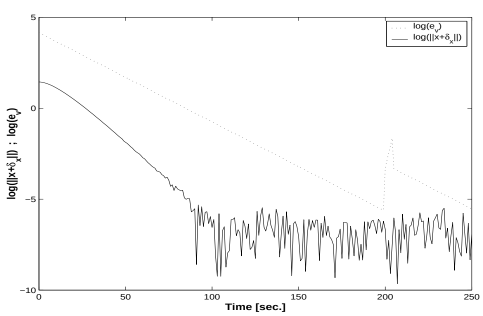

Fig. 5. $log||x(k)+\delta_x(k)||$ and $log(e_v(k))$ for second order system.

and $log(||z(k)||)$ during the "unavailability" interval. Fig.5 represents the logarithm of the upper bound ($e_v(k)$) given by (A.7) and (A.9) for each "availability" and "unavailability" interval, respectively, and $log||x(k)+\delta_x(k)||$. In particular, from this last figure it is possible to conclude of the conservativeness of the sufficient conditions (note that it is the $log(||.||)$ being represented). The next example will highlight the sufficient character of the theorems' conditions.

Example 2 The same system as in the previous example, with $\sigma_x = 2e-3$, but now the theorems' conditions are violated in the sense that the time relations between "availability" and "unavailability" are not respected. Nevertheless, it can be seen

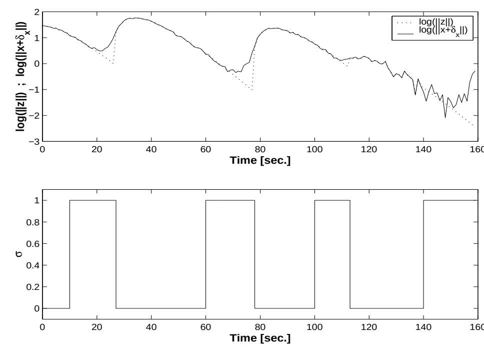

Fig. 6. $log||x(k)+\delta_x(k)||$ and $log||z(k)||$ for second order system violating the sufficient conditions (top). T_{u_i} intervals, $\sigma = 1$ (bottom).

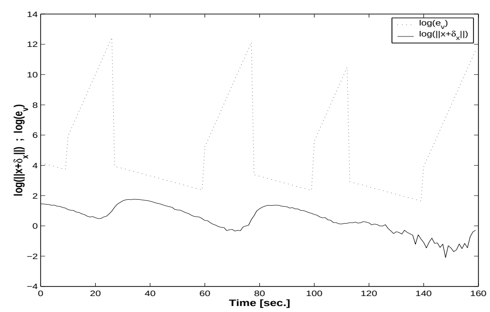

Fig. 7. $log||x(k)+\delta_x(k)||$ and $log(e_v(k))$ for second order system.

(Fig.6) that the system as uniform exponential stability and that $\|x(T_{i+1}-1)\|$, for $i = 1, 3, 5, \ldots$, is a monotonic descent series. Fig.6 and Fig.7 are equivalent to Fig.3 and Fig.5 in terms of signals represented.

It is interesting to note that in a recent work (Zhang et al., 2001), a similar conservative theoretical result regarding uniform exponential stability is reported, showing that longer intervals of "unavailability" can be reached in practice and that these theoretical results might be too conservative for practical purposes.

5. CONCLUSIONS

The paper presents and proves two sufficient conditions that allow a time analysis of sensor "unavailability" (interrupted observations) intervals, bounding these intervals in order to state that the plant of the system represented in Fig.1, when controlled in closed loop, is uniformly exponentially stable. These results are proved under the existence of modelling uncertainties and plant state disturbances, and for an exponentially bounded unstable plant. They are particularly related with practical situations of Teleoperation, when communication channels, through which feedback information passes, are not completely reliable and sensor measurements may not be available to the controller during some intervals of time, raising systems stability problems.

The paper includes illustrative simulations with a second order linear system (open loop unstable) and shows graphically the application of the two theorems, highlighting their sufficient condition conservative character.

6. REFERENCES

Halberg, E., I. Kaminer, A. Pascoal (1999). Development of a flight test system for unmanned air vehicle. *IEEE Control Systems*, 19(1), pp. 55-65.

Overstreet, J.W., A. Tzes (1999). An Internet-based real-time control engineering laboratory. *IEEE Control Systems*, 19(5), pp. 19-34.

Rugh, W.J. (1996). *Linear system theory.* Second edition, Prentice Hall, pp. 452-455.

Tugnait, J. (1982). Detection and estimation for abruptly changing systems. *Automatica*, 18, pp. 607-615.

Zhang, Wei, Michael S. Branicky, Stephen M. Phillips (2001). Stability of networked control systems. *IEEE Control Systems*, 21(1), pp. 84-99.

Appendix A

Throughout the Appendix the term "stability" will be used with the meaning of "uniform exponential stability", and the matrices norms are the ones induced by the Euclidean norm of vectors, being given by their largest singular value.

Consider the discrete time line represented in Fig.2. The intervals where the sensors yield correct measures are designated as T_{a_i}, with $i = 0, 2, 4, ...$, and the intervals where the observations are interrupted are designated as T_{u_i}, with $i = 1, 3, 5, ...$ Let the discrete time instant k_0 denote the beginning of a generic interval.

During the time intervals T_{a_i} the plant state $x(k)$ evolves according to

$$\begin{aligned} x(k) =& \Phi_{\delta L}(k,k_0)[z(k_0) - \delta_x(k_0)] - \\ & \sum_{j=k_0}^{k-1} \Phi_{\delta L}(k,j+1)b[L\delta_x(j) - r(j)] \end{aligned} \quad \text{(A.1)}$$

On the other hand, during the time intervals T_{u_i} the model state $\hat{x}(k)$ evolves according to

$$\hat{x}(k) = \Phi_L(k,k_0)z(k_0) + \sum_{j=k_0}^{k-1} \Phi_L(k,j+1)br(j) \quad \text{(A.2)}$$

it is also known that

$$\begin{aligned} x(k) =& \Phi_\delta(k,k_0)x(k_0) - \\ & \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1)b[L\hat{x}(j) - r(j)] \end{aligned} \quad \text{(A.3)}$$

Replacing (A.2) in (A.3), the evolution of the plant state $x(k)$ is

$$\begin{aligned} x(k) =& \\ & \left(\Phi_\delta(k,k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1)bL\Phi_L(j,k_0)\right)z(k_0) - \\ & \Phi_\delta(k,k_0)\delta_x(k_0) - \\ & \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1)b\left(\sum_{i=k_0}^{j-1} L\Phi_L(j,i+1)br(i) - r(j)\right) \end{aligned} \quad \text{(A.4)}$$

Both $r(k)$ and $\delta_x(k)$ are seen as inputs to the overall closed loop system and they do not affect the dynamics. The stability analysis is conducted only with the terms that relate $x(k)$ with $z(k_0)$ in (A.1) and (A.4). For bounded model uncertainties ($\|\delta A\| \leq \Delta$) and considering $\|\Phi(k,k_0)\| \leq \delta' \beta^{k-k_0}$, with $\beta > 1$ (this corresponds to assume an unfavorable situation), it can be proved (Rugh, 1996) that $\|\Phi_\delta(k,k_0)\| \leq \alpha\mu^{k-k_0}$, with $\mu > 1$, $\alpha = \delta'$, and $\mu = \beta + \delta'\Delta$. These means, as expected, that if the model dynamics are open loop unstable then also the plant dynamics will be open loop unstable for any bound $\Delta \geq 0$. A similar proof (Rugh, 1996) can be given for the stability of the plant in closed loop since the model in closed loop is stable, $\|\Phi_L(k,k_0)\| \leq \gamma\lambda^{k-k_0}$, with $0 \leq \lambda < 1$. Therefore, if the model uncertainties are bounded ($\|\delta A\| \leq \Delta$), then $\|\Phi_{\delta L}(k,k_0)\| \leq \gamma(\lambda + \gamma\Delta)^{k-k_0}$, with $0 \leq \Delta < (1-\lambda)/\gamma$.

An upper bound for each of the following expressions must be found

$$\|x(k)\| = \|\Phi_{\delta L}(k,k_0)z(k_0)\| \quad \text{(A.5)}$$

$$\begin{aligned} \|x(k)\| =& \\ & \left\|\left(\Phi_\delta(k,k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1)bL\Phi_L(j,k_0)\right)z(k_0)\right\| \end{aligned} \quad \text{(A.6)}$$

Starting with (A.5)

$$\begin{aligned} \|x(k)\| &\leq \|\Phi_{\delta L}(k,k_0)\| \cdot \|z(k_0)\| \\ &\leq \gamma(\lambda + \gamma\Delta)^{k-k_0}\|z(k_0)\| \end{aligned} \quad \text{(A.7)}$$

and for (A.6)

$$\|x(k)\| \leq \Bigg\| \Phi_\delta(k,k_0) - \sum_{j=k_0}^{k-1} \Phi_\delta(k,j+1) bL \Phi_L(j,k_0) \Bigg\| \cdot \|z(k_0)\| \leq \left(\alpha\mu^{k-k_0} + \sum_{j=k_0}^{k-1} \alpha\mu^{k-j-1}\xi\varphi\gamma\lambda^{j-k_0} \right) \cdot \|z(k_0)\| \quad \text{(A.8)}$$

After some calculations

$$\|x(k)\| \leq \alpha\mu^{k-k_0} \left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right) \|z(k_0)\| \quad \text{(A.9)}$$

Consider now that discrete time k is inside $T_{u_i} = [T_i, T_{i+1} - 1]$. Through the terms in (A.1) and (A.4) that relate $x(k)$ with $z(k_0)$, the plant state's $x(k)$ relation with the initial condition x_0, at this time instant, is given by

$$x(k) = \left(\Phi_\delta(k,T_i) - \sum_{j=T_i}^{k-1} \Phi_\delta(k,j+1) bL \Phi_L(j,T_i) \right) \cdot \Phi_{\delta L}(T_i - 1, T_{i-1}) \ldots \left(\Phi_\delta(T_2 - 1, T_1) - \sum_{j=T_1}^{T_2-2} \Phi_\delta(T_2 - 1, j+1) bL \Phi_L(j,T_1) \right) \cdot \Phi_{\delta L}(T_1 - 1, 0) \cdot z(0) \quad \text{(A.10)}$$

Therefore

$$\|x(k)\| \leq \Bigg\| \left(\Phi_\delta(k,T_i) - \sum_{j=T_i}^{k-1} \Phi_\delta(k,j+1) bL \Phi_L(j,T_i) \right) \cdot \Phi_{\delta L}(T_i - 1, T_{i-1}) \ldots \left(\Phi_\delta(T_2 - 1, T_1) - \sum_{j=T_1}^{T_2-2} \Phi_\delta(T_2 - 1, j+1) bL \Phi_L(j,T_1) \right) \cdot \Phi_{\delta L}(T_1 - 1, 0) \Bigg\| \cdot \|x_0 + \delta_x(0)\| \quad \text{(A.11)}$$

An upper bound for (A.11) is obtained through the use of (A.7) and (A.9)

$$\alpha\left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right) \mu^{k-T_i} \cdot \gamma(\lambda + \gamma\Delta)^{(T_i - 1) - T_{i-1}} \ldots \alpha\left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right) \mu^{(T_2-1)-T_1} \cdot \gamma(\lambda + \gamma\Delta)^{T_1 - 1} = \left[\alpha\left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right)\gamma\right]^{\frac{i+1}{2}} \mu^{T_u} \cdot (\lambda + \gamma\Delta)^{T_a} \quad \text{(A.12)}$$

In order for the controlled system to be stable, it must be

$$\left[\alpha\left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right)\gamma\right]^{\frac{i+1}{2}} \mu^{T_u} (\lambda + \gamma\Delta)^{T_a} < 1 \quad \text{(A.13)}$$

which results in (recall that $\alpha = \delta'$ and $\mu = \beta + \delta'\Delta$)

$$T_u < -\frac{i+1}{2} \frac{\log\left(\delta'\left(1 + \frac{\xi\varphi\gamma}{\beta + \delta'\Delta - \lambda}\right)\gamma\right)}{\log(\beta + \delta'\Delta)} - T_a \frac{\log(\lambda + \gamma\Delta)}{\log(\beta + \delta'\Delta)} \quad \text{(A.14)}$$

q.e.d.

Note that since $(\beta + \delta'\Delta) > 1$ and $0 \leq (\lambda + \gamma\Delta) < 1$, then T_u has a crescent linear relation with T_a.

Appendix B

Consider the Euclidean norm of $x(k)$ at discrete times $k = T_{i+1} - 1$, and $k = T_{i-1} - 1$, at the end of the “unavailability” intervals T_{u_i} and $T_{u_{i-2}}$, respectively. In order for $\|x(T_{i+1} - 1)\|$, for $i = 1, 3, 5, ...$, to be a monotonic descent series, it should verify

$$\frac{\|x(T_{i+1} - 1)\|}{\|x(T_{i-1} - 1)\|} < 1 \quad \text{(B.1)}$$

or, equivalently

$$\Bigg\| \left(\Phi_\delta(T_{i+1} - 1, T_i) - \sum_{j=T_i}^{T_{i+1}-2} \Phi_\delta(T_{i+1} - 1, j+1) bL \Phi_L(j,T_i) \right) \cdot \Phi_{\delta L}(T_i - 1, T_{i-1}) \Bigg\| < 1 \quad \text{(B.2)}$$

Finally

$$\alpha\left(1 + \frac{\xi\varphi\gamma}{\mu - \lambda}\right) \mu^{T_{u_i}} \cdot \gamma(\lambda + \gamma\Delta)^{T_{a_{i-1}}} < 1 \quad \text{(B.3)}$$

and (recall that $\alpha = \delta'$ and $\mu = \beta + \delta'\Delta$)

$$T_{u_i} < -\frac{\log\left(\delta'\left(1 + \frac{\xi\varphi\gamma}{\beta + \delta'\Delta - \lambda}\right)\gamma\right)}{\log(\beta + \delta'\Delta)} - T_{a_{i-1}} \frac{\log(\lambda + \gamma\Delta)}{\log(\beta + \delta'\Delta)} \quad \text{(B.4)}$$

q.e.d.