The Inventory Problem: A Dynamic Programming Approach.

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Abstract— **This paper concerns an example from inventory control which is studied in detail. The presence of an end-point constraint causes the value function to be discontinuous. Optimal controls are derived and an expression for the lower semi-continuous value function is given. This is confirmed to be the value function as it is shown to satisfy a set of generalised Bensoussan-Lions type quasi-variational inequalities, suitably interpreted for non-differentiable, extended valued functions.**

Keywords— **Optimal Control, Operations Research, Dynamic Programming, Impulse Control, Hybrid Control.**

I. Introduction

THIS paper concerns a generalisation of the inventory problem

$$(Q_{0,x_0}) \begin{cases} \text{Minimise } \int_0^T \tilde{L}(t,x(t))dt + N(v) \\ \text{subject to} \\ dx(t)/dt \ = \ \tilde{f}(t,x(t)) + dv(t)/dt\,, \\ x(0) \ = \ x_0\,, \end{cases}$$

investigated by Aubin [1], in which the differential equation $\dot{x} = \tilde{f}(t,x)$ describes the evolution of a vector valued stock. Impulse control action can be applied instantly to replenish stock at discrete times. The jump set $\mathcal{D}$ is taken to be $\mathcal{D} = \Re^n_+$ (the positive orthant of $\Re^n$). Control action incurs a unit transaction cost which is independent of the size of the stock increase. The cost function is chosen to penalise deviations from the vector of desired stock levels over the time horizon $[0,T]$.

Bensoussan and Lions [5] established the link between the value functions for impulse-controlled systems and certain optimality conditions known as the Bensoussan-Lions Quasi-Variational Inequalities (QVI) [5]. Subsequent research in a deterministic framework allowed for both 'ordinary' and impulse control action, as in this paper; the goal was to characterise value functions as unique continuous uniformly bounded viscosity solutions of Bensoussan-Lions type quasi-variational inequalities (QVI)'s (See [4], [6] and [10]. These references concern infinite time horizon problems, with discounted cost, not the finite time interval problem of this paper.)

A distinctive feature of our formulation of the generalised inventory problem is that it allows an endpoint constraint on state trajectories. In consequence, we can expect the value function to be discontinuous and even to have domain a strict subset of the 'initial conditions' space. Viability theory can be used to link value functions and generalised lower semi-continuous solutions to (QVI), even in this general setting. Analytical tools have been developed for this purpose by Aubin and his co-workers (see [2] and [3]).

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Necessary and sufficient conditions for optimality are stated for cases when the value function is lower semi-continuous and bounded below.

The emphasis in this paper is not so much on new optimality conditions (indeed the key characterisation of the value function for hybrid systems is easily derived, using methods of Aubin [3] and Frankowska [7]), rather it is on the solution to a specific inventory problem, illustrating the use of the analytic machinery. The value function is extended valued and fails to be continuously differentiable on the interior of its effective domain. Our analysis provides a candidate for the value function, and confirms that it is so by showing that it is the generalised, lower semi-continuous solution to (QVI).

II. The Generalised Inventory Problem

The generalised inventory problem is as follows:

$$(P_{0,x_0}) \begin{cases} \text{Minimise } \int_0^T L(t,x(t),u(t))dt + g(x(T)) + C(v) \\ \text{subject to} \\ dx(t)/dt \ = \ f(t,x(t),u(t)) + dv(t)/dt \quad \text{a.e. } t \in [0,T]\,, \\ u(t) \ \in \ \Omega \quad \text{a.e. } t \in [0,T]\,, \\ x(t_0) \ = \ x_0\,, \end{cases}$$

the data for which comprise a non-negative number T, functions $L : \Re \times \Re^n \times \Re^m \to \Re$, $f : \Re \times \Re^n \times \Re^m \to \Re^n$ and $d : \Re^n \to \Re$, an extended valued function $g : \Re^n \to \Re \cup \{+\infty\}$, sets $\Omega \subset \Re^m$ and $\mathcal{D} \subset \Re^n$ and a point $x_0 \in \Re^n$.

A control policy (u,v) on the interval $[a,b] \subset [0,T]$ comprises a measurable function $u : [a,b] \to \Re^m$ satisfying

$$u(t) \in \Omega \qquad \text{a.e. } t \in [a,b]$$

and an *impulse control* v

$$v \ = \ \{t_1,\ldots,t_{N(v)},\xi_1,\ldots,\xi_{N(v)}\}$$

described by the number of impulses, a non-negative integer $N(v)$, the impulse times $t_1,\ldots,t_{N(v)}$ which are real numbers such that $a \le t_1 \le \cdots \le t_{N(v)} \le b$, and the impulses $\xi_1,\ldots,\xi_{N(v)}$ at these times, which are vectors in $\Re^n$ such that $\xi_i \in \mathcal{D}$ for $i = 1,\ldots,N(v)$. Notice that we allow jump times to be coincident, since it may be favourable to execute a large jump, which can be implemented as a sum of jumps in the *jump set* $\mathcal{D}$.

We define a state trajectory corresponding to (u,v) to be a piecewise Lipschitz continuous function $y : [a,b] \to \Re^n$, continuous from the right on (a, b), satisfying

$$\dot{x}(t) \;=\; f(t, x(t), u(t)) \qquad \text{a.e. } t \in [a, b],$$

whose points of discontinuity are contained in the set $\{t_1, \ldots, t_{N(v)}\}$ and are such that

$$x(t^+) - x(t^-) \;=\; \sum_{\{j : t_j = t\}} \xi_j \,.$$

for all $t \in \{t_1, \ldots, t_{N(v)}\}$. Here, $x(t^+)$ and $x(t^-)$ denote the limit from the right and left respectively. (We interpret $t^- = a$ if $t = a$ and $t^+ = b$ if $t = b$.) A process (y, u, v) on $[a, b]$ comprises a control policy (u, v) on $[a, b]$ and an associated state trajectory y. The underlying time interval, for a control policy, state trajectory, etc., is taken to be $[0, T]$, if not otherwise specified.

Accordingly, the evolution of the state between jump times is governed by a differential equation with conventional control term u, which we are free to choose. The evolution of the system can also be controlled, however, by applying a finite number of impulses over the relevant time interval, each of which causes a jump in the state variable.

The optimal control problem is to minimise the cost function

$$J_{0,x_0}(u, v) \;=\; \int_0^T L(t, x(t), u(t))dt + g(x(T)) + C(v)$$

over control policies (u, v). Here x is the state trajectory corresponding to (u, v), for which $x(0) = x_0$. The term $C(v)$ on the right side denotes the *cost* of the impulse control, namely

$$C(v) \;=\; \sum_{i=1}^{N(v)} d(\xi_i)\,.$$

A control policy for $(P_{0,t})$ which minimises the cost is called an optimal control policy. An associated process is called an optimal process.

III. Characterisation of the Value Function

Consider the generalised inventory problem (P_{0,x_0}). We shall invoke the following hypotheses, in which $\tilde{f}(t, x, u) = (f(t, x, u), L(t, x, u))$.

(H1): Ω is a compact set,

(H2): $\tilde{f}$ is continuous and there exists $k > 0$, $c > 0$ such that

$$\left|\tilde{f}(t, x, u) - \tilde{f}(t, x', u)\right| \le k|x - x'| \quad \text{and} \quad |\tilde{f}(t, x, u)| \le c$$

for all $t \in [0, T]$, $x, x' \in \Re^n$, $u \in \Omega$,

(H3): L is bounded below on $[0, T] \times \Re^n \times R^m$ and, for each $(t, x) \in \Re \times \Re^n$, the set

$$\big\{(v, \alpha) \in \Re^n \times \Re \,:\, v = f(t, x, u),\ \alpha \ge L(t, x, u) \quad \text{for some } u \in \Omega\big\}$$

is convex,

(H4): d is continuous and there exists $r > 0$ such that $d(x) > r$ for all $x \in \Re^n$,

(H5): g is a lower semi-continuous extended valued function that is bounded below.

$$g(x) \;\le\; \min_{\xi \in \mathcal{D}} \{g(x + \xi) + d(\xi)\} \quad \text{for all } x \in \Re^n\,.$$

(H6): $\mathcal{D}$ is compact.

Comments on the Hypotheses *(H1) and (H2) ensure that, corresponding to any control policy and initial state, there exists a unique state trajectory. (H3)–(H4) and (H6), together with the condition that g is lower semi-continuous and bounded below, ensure the existence of a minimiser for* (P_{0,x_0}). *The role of (H5) is to exclude the possibility of a jump in the optimal policy at the final time.*

Embed (P_{0,x_0}) in the family of problems $\big\{(P_{t,x}) : (t, x) \in [0, T] \times \Re^n\big\}$. Here $(P_{t,x})$ denotes a modified version of (P_{0,x_0}), in which the initial data (t, x) replaces $(0, x_0)$.

The value function $V : [0, T] \times \Re^n \to \Re \cup +\{\infty\}$ is

$$V(t, x) \quad = \quad \inf(P_{t,x}) \quad \text{for all } (t, x) \in [0, T] \times \Re^n\,.$$

(The right side denotes 'infimum cost of $(P_{t,x})$'.)

Define the Hamiltonian function $\mathcal{H} : [0, T] \times \Re^n \times \Re^n$:

$$\mathcal{H}(t, x, p) \;=\; \min_{u \in \Omega} \{p \cdot f(t, x, u) - L(t, x, u)\}\,.$$

The value function is linked with functions which are solutions, in some appropriate sense, to a Bensoussan-Lions type quasi-variational inequality (QVI) for problem (P_{0,x_0}), namely functions ϕ satisfying the conditions:

$$\phi_t - \mathcal{H}(t, x, -\phi_x) \;\ge\; 0 \text{ for all } (t, x) \in (0, T) \times \Re^n \tag{1}$$

$$\phi(t, x) \;\le\; \min\nolimits_{\xi \in \mathcal{D}} \big\{\phi(t, x + \xi) + d(\xi)\big\} \quad \text{for all } (t, x) \in [0, T] \times \Re^n \tag{2}$$

$$(\phi_t - \mathcal{H}(t, x, -\phi_x)).(\phi - \min\nolimits_{\xi \in \mathcal{D}} \{\phi + d\}) = 0 \quad \text{for all } (t, x) \in (0, T) \times \Re^n \tag{3}$$

$$\phi(T, x) \;=\; g(x) \qquad \text{for all } x \in \Re^n\,. \tag{4}$$

Under hypotheses (H1)–(H6) the value function may fail to be continuously differentiable. The value function is lower semi-continuous in these circumstances, however. To exploit this fact, it is necessary to interpret lower semi-continuous functions which are said to satisfy (QVI).

There are a number of ways to do this. Our interpretation is based on the notion of the proximal sub-differential. Given an open set $\mathcal{O} \subset \Re^k$, a function $\psi : \mathcal{O} \to R \cup \{+\infty\}$ and a point $\bar{y} \in \mathcal{O}$ such that $\psi(\bar{y}) < +\infty$, the *proximal sub-differential of* ψ *at* $\bar{y}$, written $\partial^P \psi(\bar{y})$, is

$$\begin{aligned}\partial^P \psi(\bar{y}) \quad := \quad & \{\eta \in \Re^k \,:\, \text{there exists } M > 0,\ \epsilon > 0 \\ & \text{such that } \psi(y) - \psi(\bar{y}) \;\ge\; \eta \cdot (y - \bar{y}) \\ & -M|y - \bar{y}|^2 \text{ for all } y \in \bar{y} + \epsilon\mathcal{B}.\}\,.\end{aligned}$$

Definition 1: A function $\phi : [0, T] \times \Re^n \to \Re \cup \{+\infty\}$ is said to be a lower semi-continuous solution of (QVI), if it is lower semi-continuous and bounded below, and the following conditions are satisfied

(a) for each $(t, x) \in (0, T) \times \Re^n$

$$\xi_0 - \mathcal{H}(t, x, -\xi_1) \;\ge\; 0 \quad \forall\, (\xi_0, \xi_1) \in \partial^P \phi(t, x)\,,$$

(b) for each $(t,x) \in [0,T] \times \Re^n$ such that $\phi(t,x) < +\infty$,

$$\phi(t,x) \le \min_{\xi \in \mathcal{D}} \{\phi(t, x+\xi) + d(\xi)\}.$$

(c) for each $(t,x) \in (0,T) \times \Re^n$ such that

$$\phi(t,x) < +\infty \quad \text{and} \quad \phi(t,x) < \min_{\xi \in \mathcal{D}} \{\phi(t, x+\xi) + d(\xi)\}$$

we have

$$\xi_0 - \mathcal{H}(t,x,-\xi_1) = 0 \qquad \forall\, (\xi_0,\xi_1) \in \partial^P \phi(t,x),$$

(d) for each $x \in \Re^n$

$$\liminf_{t' \uparrow T, x' \to x} \phi(t',x') = \phi(T,x) = g(x),$$

(e) for each $x \in \Re^n$ such that

$$\phi(0,x) < \min_{\xi \in \mathcal{D}} \{\phi(0, x+\xi) + d(\xi)\}$$

we have

$$\phi(0,x) = \liminf_{t' \downarrow 0, x' \to x} \phi(t',x').$$

Notice that, when ϕ is of class C^1, then conditions (a)–(e) reduce to conditions (1)–(4). Thus, lower semi-continuous solutions, as we define them, reduce to classical-sense solutions for sufficiently regular functions. (We use here the fact that, for a C^1 function ψ, we have $\partial^P \psi(\bar{y}) = \{\nabla \psi(\bar{y})\}$.)

The following theorem, which is simply proved, using methods of Aubin [3] and Frankowska [7], gives conditions under which the set of lower semi-continuous solutions to (QVI) precisely captures the value function for problem (P_{0,x_0}):

Theorem 1: Assume (H1)–(H6). Then the value function $V : [0,T] \times \Re^n \to \Re \cup \{+\infty\}$ is the unique lower semi-continuous function to (QVI).

Theorem 1 provides a necessary and sufficient condition for a lower semi-continuous function to be the value function: it must be a lower semi-continuous solution to (QVI). If we are content with merely a sufficient condition along these lines, namely a 'verification theorem', we can relax the hypotheses (dropping the requirement that $\mathcal{D}$ is compact) and state the condition in terms of a function satisfying just some of the defining conditions of lower semi-continuous solutions to (QVI):

Proposition 1: Assume (H1)–(H5). Let (x,u,v) be a policy for (P_{0,x_0}). Suppose there exists a lower semi-continuous function $\phi : [0,T] \times \Re^n \to \Re \cup \{+\infty\}$ such that

(a′) for each $(t,x) \in (0,T) \times \Re^n$

$$\xi_0 - \mathcal{H}(t,x,-\xi_1) \ge 0 \quad \forall\, (\xi_0,\xi_1) \in \partial^P \phi(t,x),$$

(b′) for each $(t,x) \in [0,T] \times \Re^n$ such that $\phi(t,x) < +\infty$,

$$\phi(t,x) \le \min_{\xi \in \mathcal{D}} \{\phi(t, x+\xi) + d(\xi)\}.$$

(d′) for each $x \in \Re^n$

$$\liminf_{t' \uparrow T, x' \to x} \phi(t',x') = \phi(T,x) = g(x).$$

Assume, furthermore, that

$$\phi(0,x_0) = \int_0^T L(t,x(t),u(t))dt + g(x(T)) + C(v). \quad (5)$$

Then (x,u,v) is an optimal process, and $\phi(0,x_0)$ is the minimum cost for (P_{0,x_0}).

IV. Analysis of a Specific Inventory Problem

In this section we illustrate the application of the preceding theory by using it to solve a special case of the inventory problem, namely:

$$(E_{0,x_0}) \quad \begin{cases} \text{Minimise } \int_0^T |x(t)|dt + N(v) \\ \text{subject to} \\ dx(t)/dt = -1 + dv(t)/dt \\ x(0) = x_0 \quad \text{and} \quad x(T) \le 0. \end{cases}$$

Here, a control policy v is an impulse control

$$v = \{t_1, \ldots, t_{N(v)}, \xi_1, \ldots, \xi_{N(v)}\},$$

giving rise to jumps $\xi_1, \ldots, \xi_{N(v)}$ in the state trajectory at times $t_1, \ldots, t_{N(v)}$ respectively. The jumps are required to satisfy:

$$\xi_i \ge 0 \quad \text{for } i = 1, \ldots, N(v).$$

There is no conventional control component. In the present context, we omit reference to a conventional control and denote a process (x,v).

For this problem, the cost function is the sum of two terms, both of which we want to keep small. The first is the average stock level deviation. The other is the sum of transaction charges for restocking; each intervention carries a flat rate charge '1', independent of the amount of new stock. There must be no excess stock at the end of the time period.

(E_{0,x_0}) will be recognised as a special case of (P_{0,x_0}), in which $n = 1$,

$$L(x,u) = |x|,\ f(.,.,.) \equiv -1,\ d(.) \equiv 1,\ \mathcal{D} = [0,\infty)$$

and

$$g(x) = \begin{cases} +\infty & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0. \end{cases}$$

Define the functions $V^N : [0,T] \times \Re \to \Re \cup \{+\infty\}$, $N = 0,1,\ldots$, as follows.

$$V^0(t,x) = \begin{cases} +\infty & \text{if} \quad x > (T-t) \\ \frac{((T-t)-x)^2}{2} + \frac{x^2}{2} & \text{if} \quad (T-t) \ge x \ge 0 \\ \frac{(T-t)^2}{2} - x(T-t) & \text{if} \quad 0 > x \end{cases} \quad (6)$$

and, for $N \geq 1$,

$$V^N(t,x) = \begin{cases} +\infty & \text{if} \quad x > (T-t) \\ N + \frac{((T-t)-x)^2}{2(2N+1)} + \frac{x^2}{2} & \text{if } (T-t) \geq x \geq 0 \\ N + \frac{((T-t)-x)^2}{2(2N+1)} - \frac{x^2}{2} & \text{if } 0 > x \geq -\frac{(T-t)}{2N} \\ N + \frac{(T-t)^2}{4N} & \text{if } -\frac{(T-t)}{2N} > x \,. \end{cases} \tag{7}$$

Now define $V : [0,T] \times \Re \to \Re \cup \{+\infty\}$:

$$V(t,x) = \inf_{N=0,1,\dots} V^N(t,x) \quad \text{for } (t,x) \in [0,T] \times \Re \,. \tag{8}$$

Proposition 2: **(The Value Function)** Take V to be the function defined by (8). We have

$$\inf(E_{t,x}) \quad = \quad V(t,x) \quad \text{for all } (t,x) \in [0,T] \times \Re \,,$$

i.e., V is the value function for (E_{0,x_0}).

(An Optimal Policy) There exists a non-negative integer N such that

$$\inf(E_{0,x_0}) = V^N(0,x_0) \,.$$

Let $\bar{N}$ be the smallest such integer.

If $\bar{N} = 0$, then $v \equiv 0$ is an optimal impulse control (for $E_{0,x}$). If $\bar{N} > 0$, then $x \leq (T-t)$ and

$$v \quad = \quad \{t_1, \dots, t_{\bar{N}}, \xi_1, \dots, \xi_{\bar{N}}\}$$

is an optimal impulse control, where $t_1, \dots, t_{\bar{N}}, \xi_1, \dots, \xi_{\bar{N}}$ are defined as follows.

A: $-\frac{(T-t)}{2\bar{N}} \leq x \leq (T-t)$. In this case,

$$\begin{aligned} t_j &= \frac{2\bar{N}x + T - t}{2\bar{N}+1} + (j-1) \times \frac{2(T-t-x)}{2\bar{N}+1} \\ &\qquad\qquad \text{for } j = 1, 2, \dots, \bar{N} \,, \\ \xi_j &= \frac{2(T-t-x)}{2\bar{N}+1} \,. \end{aligned}$$

B: $x < -\frac{(T-t)}{2\bar{N}}$. In this case,

$$\begin{aligned} t_j &= (j-1) \times \frac{(T-t)}{\bar{N}} \quad \text{for } j = 1, 2, \dots, \bar{N} \,, \\ \xi_1 &= \left(|x| + \frac{(T-t)}{\bar{N}}\right) \quad \text{and} \quad \xi_j = \frac{(T-t)}{\bar{N}} \\ &\qquad\qquad \text{for } j = 2, 3, \dots, \bar{N} \,. \end{aligned}$$

Comments

(a): Minimising processes are not unique. For $x_0 > T$, it is not possible to satisfy the constraints of (E_{0,x_0}). So, in this case, all processes have the same cost '$+\infty$' and are therefore optimal, in a trivial sense. Even if $x_0 < T$, for certain values of T and x_0 there are two options for achieving the minimum cost (jumping a greater/lesser amount, fewer/more times respectively). The control described in Proposition 2 is the optimal control involving the least number of jumps.

(b): A detailed analysis of the above representations of the V^N's provides explicit formulae for the minimising N in equation (8). These formulae are rather complicated and are omitted from this paper.

An optimal state trajectory is illustrated in Figure 1, for the case $T = 9$ and $x_0 = 0$. Figure 2 provides a plot of the value function.

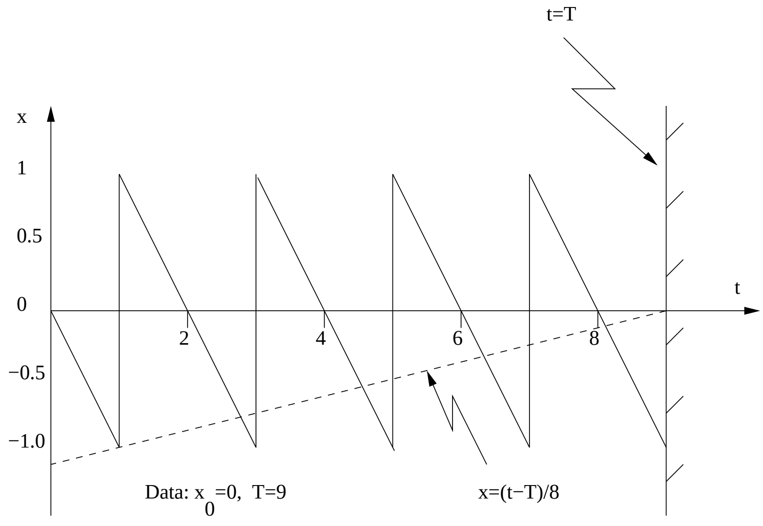

Fig. 1. An optimal state trajectory

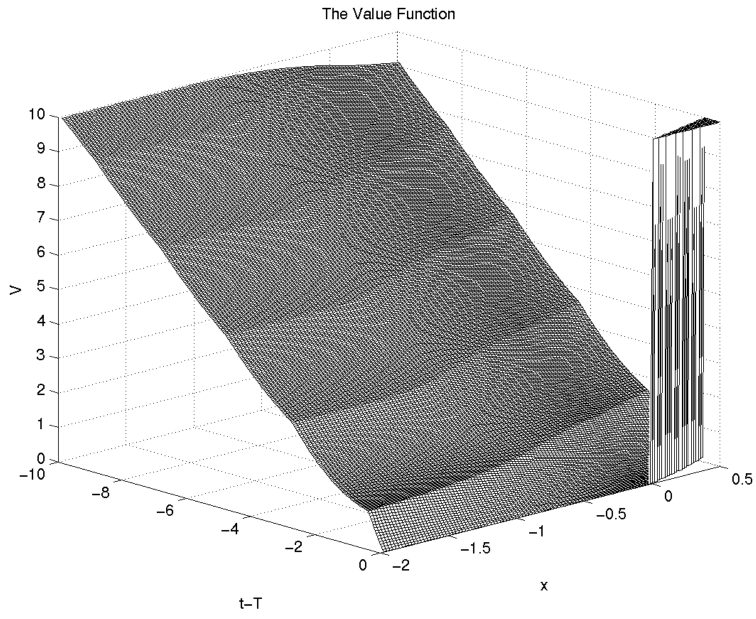

Fig. 2. The Value Function

Proof of the proposition is broken down into stages, the conclusions of each of which is summarised as a Lemma. First, we state, without proof, some properties of the V^N's, all of which are straightforward consequences of the defining relationships (6) and (7). Define

$$\mathcal{A} \;=\; \{(t,x) \in [0,T] \times \Re \,:\, x \leq T-t\} \,.$$

Lemma 1: Fix $t \in [0,T]$ and an integer $N \geq 0$,
(a) $\mathrm{dom}\,(V^N) = \mathcal{A}$. $V^N(.,.)$ is the restriction to $\mathcal{A}$ of a continuously differentiable function on $\Re \times \Re$.
(b) The restriction of $V^N(t,.)$ to $[0,\infty)$ is convex.
(c) Define

$$y(N,t) \quad := \quad \frac{T-t}{2(N+1)}\,.$$

Then $V^N(t,.)$ is decreasing on $\big(-\infty, y(N,t)\big)$, increasing on $\big(y(N,t),\infty\big)$ and

$$V^N\big(t,y(N,t)\big) \quad = \quad N + \frac{(T-t)^2}{4(N+1)}.$$

(d) $V^N(t,.)$ is constant on $\big(-\infty, -\frac{(T-t)}{2N}\big)$, if $N > 0$.

Lemma 2: (a) V is a lower semi-continuous function, bounded from below, and $\mathrm{dom}\,V = \mathcal{A}$. The restriction of V to $\mathcal{A}$ is locally Lipschitz continuous.
(b) Take any bounded subset $D \subset \mathcal{A}$. Then there exists a finite index set $J_D \subset \{0,1,2,\ldots\}$ with the property: corresponding to any point $(t',x') \in D$, there exists $N \in J_D$ such that

$$V(t',x') \quad = \quad V^N(t',x')\,. \quad (9)$$

Furthermore, (9) implies $N \geq 1$, if $x < -(T-t)^{-1} - \frac{(T-t)}{4}$.
(c) $V(t,.)$ is decreasing on $(-\infty,0]$.
(d) $V(T,x) = \Psi_{(-\infty,0]}(x)$ for all $x \in \Re$.

Proof: The first part of (b) is true because the V^N's are uniformly continuous on D and

$$\lim_{N\to\infty} \inf\{V^N(t,x)\,:\,(t,x) \in [0,T]\times\Re^n\} = +\infty.$$

Examining the formula for V^0 and V^1 we see that, if $x < -(T-t)^{-1} - \frac{T-t}{4}$, then $V^1(t,x) < V^0(t,x)$. It follows that $V(t,x) \leq V^1(t,x) < V^0(t,x)$. So we must have $N \geq 1$. (b) is proved.
Property (a) follows from (b) and Lemma 1 (a). (c) is true because $V(t,.)$ inherits the monotonicity properties of the $V^N(t,.)$'s on $(-\infty,0]$. (d) follows from (6) and (7). ■

Lemma 3: Take $t \in [0,T)$ and $x,y \in \Re$ such that $x \leq y$. Then

$$V(t,x) \;\leq\; V(t,y) + 1\,.$$

Proof: We can assume that $y \leq T-t$ since, otherwise, $V(t,y) = +\infty$ and the assertion is automatically true. Suppose then that $x \leq y \leq (T-t)$.

A: $0 \leq x \leq T-t$. In this case, in view of Lemma 2 (b), there exists an integer $N \geq 0$ such that $V(t,x) = V^N(t,x)$. We have

$$\begin{aligned} V(t,x) - V(t,y) &\leq V^{N+1}(t,x) - V^N(t,y) \\ &\leq \max\big\{V^{N+1}(t,0) - V^N(t,y), \\ &\qquad\quad V^{N+1}(t,y) - V^N(t,y)\big\}. \end{aligned}$$

In the second line, we have used the fact that $V^{N+1}(t,.)$ is convex on $[0,y]$ (Lemma 1 (b)) and therefore achieves its maximum at an extreme point of the interval $[0,y]$, namely 0 or y. But, by Lemma 1 (c),

$$\begin{aligned} &V^{N+1}(t,0) - V^N(t,y) \\ &\quad\leq\; N+1+\frac{(T-t)^2}{2(2N+3)} - \min_{0\leq y'\leq T-t} V^N(T,y) \\ &\quad=\; N+1+\frac{(T-t)^2}{2(2N+3)} - N - \frac{(T-t)^2}{4N+4} \;<\; 1\,. \end{aligned}$$

Also

$$\begin{aligned} V^{N+1}(t,y) - V^N(t,y) \;&=\; N+1+\frac{|(T-t)-y|^2}{4N+6} - N \\ &\quad -\frac{|(T-t)-y|^2}{4N+4} \;<\; 1\,. \end{aligned}$$

It follows that $V(t,x) - V(t,y) \;\leq\; 1$.

B: $x < 0$. In this case, set $x' = \min\{-\frac{T-t}{2}, -(T-t)^{-1} - \frac{T-t}{4}\}$. By Lemma 1 (d) and Lemma 2 (b) and (c)

$V(t,x) \leq V(t,x')$ and $V(t,x') = V^N(t,x')$ for some $N \geq 1$.

Since $x' \leq -\frac{(T-t)}{2}$ and N is 'minimising'

$$\begin{aligned} V^N(t,x') - V^{N'}(t,x') \;&=\; N + \frac{(T-t)^2}{4N} - N' - \frac{(T-t)^2}{4N'} \\ &\leq\; 0 \qquad \text{for all } N' \geq 1\,. \qquad (10) \end{aligned}$$

Let $M \geq 0$ be an integer such that $V(t,y) = V^M(t,y)$. Then by Lemma 1 (c)

$$\begin{aligned} V(t,y) \;&\geq\; V^M\big(t,y(t,M)\big) \\ &=\; M + \frac{(T-t)^2}{4(M+1)}\,. \end{aligned}$$

The proof is completed by noting that, in view of (10), we can now deduce that

$$\begin{aligned} V(t,x) - V(t,y) \;&\leq\; V^N(t,x') - V^M\big(t,y(t,M)\big) \\ &=\; N + \frac{(T-t)^2}{4N} - M - \frac{(T-t)^2}{4(M+1)} \\ &\leq\; 1\,. \end{aligned}$$

■

Lemma 4: For each $(t,x) \in (0,T)\times\Re$

$$\xi_0 - \mathcal{H}(\S, -\xi_\infty) \quad\geq\quad 0 \qquad \text{for all } (\xi_0,\xi_1) \in \partial^P V(t,x)\,.$$

Proof: Take any $(t,x) \in (0,T)\times\Re$. We can assume that $(t,x) \in \mathcal{A}$, i.e. $x \leq T-t$, since, otherwise, $V(t,x) = +\infty$ and there is nothing to prove. We know $V(t,x) = V^N(t,x)$ for some $N \geq 0$. Making use of the formulae for V^N above we can show, by means of simple calculations, that

$$\tfrac{\partial}{\partial t}V^N(t,x) - \tfrac{\partial}{\partial x}V^N(t,x) + |x| \geq 0\,. \quad (11)$$

(If (t,x) is a boundary point of $\mathcal{A}$, the derivatives in the formula are taken to be limits of values at neighbouring interior points.)

Take any $(\xi_0, \xi_1) \in \partial^P V(t,x)$. Then there exists $M > 0$ and $\epsilon > 0$ such that

$$\begin{aligned} V(t',x') - V(t',x) \quad &\geq \quad (\xi_0,\xi_1)\cdot((t',x')-(t,x)) \\ &\qquad -M|(t',x')-(t,x)|^2 \end{aligned}$$

for all $(t',x') \in (t,x) + \epsilon\mathcal{B}$. Since $V(t,x) = V^N(t,x)$ and $V^N(t',x') \geq V(t,x)$

$$\begin{aligned} V^N(t',x') - V^N(t,x) \quad &\geq \quad (\xi_0,\xi_1)\cdot((t',x')-(t,x)) \\ &\qquad -M|(t',x')-(t,x)|^2 \end{aligned}$$

for all $(t',x') \in (t,x) + \epsilon\mathcal{B}$. Since V^N is the restriction of a continuously differentiable function to $\mathcal{D}$, and, in the event (t,x) is a boundary point of $\mathcal{A}$, the normal cone to $\mathcal{A}$ at (t,x) is $\{\alpha(1,1) : \alpha \geq 0\}$, we deduce that

$$(\xi_0,\xi_1) \quad = \quad \Big(\frac{\partial V^N}{\partial t}, \frac{\partial V^N}{\partial x}\Big) + \alpha(1,1)$$

for some $\alpha \geq 0$. It follows from Lemma 4 (a) and (b) that

$$\xi_0 - \mathcal{H}(x,-\xi_1)\Big(= \frac{\partial V^N}{\partial t}(t,x) + \alpha - \frac{\partial V^N}{\partial x}(t,x) - \alpha + |x|\Big) \geq 0\,.$$

■

Proof of Proposition 2 We show that the impulse control v described in Proposition 1 is a minimiser. Here, we make use of the sufficient condition of optimality provided by Proposition 1, all hypotheses for the application of which are satisfied by the data for problem E_{0,x_0}.

Identify the function ϕ of Proposition 1 with V, defined by (8). Lemmas 3 and 4 establish that ϕ satisfies hypotheses (a′) and (b′) of Proposition 1.

Lemma 2 (d) tells us that $V(T,.) = g(.)$. Since, however, the restriction of V to $\mathrm{dom}\,V (= \{(t,x) : x \leq T - t\})$ is continuous, we have

$$\lim_{t'\uparrow T, x'\to x} V(t',x') = V(T,x) = g(x) = \tilde{g}(x) \quad \text{for all } x \in \Re\,.$$

We see that hypothesis (d′) of Proposition 1 is also satisfied. Let N be a non-negative integer such that $V(0,x_0) = V^N(0,x_0)$. Simple calculations, treating each of the different cases, '$x_0 > T$', '$T > x_0 > -(T)/2N$' and $-T/2N > x_0$, reveal that

$$V(0,x_0) \;=\; V^N(0,x_0) \;=\; \int_0^T |x(t)|dt + g(x(T)) + N(v)\,.$$

But this is (5) of Proposition 1. It follows from Proposition 1 that (x,v) is a minimiser and $V(0,x_0)$ is the minimum cost for (E_{0,x_0}).

It remains to note that, for any $(t,x) \in [0,T] \times \Re$, the preceding analysis with T and x_0 replaced by $T-t$ and x respectively, establishes that $V(t,x)$ is the minimum cost for $(E_{t,x})$. We have shown that V is the value function for (E_{0,x_0}).

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