Separation Principle for a Class of Non-linear Systems

Václav Černý, Josef Hrušák

***Abstract*— It is known that the separation principle was deduced in the linear systems theory from both the optimal and asymptotic points of view. But, it has not sufficiently been solved in non-linear case yet. The method proposed in this paper represents a new approach to the solution of the separation principle problem for a certain class of non-linear systems. As the theoretical basis of the approach the well known dissipative systems theory has been chosen. The Lyapunov's stability theory is the other basic point of the method.**

I. Introduction

This paper deals with a new approach to the solution of a separation principle problem for a certain class of non-linear systems. It is based on so called the *dissipation normal form* and consists in combining two methods. In both the methods the dissipation normal form is used. One of them solves the stabilization problem of non-linear systems. The dissipation normal form is used here in such a way that the structure of the representation of a closed loop system is only chosen in this form. An appropriate controller is proposed so that the demands required of the behaviour of a closed loop system are implemented. The other method solves the state reconstruction problem of non-linear systems. The dissipation normal form is used here in such a way that the structure of the representation of an error system is only chosen in this form. By means of integrating the stabilization and state reconstruction methods mentioned above the solution of the separation principle problem for a certain class of non-linear systems is found. It is in general embodied in the proposal of a *compensation function* that guarantees the asymptotical stability of a resulting closed loop system after applying the separation principle if it is put to a proposed controller as an addition. This method is similar to the method described in [1]. The problem is solved there with the help of a damping function added to a proposed controller as well. Then, the damping function guarantees the asymptotical stability of a closed loop system where the separation principle was used. Other approaches to the solution of the separation principle problem for non-linear systems present for example methods described in [2], [3], [4], [5]. Some comparisons with the method mentioned in [4], [5] are performed in this paper. It is shown that a certain similarity can be also found there.

II. Problem Formulation

Consider the representation $R(S)$ of a system S in the form:

$$\begin{aligned} \frac{dx(t)}{dt} &= f[x(t), u(t)] && (1)\\ y(t) &= h[x(t)], && (2) \end{aligned}$$

where $x(t) \in R^n$ is a state vector, $u(t) \in R^1$ is an input, $y(t) \in R^1$ is an output and $f[x(t), u(t)] \in C^\infty : R^n \times R^1 \to R^n$, $h[x(t)] \in C^\infty : R^n \to R^1$ are non-linear mappings. The state vector $x(t)$ is supposed not to be accessible for measurement.

Assume that the representation $R(S)$ (1), (2) is controllable and observable.

Our aim is to propose a controller

$$u(t) = L[x(t)] \qquad (3)$$

and an observer

$$\hat{R}(S) : \frac{d\hat{x}(t)}{dt} = \hat{f}[\hat{x}(t), u(t), y(t)] \qquad (4)$$

in such a way that the closed loop system S_{cl} containing the original system S (1), (2), the controller (3) and the observer (4):

$$\begin{aligned} R(S_{cl}) : \frac{dx(t)}{dt} &= f\{x(t), L[\hat{x}(t)]\} && (5)\\ \frac{d\hat{x}(t)}{dt} &= \hat{f}\{\hat{x}(t), L[\hat{x}(t)], h[x(t)]\} && (6)\\ y(t) &= h[x(t)] && (7) \end{aligned}$$

is asymptotically stable. It means that

$$\begin{aligned} V[x(t), \hat{x}(t)] &> 0 \text{ for } x(t) \neq x_e,\ \hat{x}(t) \neq \hat{x}_e && (8)\\ V[x(t), \hat{x}(t)] &= 0 \text{ for } x(t) = x_e,\ \hat{x}(t) = \hat{x}_e && (9)\\ \frac{dV[x(t), \hat{x}(t)]}{dt} &< 0 \text{ for } x(t) \neq x_e,\ \hat{x}(t) \neq \hat{x}_e && (10)\\ \frac{dV[x(t), \hat{x}(t)]}{dt} &= 0 \text{ for } x(t) = x_e,\ \hat{x}(t) = \hat{x}_e, && (11) \end{aligned}$$

where $V[x(t), \hat{x}(t)] : R^n \times R^{\hat{n}} \to R$ is a Lyapunov function related to the representation $R(S_{cl})$ of a closed loop system S_{cl} (5), (6), (7) and x_e, $\hat{x}_e$ is its equilibrium state for which it holds that

$$\frac{dx_e}{dt} = 0,\ \frac{d\hat{x}_e}{dt} = 0. \qquad (12)$$

III. Dissipation Normal Form

Definition 1: Consider the representation $R(S)$ of a system S and assume that there exists an accumulation function $W[x(t)]$ defined on a domain $\Omega \subset R^n$. The representation $R(S)$ will be called the *dissipation normal form* if the accumulation function $W[x(t)]$ fulfills the following conditions:

$$\begin{aligned} W[x(t)] &= \|x(t)\|^2 && (13)\\ L_f\{W[x(t)]\} &= \beta[y(t)] \leq 0. && (14) \end{aligned}$$

Remark 1: The accumulation function $W[x(t)]$ represents measure of the signal energy stored in a system S at a time instant *t*. There is an obvious connection with a Lyapunov function $V[x(t)]$. The accumulation function $W[x(t)]$ is also related to the available storage [6] and the non-linear function $\beta[y(t)]$ corresponds to the Rayleigh function [7].

A. Structural Asymptotical Stability

The following theorem will be used later for guaranteeing the asymptotical stability of the closed loop system S_{cl} (5), (6), (7).

Theorem 1: Let $k_2, \ldots, k_n \in R$; $k_2, \ldots, k_n \neq 0$ are constants and $\varphi_1[x_1(t)]$, $\alpha[x_1(t)]$ are non-linear functions which satisfy the following conditions: $\varphi_1[x_1(t)] < 0$ for $x_1(t) \neq 0$, $\exists\ \alpha^{-1}[y(t)]$ and $\alpha[x_1(t)] = 0 \Longleftrightarrow x_1(t) = 0$.

If a representation $R(S)$ has the structure [8]:

$$\frac{dx(t)}{dt} = \begin{bmatrix} \varphi_1[x_1(t)] & k_2 & 0 & . & 0 \\ -k_2 & 0 & k_3 & . & . \\ 0 & -k_3 & . & . & 0 \\ . & . & . & . & k_n \\ 0 & . & 0 & -k_n & 0 \end{bmatrix} x(t) \tag{15}$$

$$y(t) = \alpha[x_1(t)], \tag{16}$$

then the only equilibrium state $x_e = 0$ is asymptotically stable and the corresponding accumulation function $W[x(t)]$ fulfills the conditions (13), (14) for any $\varphi_1[x_1(t)]$, $\alpha[x_1(t)]$ and $k_2, \ldots, k_n$.

Proof: Assume that a representation $R(S)$ has the form (15), (16) and consider the accumulation function $W[x(t)] = \|x(t)\|^2$.

1) The relation (15) implies that

$$\frac{dx(t)}{dt} = 0 \ \Leftrightarrow \ x(t) = 0. \tag{17}$$

Hence, $x(t) = x_e = 0$ is the only equilibrium state of the representation $R(S)$.

2) It holds that

$$W[x(t)] > 0 \ \text{ for } \ x(t) \neq 0 \tag{18}$$

$$W[x(t)] = 0 \ \text{ for } \ x(t) = 0 \tag{19}$$

$$\begin{aligned} L_f\{W[x(t)]\} &= 2x_1^2(t)\varphi_1[x_1(t)] = \\ &= 2\{\alpha^{-1}[y(t)]\}^2\varphi_1\{\alpha^{-1}[y(t)]\} = \\ &= \beta[y(t)] < 0 \ \text{ for } \ x(t) \neq 0 \end{aligned} \tag{20}$$

$$\begin{aligned} L_f\{W[x(t)]\} &= 2x_1^2(t)\varphi_1[x_1(t)] = \\ &= 2\{\alpha^{-1}[y(t)]\}^2\varphi_1\{\alpha^{-1}[y(t)]\} = \\ &= \beta[y(t)] = 0 \ \text{ for } \ x(t) = 0. \end{aligned} \tag{21}$$

It follows from the relations (18), (19), (20), (21) that the accumulation function $W[x(t)]$ is a Lyapunov function. Thus, the equilibrium state $x_e = 0$ is asymptotically stable. It is also obvious that the accumulation function $W[x(t)]$ fulfills the conditions (13), (14) for any $\varphi_1[x_1(t)]$, $\alpha[x_1(t)]$ and $k_2, \ldots, k_n$.

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Remark 2: The relation (20) implies that

$$\varphi_1[x_1(t)] < 0 \ \text{ for } \ x_1(t) \neq 0 \tag{22}$$

is a necessary and sufficient condition for the structural asymptotical stability of a system S.

Remark 3: If the accumulation function $W[x(t)]$ is defined on the whole state space R^n and the relations (18), (19), (20), (21) hold, then a system S is globally asymptotically stable.

Remark 4: The structure of the dissipation normal form is related to the Schwarz matrix [9] and can be seen as the generalization of a corresponding linear system representation.

B. Structural Observability

It holds that

$$\begin{aligned} det\ H_o[x(t)] &= det\ \frac{\partial}{\partial x(t)} \begin{bmatrix} \alpha[x_1(t)] \\ L_f\{\alpha[x_1(t)]\} \\ \vdots \\ L_f^{n-1}\{\alpha[x_1(t)]\} \end{bmatrix} = \\ &= k_2^{n-1} \cdot k_3^{n-2} \cdot \ldots \cdot k_n \cdot \{\frac{d\alpha[x_1(t)]}{dx_1(t)}\}^n, \end{aligned} \tag{23}$$

where $H_o[x(t)]$ is a generalized observability matrix.

It follows from the relation (23) that the conditions $k_2, \ldots, k_n \neq 0$, $\exists\ \alpha^{-1}[y(t)]$ and $\alpha[x_1(t)] = 0 \Longleftrightarrow x_1(t) = 0$ are necessary and sufficient conditions for structural observability of the dissipation normal form.

IV. Non-linear Observer Design Based on Dissipation Normal Form

A. Problem Formulation

Consider the representation $R(S)$ of a system S in the form (1), (2).

Our aim is to design an observer $\hat{R}(S)$:

$$\hat{R}(S) : \frac{d\hat{x}(t)}{dt} = \hat{f}[\hat{x}(t), u(t), y(t)] \tag{24}$$

which will generate the asymptotic estimate $\hat{x}(t)$ of the state vector $x(t)$ using the input $u(t)$ and the output $y(t)$ in such a way that the following two demands will be satisfied.

The first one is the *state error invariance condition*:

$$\begin{aligned} \tilde{R}(S) : \frac{d\tilde{x}(t)}{dt} &= \tilde{f}[\tilde{x}(t), x(t), \hat{x}(t), u(t), y(t)] = \\ &= \tilde{f}[\tilde{x}(t)], \end{aligned} \tag{25}$$

where $\tilde{x}(t)$ is a state error defined as

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{26}$$

The second one is the *state error convergence condition* to zero

$$\lim_{t\to\infty} \tilde{x}(t) = \lim_{t\to\infty} [x(t) - \hat{x}(t)] = 0 \tag{27}$$

corresponding to the asymptotical stability of the state error system $\tilde{R}(S)$ (25).

B. Problem Solution

This method consists in the prior choice of the structure of a state error system representation selected in order to fulfill structurally the two demands mentioned above. The structure of the state error system representation is chosen in the *dissipation normal form*:

$$\begin{aligned}\tilde{R}^*(S) \quad &: \quad \frac{d\tilde{x}^*(t)}{dt} = \\ &= \omega_0 \begin{bmatrix} \delta_1^*[\tilde{x}_1^*(t)] & \delta_2^* & 0 & . & 0 \\ -\delta_2^* & 0 & \delta_3^* & . & . \\ 0 & -\delta_3^* & . & . & 0 \\ . & . & . & . & \delta_n^* \\ 0 & . & 0 & -\delta_n^* & 0 \end{bmatrix} \tilde{x}^*(t),\end{aligned} \tag{28}$$

where $\delta_1^*[\tilde{x}_1^*(t)]$, ω_0, $\delta_2^*, \dots, \delta_n^*$ are design parameters.

It holds that

$$L_{\tilde{f}^*}\{\tilde{V}^*[\tilde{x}^*(t)]\} = L_{\tilde{f}^*}\{\|\tilde{x}^*(t)\|^2\} = 2\omega_0 \tilde{x}_1^{*^2}(t)\delta_1^*[\tilde{x}_1^*(t)], \tag{29}$$

where $\tilde{V}^*[\tilde{x}^*(t)] = \|\tilde{x}^*(t)\|^2$ is the Lyapunov function related to the representation $\tilde{R}^*(S)$ (28). The relations (28), (29) imply that both the *state error invariance condition* and the *state error convergence condition* to zero are satisfied *structurally* if the design parameters are properly chosen. If $\omega_0 > 0$ and $\delta_1^*[\tilde{x}_1^*(t)] < 0$ for all $\tilde{x}_1^*(t)$, then the state error system is globally asymptotically stable. If $\omega_0 > 0$ and $\delta_1^*[\tilde{x}_1^*(t)] < 0$ only for $\tilde{x}_1^*(t) \in \langle \tilde{x}_1^{*^1}, \tilde{x}_1^{*^2} \rangle$, $|\tilde{x}_1^{*^2} - \tilde{x}_1^{*^1}| = \sigma \neq 0$, then the state error system is semi-globally asymptotically stable over a finite area of the state space R^n. The constant ω_0 represents a time scale transformation and therefore it affects the convergence rate. The non-linear function $\delta_1^*[\tilde{x}_1^*(t)]$ describes in what way the system energy dissipates and therefore it specifies the convergence mode. It is clear from the relation (29) that the constants $\delta_2^*, \dots, \delta_n^* \neq 0$ do not have any effect on either rate or mode of convergence. From this point of view, they can in principle be chosen in an arbitrary way.

Remark 5: In fact, $\delta_2^*, \dots, \delta_n^*$ can be non-linear functions in general: $\delta_2^* = \delta_2^*[\tilde{x}^*(t), \hat{x}^*(t), x^*(t), u(t), y(t), t], \dots, \delta_n^* = \delta_n^*[\tilde{x}^*(t), \hat{x}^*(t), x^*(t), u(t), y(t), t]$. Nevertheless, this complication is not necessary. It has already been said that they have no effect on either rate or mode of convergence. Because of this, they are chosen without loss of generality as constants.

Remark 6: If $\omega_0 \to \infty$ ($\frac{1}{\omega_0} \to 0$), then an appropriate observer corresponds to the high-gain observer [10], [4], [5].

Further, the original representation $R(S)$ of a system S (1), (2) is supposed to be transformed into a proper state equivalent canonical form. Then, substituting to the relation for the state error (26) we get an observer structure.

The parametrization of the observer is performed via the generalized observability normal form and consists in general in solving a system of differential equations, which is the consequence of the validity of a certain structural condition unwinding from an equivalence relation.

Finally, the proposed observer is transformed into original coordinates.

More information about this method can be found in [11], [12], [13], [14], [15], [16].

V. Stabilization of Non-Linear Systems Based on Metric Equivalence

A. Problem Formulation

Consider the representation $R(S)$ of a system S in the form (1), (2).

Our aim is to propose a controller

$$u(t) = L[x(t)] \tag{30}$$

in such a way that a closed loop system S_{cl}:

$$\begin{aligned} R(S_{cl}) : \frac{dx(t)}{dt} &= f\{x(t), L[x(t)]\} \quad (31) \\ y(t) &= h[x(t)] \quad (32) \end{aligned}$$

is asymptotically stable.

B. Problem Solution

We would like to use the Lyapunov's stability theory for solving the stabilization problem mentioned above. However, the application of the theory for stabilization problem solving is quite complicated because a Lyapunov function $V[x(t)]$ related to the given original representation $R(S)$ of a system S (1), (2) is not explicitly known in general. Therefore, the dissipation normal form will be used for the synthesis of a control law because its Lyapunov function is explicitly known.

Choose the representation $R^*(S_{cl})$ of a closed loop system S_{cl} in the dissipation normal form:

$$\begin{aligned}R^*(S_{cl}) \quad &: \quad \frac{dx^*(t)}{dt} = \\ &= \nu_0 \begin{bmatrix} f_1^*[x_1^*(t)] & f_2^* & 0 & . & 0 \\ f_2^* & 0 & f_3^* & . & . \\ 0 & f_3^* & . & . & 0 \\ . & . & . & . & f_n^* \\ 0 & . & 0 & f_n^* & 0 \end{bmatrix} x^*(t)\end{aligned} \tag{33}$$

$$y = x_1^*(t), \tag{34}$$

where $f_1^*[x_1^*(t)]$, ν_0, $f_2^*, \dots, f_n^*$ are design parameters. They have the same influence on closed loop system behaviour as mentioned in the section IV-B.

Remark 7: (metric equivalence vs. state equivalence) The structure of the representation $R^*(S_{cl})$ of a closed loop system S_{cl} (33), (34) is only one of possible structures which conform to the conditions (13), (14). We will obtain another one if we use an orthonormal transformation applied to the relations (33), (34). We chose this one, but we could still choose another one. The reason we selected this form is that it has a structural asymptotical stability property and therefore certain measure of robustness is held.

Further, suppose that the original representation $R(S)$ of a system S (1), (2) can be transformed to the following form:

$$\bar{R}(S): \frac{d}{dt}\begin{bmatrix} \bar{x}_1(t) \\ \vdots \\ \bar{x}_{n-1}(t) \\ \bar{x}_n(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \\ \mu[\bar{x}(t), u(t)] \end{bmatrix} \quad (35)$$

$$y(t) = \bar{x}_1(t), \quad (36)$$

where $\mu[\bar{x}(t), u(t)]$ is a non-linear function.

Remark 8: Conditions for the existence of an appropriate transformation $\bar{x}(t) = T[x(t), u(t)]$ are controllability and observability of the representation $R(S)$ (1), (2).

Then, the controller $u(t) = L[\bar{x}(t)]$ is proposed with using an equivalence relation and specified by the following term:

$$u(t) = L[\bar{x}(t)] = \mu^{-1}\{\bar{x}(t), \eta[\bar{x}(t)]\}, \quad (37)$$

where $\eta[\bar{x}(t)] = L^n_{f^*}[x^*_1(t)]$ for $x^*(t) = T^{-1}[\bar{x}(t)]$.

Finally, the proposed controller is transformed into original coordinates.

More information about this method can be found in [13], [17],[18], [19], [16].

VI. Separation principle – compensation function

Consider for now that

$$\begin{aligned} u(t) &= L[\bar{x}(t)]|_{\bar{x}(t)=\hat{\bar{x}}(t)} = L[\hat{\bar{x}}(t)] = \\ &= \mu^{-1}\{\hat{\bar{x}}(t), \eta[\hat{\bar{x}}(t)]\}. \end{aligned} \quad (38)$$

Then, the representation $\bar{R}(S_{cl})$ of a closed loop system S_{cl} has the form:

$$\bar{R}(S_{cl}): \frac{d}{dt}\begin{bmatrix} \bar{x}_1(t) \\ \vdots \\ \bar{x}_{n-1}(t) \\ \bar{x}_n(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \\ \gamma[\bar{x}(t), \hat{\bar{x}}(t)] \end{bmatrix} \quad (39)$$

$$y(t) = \bar{x}_1(t), \quad (40)$$

where $\gamma[\bar{x}(t), \hat{\bar{x}}(t)] = \mu\{\bar{x}(t), \mu^{-1}\{\hat{\bar{x}}(t), \eta[\hat{\bar{x}}(t)]\}\}$.

If the methods for non-linear observer design mentioned in the chapter IV and stabilization of non-linear systems mentioned in the chapter V are combined, then the representation $\bar{R}(S_{cl})$ of a closed loop system S_{cl} (39), (40) can be transformed back to the dissipation normal form:

$$\begin{aligned} R^*(S_{cl}) \quad &: \quad \frac{d}{dt}\begin{bmatrix} x^*_1(t) \\ x^*_2(t) \\ \vdots \\ x^*_{n-1}(t) \\ x^*_n(t) \end{bmatrix} = \\ &= \begin{bmatrix} f^*_1[x^*_1(t)]x^*_1(t) + f^*_2 x^*_2(t) \\ -f^*_2 x^*_1(t) + f^*_3 x^*_3(t) \\ \vdots \\ -f^*_{n-1}x^*_{n-2}(t) + f^*_n x^*_n(t) \\ \zeta[x^*(t), \hat{x}^*(t)] \end{bmatrix} \end{aligned} \quad (41)$$

$$y(t) = x^*_1(t), \quad (42)$$

where

$$\zeta[x^*(t), \hat{x}^*(t)] = \sum_{i=1}^{n-1} \frac{\partial T^{-1}_n[\bar{x}(t)]}{\partial \bar{x}_i(t)} L^i_{f^*}[x^*_1(t)] + \frac{1}{f^*_n}\gamma[\bar{x}(t), \hat{\bar{x}}(t)] \quad (43)$$

for $\bar{x}(t) = T[x^*(t)]$ and $\hat{\bar{x}}(t) = \hat{T}[\hat{x}^*(t)]$.

The representation $R^*(S_{cl})$ of a closed loop system S_{cl} is the dissipation normal form if and only if

$$\begin{aligned} \zeta[x^*(t), \hat{x}^*(t)] &= \sum_{i=1}^{n-1} \frac{\partial T^{-1}_n[\bar{x}(t)]}{\partial \bar{x}_i(t)} L^i_{f^*}[x^*_1(t)] + \\ &+ \frac{1}{f^*_n}\gamma[\bar{x}(t), \hat{\bar{x}}(t)] = -f^*_n x^*_{n-1}(t) \end{aligned} \quad (44)$$

for $\bar{x}(t) = T[x^*(t)]$ and $\hat{\bar{x}}(t) = \hat{T}[\hat{x}^*(t)]$.

Unfortunately, the condition (44) can not be fulfilled in any way.

Suppose for now that $\gamma[\bar{x}(t), \hat{\bar{x}}(t)] = \gamma_1[\bar{x}(t)] + \gamma_2[\hat{\bar{x}}(t)]$. It means that $\mu[\bar{x}(t), u(t)] = \mu_1[\bar{x}(t)] + u(t)$.

Then

$$\begin{aligned} \zeta[x^*(t), \hat{x}^*(t)] &= \sum_{i=1}^{n-1} \frac{\partial T^{-1}_n[\bar{x}(t)]}{\partial \bar{x}_i(t)} L^i_{f^*}[x^*_1(t)] + \\ &+ \frac{1}{f^*_n}\gamma_1[\bar{x}(t)] + \frac{1}{f^*_n}\gamma_2[\hat{\bar{x}}(t)] \end{aligned} \quad (45)$$

for $\bar{x}(t) = T[x^*(t)]$ and $\hat{\bar{x}}(t) = \hat{T}[\hat{x}^*(t)]$.

The representation $R^*(S_{cl})$ of a closed loop system S_{cl} is the dissipation normal form if and only if

$$\begin{aligned} \zeta[x^*(t), \hat{x}^*(t)] &= \sum_{i=1}^{n-1} \frac{\partial T^{-1}_n[\bar{x}(t)]}{\partial \bar{x}_i(t)} L^i_{f^*}[x^*_1(t)] + \\ &+ \frac{1}{f^*_n}\gamma_1[\bar{x}(t)] + \frac{1}{f^*_n}\gamma_2[\hat{\bar{x}}(t)] = -f^*_n x^*_{n-1}(t) \end{aligned} \quad (46)$$

for $\bar{x}(t) = T[x^*(t)]$ and $\hat{\bar{x}}(t) = \hat{T}[\hat{x}^*(t)]$.

The condition (46) holds if and only if

1) the original representation $R(S)$ of a system S (1), (2) can be transformed to the dissipation normal form:

$$R^*(S): \frac{dx^*(t)}{dt} = a^*[x^*(t)] + b^*[x^*(t)]u(t) \quad (47)$$

$$y(t) = c^*[x^*_1(t)], \quad (48)$$

where

$$a^*[x^*(t)] = \begin{bmatrix} a^*_1[x^*_1(t)] & a^*_2 & 0 & . & 0 \\ -a^*_2 & 0 & a^*_3 & . & . \\ 0 & -a^*_3 & . & . & 0 \\ . & . & . & . & a^*_n \\ 0 & . & 0 & -a^*_n & 0 \end{bmatrix} x^*(t)$$

2) $f^*_2 = a^*_2, \cdots, f^*_n = a^*_n$

3) a certain function

$$\varrho[\hat{\bar{x}}(t)] = -\frac{1}{a^*_n}\gamma_2[\hat{\bar{x}}(t)], \quad (49)$$

where

$$\gamma_2[\hat{\bar{x}}(t)] = \mu_1[\hat{\bar{x}}(t)] - L^n_{f^*}[x^*_1(t)] \quad (50)$$

for $x^*(t) = T^{-1}[\bar{x}(t)]$ and $\bar{x}(t) = \hat{\bar{x}}(t)$.

The function $\varrho[\hat{\bar{x}}(t)]$ is the compensation function added to the proposed controller:

$$u(t) = L[\hat{\bar{x}}(t)] + \varrho[\hat{\bar{x}}(t)]. \tag{51}$$

Finally, if the condition (46) holds, then the asymptotical stability of a closed loop system S_{cl} (5), (6), (7) is guaranteed.

Remark 9: The formula for the function $\varrho[\cdot](\gamma_2[\cdot])$ does not depend on observer equations. It means that it can be used another method for non-linear observer design proposing an asymptotic observer, not necessarily the mentioned one.

VII. Illustrative example

Consider the second order non-linear system S (van der Poll equation):

$$\begin{aligned} R(S): \frac{dx_1(t)}{dt} &= x_2(t) && (52)\\ \frac{dx_2(t)}{dt} &= -\varepsilon[\psi - \beta x_1^2(t)]x_2(t) - \\ &- Kx_1(t) + u(t) && (53)\\ y(t) &= x_1(t), && (54) \end{aligned}$$

where $\varepsilon = -2$, $\psi = 4$, $\beta = 2$, $K = 4$ are the system parameters. The response of the system S to certain initial conditions for $u(t) = 0$ is shown on the fig. 1.

At first we propose a controller using the method mentioned in the section V so that the closed loop system is asymptotically stable. Then, the controller is the following:

$$u(t) = -(0.3+\varepsilon\beta)x_1^2(t)x_2(t)+(K-1)x_1(t)+(\varepsilon\psi-2)x_2(t). \tag{55}$$

The response of the closed loop system S_{cl} to the initial conditions is shown on the fig. 2.

At second we propose an observer using the method mentioned in the section IV so that its state error system is asymptotically stable. Then, the observer is the following:

$$\begin{aligned} \hat{R}(S): \frac{d\hat{x}_1(t)}{dt} &= \hat{x}_2(t) - \varepsilon[\frac{\beta}{3}\hat{x}_1^2(t) - \psi]\hat{x}_1(t) + \\ &+ \frac{\varepsilon\beta}{3}y^3(t) - \varepsilon\psi y(t) + 0.5\omega_0[y(t) - \hat{x}_1(t)] && (56)\\ \frac{d\hat{x}_2(t)}{dt} &= -\varepsilon[\psi - \beta\hat{x}_1^2(t)]\hat{x}_2(t) - K\hat{x}_1(t) + u(t) + \\ &+ [\varepsilon\beta\hat{x}_1^2(t) - \varepsilon\psi]\{-\varepsilon[\frac{\beta}{3}\hat{x}_1^2(t) - \psi]\hat{x}_1(t) + \\ &+ \frac{\varepsilon\beta}{3}y^3(t) - \varepsilon\psi y(t) + 0.5\omega_0[y(t) - \hat{x}_1(t)]\}, && (57) \end{aligned}$$

where $\omega_0 = 2$. The state error course of the observer is shown on the fig. 3, 4.

The response of the closed loop system S_{cl} containing the original system S, the controller and the observer is shown on the fig. 5. The output of the resulting closed loop system S_{cl} diverges and goes to infinity at a finite time.

It can be shown that it is possible to transform the given representation $R(S)$ of the system S into the dissipation normal form:

$$R^*(S): \frac{dx^*(t)}{dt} = \begin{bmatrix} a_1^*[x_1^*(t)] & a_2^* \\ -a_2^* & 0 \end{bmatrix} x^*(t) + \begin{bmatrix} 0 \\ b_2^* \end{bmatrix} u(t) \tag{58}$$

$$y(t) = c^*[x_1^*(t)] \tag{59}$$

with $a_1^*[x_1^*(t)] = \frac{\varepsilon\beta}{3}{x_1^*}^3(t) - \varepsilon\alpha x_1^*(t)$, $a_2^* = 2$, $c^*[x_1^*(t)] = x_1^*(t)$ and $b_2^* = \frac{1}{2}$.

Now we compute a compensation function and add it to the proposed controller. The compensation function is the following:

$$\varrho[\hat{x}(t)] = -(0.3 + \varepsilon\beta)\hat{x}_1^2(t)\hat{x}_2(t) + (\varepsilon\psi - 2)\hat{x}_2(t). \tag{60}$$

In the consequence of $f_2^* = a_2^*$ the controller is changed to the form:

$$u(t) = -(0.3 + \varepsilon\beta)\hat{x}_1^2(t)\hat{x}_2(t) + (\varepsilon\psi - 2)\hat{x}_2(t). \tag{61}$$

The response of the resulting closed loop system S_{cl} is shown on the fig. 6. The compensation function stabilized the closed loop system S_{cl} when the separation principle was used.

Remark 10: It can be seen that the compensation function has the same form as the controller.

VIII. Acknowledgments

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IX. Conclusions

The separation principle for a certain class of non-linear systems has been successfully deduced. It consists in guaranteing the asymptotical stability of the closed loop system where the separation technique was used. The method mentioned in the paper is exact and does not require any system linearization in the sense that the system to be stabilized is replaced by a linear one. In comparison with the method described in [4], [5] it is analytical. It means that no numerical approach is used. However, a certain similarity can be found (see the section IV-B). The problem is that the dissipation normal form is not general enough. The extension of the method mentioned in the paper could consist in generalizing this form.

References

[1] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. 605 Third Avenue, New York, NY 10158-0012, USA: John Wiley & Sons, Inc., 1995.

[2] R. Marino and P. Tomei, "Dynamic output feedback linearization and global stabilization," *Systems & Control Letters*, vol. 17, pp. 115–121, 1991.

[3] J. Tsinias, "A generalization of Vidyasagar's theorem on stabilizability using state detection," *Systems & Control Letters*, vol. 17, pp. 37–42, 1991.

[4] A. N. Atassi and H. K. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *IEEE Trans. Automat. Control*, vol. 44, no. 9, pp. 1672–1687, 1999.

[5] ——, "A separation principle for the control of a class of nonlinear systems," *IEEE Trans. Automat. Control*, vol. 46, no. 5, pp. 742–746, 2001.

[6] J. C. Willems, “Dissipative dynamical systems - part I: General theory,” *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–350, 1972.
[7] J. W. Rayleigh, *The Theory of Sound: Vols. I and II.* Dover (1945 edition), 1887.
[8] J. Hrušák, “The isometric transformation method and some of its applications,” Ph.D. dissertation, KKY VŠSE, Pilsen, Czech Republic, 1971, (in Czech only).
[9] H. R. Schwarz, “Ein verfahren zur stabilitatsfrage bei matrizen eigenwertproblemen,” *Math. Phys.*, vol. 7, pp. 473–500, 1956.
[10] F. Esfandiari and H. K. Khalil, “Output feedback stabilization of fully linearizable systems,” *Int. J. Control*, vol. 56, pp. 1007–1037, 1992.
[11] V. Černý, “Asymptotic state reconstruction of non-linear systems,” Department of Cybernetics, Faculty of Applied Sciences, University of West Bohemia, Czech Republic,” Master thesis, 1998.
[12] J. Hrušák and V. Černý, “Nonlinear state reconstruction of linear systems,” in *Proceedings of the 4th International Scientific-Technical Conference Process Control 2000*, Kouty nad Desnou, Czech Republic, June,11.–14. 2000.
[13] V. Černý, “Separation principle for non-linear systems,” Department of Cybernetics, Faculty of Applied Sciences, University of West Bohemia, Czech Republic,” Doctor thesis, 2000.
[14] J. Hrušák and V. Černý, “Generalization of nonlinear observer design via dissipation normal form,” in *Proceedings of the 11th IFAC International Conference CAO 2000*, St. Petersburg, Russia, July,3.–6. 2000.
[15] V. Černý and J. Hrušák, “Non-linear observer design via dissipation normal form,” in *Proceedings of International Scientific Conference of FME*, Ostrava, Czech Republic, September,5.–7. 2000.
[16] V. Černý, “Stabilization, separation and state reconstruction of non-linear systems,” Ph.D. dissertation, University of West Bohemia, Faculty of Applied Sciences, Department of Cybernetics, Univerzitní 8, 306 14 Pilsen, Czech Republic, 2002, (in Czech only).
[17] V. Černý, A. Fialová, and J. Hrušák, “Stabilization of non-linear systems based on metric equivalence,” in *Proceedings of the 9th Mediterranean Conference on Control and Automation MED 2001*, Dubrovnik, Croatia, June,27.–29. 2001.
[18] A. Fialová and V. Černý, “Synthesis of control law for non-linear systems in critical case,” in *Proceedings of the 1st IFAC/IEEE Symposium on System Structure and Control 2001*, Prague, Czech Republic, August,29.–31. 2001.
[19] V. Černý and J. Hrušák, “Synthesis of control law for non-linear systems based on metric equivalence,” in *Proceedings of the IASTED International Conference Control and Applications CA 2002*, Cancun, Mexico, May,20.–22. 2002.

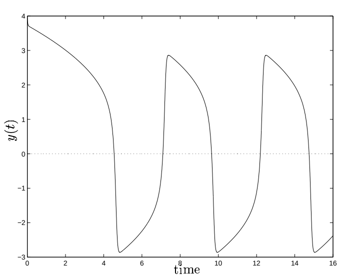

Fig. 1. The response of the system S

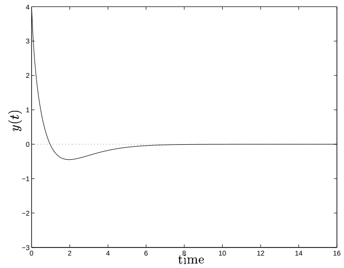

Fig. 2. The response of the closed loop system S_{cl}

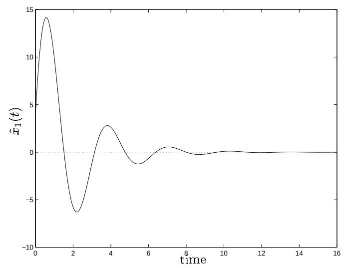

Fig. 3. The course of the first component of the state error

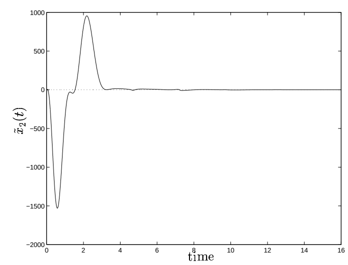

Fig. 4. The course of the second component of the state error

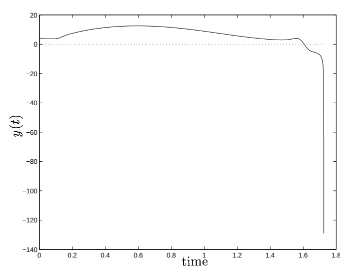

Fig. 5. The response of the resulting closed loop system S_{cl} without a compensation function

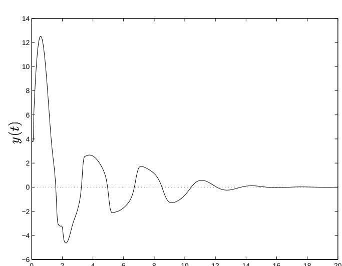

Fig. 6. The response of the resulting closed loop system S_{cl} with the compensation function