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Relationship between infinite eigenvalue assignment for singular systems and solvability of polynomial matrix equations

Abstract - Two associated problems: the problem of infinite eigenvalue assignment and the problem of solvability of polynomial matrix equations are considered. Necessary and sufficient conditions for the existence of solutions to the problems are established. The relationship between the problems are discussed and some applications from the field of the perfect observers design for singular linear systems are presented.

1. INTRODUCTION

It is well-known [1,7,11,6,9] that if a pair (A,B) of standard linear system $\dot{x} = Ax + Bu$ is controllable then there exist a state-feedback gain matrix K such that $\det[I_n s - A + BK] = p(s)$, where $p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$ is a given arbitrary n degree polynomial. By changing K we may modify arbitrarily only the coefficients $a_0, a_1, ..., a_{n-1}$ but we are not able to change the degree n of the polynomial which is determined by the matrix $I_n s$. In singular linear systems we are also able to change the degree of the closed-loop characteristic polynomials by suitable choice of the state-feedback matrix K. The problem of finding of a state-feedback matrix K such that $\det[Es - A + BK] = \alpha \neq 0$ (α is independent of s) has been considered in [3,2].

The polynomial equation approach to linear control systems has been considered in many papers and books [8-10,6].

In this paper a new approach to solve the problems will be proposed. The problem of infinite eigenvalue assignment is closely related with the problem of finding a solution $X = I_n$, $Y = K$ to the polynomial matrix equation $[Es - A]X + BY = U(s)$ for an unimodular matrix $U(s)$ with $\det U(s) = \alpha$.

Necessary and sufficient conditions for the existence of a solution (X, Y) to the polynomial matrix equation will be established. The relationship between the problems will be discussed and some applications from the field of the perfect observers design for singular linear systems will be presented.

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2. PROBLEM FORMULATION

Let $R^{n\times m}$ be the set of $n \times m$ real matrices and $R^n = R^{n\times 1}$. Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$, $x \in R^n$ and $u \in R^m$ are the semistate and input vectors and $E, A \in R^{n\times n}, B \in R^{n\times m}$. The system (1) is called singular if $\det E = 0$ and it is called standard when $\det E \neq 0$.

It is assumed that $rank\, E = r < n, rank\, B = m$ and the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \quad \text{for} \quad \text{some} \quad s \in \mathbf{C} \tag{2}$$

(the field of complex numbers)

Let us consider the system (1) with the state-feedback

$$u = v - Kx \tag{3}$$

where $v \in R^m$ is a new input and $K \in R^{m\times n}$ is a gain matrix.

From (1) and (3) we have

$$E\dot{x} = (A - BK)x + Bv \tag{4}$$

Problem 1. Given matrices E, A, B of (1) and nonzero scalar α (independent of s). Find a $K \in R^{m\times n}$ such that

$$\det[Es - A + BK] = \alpha \tag{5}$$

Let $R^{n\times m}[s]$ be the set of $n \times m$ polynomial matrices in s with real coefficients and $U(s) \in R^{n\times n}[s]$ be a unimodular matrix such that $\det U(s) = \alpha$.

Then (5) can be written as

$$\det\left\{[Es-A,B]\begin{bmatrix} I_n \\ K \end{bmatrix}\right\} = \det U(s)$$ (I_n - the identity matrix)

(6)

and

$$[Es-A]X + BY = U(s)$$

(7)

where

$$X = I_n \qquad \text{and} \qquad Y = K$$

(8)

Therefore, the following problem associated with Problem 1 can be formulated as follows.

Problem 2. Given the matrices $Es-A, B$ and $U(s)$ with $\det U(s) = \alpha$. Find a solution X, Y satisfying (8) of the polynomial matrix equation (7).

In this paper necessary and sufficient conditions for the existence of solutions to the problems will be established and procedures for computation of K will be proposed. The relationship between the problems will be also discussed.

3. SOLUTION OF THE PROBLEM 1

It is well-known [1,6] that the system (1) is completely controllable if and only if

$$rank[Es-A,B] = n \quad \text{for all finite} \quad s \in \mathbf{C}$$

(9a)

and

$$rank[E,B] = n$$

(9b)

The solution of the problem 1 is based on the following lemma [2].

Lemma 1. If the condition (2) is satisfied then there exist orthogonal matrices U, V such that

$$U[Es-A]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix},$$

$$UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \begin{matrix} E_1, A_1 \in R^{n_1 \times n_1} \\ E_0, A_0 \in R^{n_0 \times n_0} \end{matrix}, B_1 \in R^{n_1 \times m}$$

(10a)

where the subsystem (E_1, A_1, B_1) is completely controllable, the pair (E_0, A_0) is regular, E_1 is upper triangular and * denotes an unimportant matrix.

Moreover the matrices E_1, A_1 and B_1 are of the forms

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & & E_{3k}s - A_{3k} \\ \hdashline 0 & 0 & \cdots\ 0 & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} E_{ij}, A_{ij} \in R^{\bar{n}_i \times \bar{n}_j},\ i, j = 1,...,k \\ B_{11} \in R^{\bar{n}_1 \times m}, \sum_{i=1}^{n} \bar{n}_i = n_1 \end{matrix}$$

(10b)

with $B_{11}, A_{21}, ..., A_{k,k-1}$ of full row rank and $E_{22}, ..., E_{kk}$ nonsingular.

Theorem 1. Let the condition (2) be satisfied and the matrices E, A, B of (1) be transformed to the forms (10). There exists a matrix K satisfying the condition (5) if and only if

i) the subsystem (E_1, A_1, B_1) is singular, i.e.

$$\det E_1 = 0$$

(11a)

ii) if $n_0 > 0$ then the degree of the polynomial $\det[E_0 s - A_0]$ is zero, i.e.

$$\deg \det[E_0 s - A_0] = 0 \quad \text{for} \quad n_0 > 0$$

(11b)

Proof. (compare with [2])

Necessity. From (5) and (10a) we have

$$\det[Es - A + BK] =$$
$$\det U^{-1} \det V^{-1} \det[E_1 s - A_1 + B_1 \bar{K}] \det[E_0 s - A_0] = \alpha$$

(12)

where $\bar{K} = KV \in R^{m \times n}$ and $\det\left[E_0 s - A_0\right] = 1$ if $n_0 = 0$.

From (12) it follows that the condition (5) holds only if the conditions (11) are satisfied.

Sufficiency. First let us consider the single-input $(m = 1)$ case. In this case we have

$$E_1 = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \hdashline 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix}, B_1 = b_1 = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(13)

where $e_{ii} \neq 0, a_{i,i-1} \neq 0$ for $i = 2,...,n_1$ and $b_{11} \neq 0$.

The condition (11a) implies that $e_{11} = 0$. Premultiplying the matrix $[E_1 s - A_1, b_1]$ by orthogonal row operations matrix P_1 it is possible to make zero the entries $e_{12}, e_{13},..., e_{1n_1}$ of E_1 since $e_{ii} \neq 0$, $i = 2,...,n_1$. By this reduction only the entries of the first row of A_1 will be modified.

$$\overline{E}_1 = P_1 E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \hdashline 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$\overline{A}_1 = P_1 A_1 = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1,n_1-1} & \overline{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix}, \overline{b}_1 = P_1 b_1 = b_1 \qquad (14)$$

Let

$$\overline{k}_1 = \frac{1}{b_{11}}\left[-\overline{a}_{11}, -\overline{a}_{12}, ..., -\overline{a}_{1,n_1-1}, 1 - \overline{a}_{1n_1}\right] \qquad (15)$$

Using (12), (14) and (15) we obtain

$$\det\left[\overline{E}_1 s - \overline{A}_1 + \overline{b}_1 \overline{k}_1\right] =$$

$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s - a_{22} & \cdots & & e_{2n_1}s - a_{2n_1} \\ 0 & -a_{31} & \cdots & & e_{3n_{1a}}s - a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & & e_{n_1 n_1}s - a_{n_1 n_1} \end{vmatrix} =$$

$$= a_{21} a_{31} \cdots a_{n_1,n_1-1} = \overline{\alpha}$$

(16)

where $\overline{\alpha} = \alpha \det U \det V \det P_1 \det[E_0 s - A_0]^{-1}$.

The considerations can be easily extended for multi-input systems, $m > 1$. In this case the matrix P_1 of the orthogonal row operations is chosen so that all entries of the first row of $\overline{E}_1 = P_1 E_1$ are zero. By this reduction only the entries of $A_{1i}, i = 1,...,k$ and B_{11} will be modified. The modified matrices will be denoted by $\overline{A}_{1i}, i = 1,...,k$ and $\overline{B}_{11}$.

Let

$$\overline{K} = \overline{B}_1^{-1}\left\{\left[\overline{A}_{11}, \overline{A}_{12}, ..., \overline{A}_{1k}\right] + \hat{E}\right\}$$

(17)

The matrix $\hat{E} \in R^{m \times n}$ in (17) is chosen so that

$$\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1} h \\ \overline{a}_{21} & * & \cdots & * & * \\ 0 & \overline{a}_{32} & \cdots & * & * \\ \hdashline 0 & 0 & \cdots & \overline{a}_{l,l-1} & * \end{bmatrix}$$

(18)

(* denotes unimportant entries)

$$h = \frac{\alpha(-1)^{l+1}}{\overline{a}_{21}\overline{a}_{32}...\overline{a}_{l,l-1} c}$$

and $c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det[E_0 s - A_0]$.

Using (12), (17) and (18) it is easy to verify that

$$\det\left[Es - A + BK\right] = c \det\left[\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K}\right] = \alpha \qquad \square$$

(19)

Remark 1. For $m > 1$ there exist many different matrices *K* satisfying the condition (5).

Remark 2. If the order of system is not high (n<5) the elementary row and column operations instead of the orthogonal operations can be used.

Example 1. For the singular system (1) with

$$E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(20)

find the gain matrix $K \in R^{2\times 4}$ such that the condition (5) is satisfied for $\alpha = 1$.

In this case the pair (*E*,*A*) is regular and the matrices (20) have already the desired forms (10) with

$E_1 = E, A_1 = A, B_1 = B$,
$n_1 = n = 4, \overline{n}_1 = 2, \overline{n}_2 = \overline{n}_3 = 1, m = 2$ and

$$E_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, E_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$E_{22} = [1], E_{23} = [-1], E_{33} = [1]$$

$$A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, A_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_{21} = [-1 \;\; 0], A_{22} = [1], A_{23} = [-1], A_{32} = [2], A_{33} = [1]$$

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using the elementary row and column operations [6,7] we obtain

$$P_1 = \begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\left[\overline{E}_1 s - \overline{A}_1, \overline{B}_1\right] = P_1\left[Es - A, B\right] =$$

$$= \begin{bmatrix} -4 & 3 & 5 & -5 & 1 & -2 \\ 1 & s-1 & -1 & 2 & 0 & 1 \\ 1 & 0 & s-1 & 1-s & 0 & 0 \\ 0 & 0 & -2 & s-1 & 0 & 0 \end{bmatrix}$$

Taking into account that in this case

$$\hat{E} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix}, \left[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}\right] =$$
$$= \begin{bmatrix} 4 & -3 & -5 & 5 \\ -1 & 1 & 1 & -2 \end{bmatrix}, \overline{B}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

and using (17) we obtain

$$K = \overline{K} = \overline{B}_1^{-1}\left\{\left[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}\right] + \hat{E}\right\} =$$
$$= \begin{bmatrix} 2 & -2 & -5 & 0 \\ -1 & 1 & 1 & -2.5 \end{bmatrix}$$

If there exist a matrix K satisfying (5) then it can be also computed by the use of the following procedure.
Compute the determinant

$$\det[Es - A + BK] = a_r s^r + a_{r-1}s^{r-1} + \cdots + a_1 s + a_0, \; r < rank\, E$$

(21)

where the coefficients $a_i = a_i(K), i = 0,1,...,r$ depend on the entries of K.
Comparison of the coefficients at the like powers of s of (21) and (5) yields

$$a_0(K) = \alpha, \; a_i(K) = 0, \; i = 1,...,r$$

(22)

Solving the equations (22) we may compute the entries of K.

4. SOLUTION OF PROBLEM 2 AND THE RELATIONSHIP BETWEEN PROBLEMS

Theorem 2. The problem 2 has a solution only if

$$rank[Es - A, B] = n \quad \text{for all finite} \quad s \in \mathbf{C}$$

(23)

and

$D = Es - U(s)$ is a real matrix independent of s.

(24)

Proof. From the equality

$$Es - A + BK = [Es - A, B]\begin{bmatrix} I_n \\ K \end{bmatrix}$$

(25)

it follows that (5) implies (23).
From (7) and (8) we have

$$Es - U(s) = A - BK = D \in R^{n\times n}$$

(26)

Therefore, the equation (7) has a solution (8) only if (24) is satisfied. □

Example 2. Consider the problems for

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha = 1$$

(27)

and

a) $U(s) = \begin{bmatrix} 1 & s \\ 0 & \alpha \end{bmatrix}$ b) $U(s) = \begin{bmatrix} s & 1 \\ -\alpha & 0 \end{bmatrix}$

The problem 1 has a solution since for $K = [k_1 \;\; k_2]$ we have

$$\det[Es - A + BK] = \begin{vmatrix} s + k_1 & k_2 - 1 \\ -1 & 0 \end{vmatrix} = k_2 - 1 = \alpha$$

for $k_2 = 1 + \alpha = 2$ and arbitrary k_1.
The problem 2 in the case a) has no solution since the condition (24) is not satisfied. The matrix

$D = Es - U(s) = \begin{bmatrix} s-1 & -s \\ 0 & -\alpha \end{bmatrix}$ is a polynomial matrix (not a real matrix).

In the case b) the condition (24) is satisfied since the matrix $D = Es - U(s) = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}$ is real.

The Problem 2 has the solution $K = [0 \;\; 2]$ since

$$Es - A + BK = \begin{bmatrix} s+k_1 & k_2 - 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s & 1 \\ -\alpha & 0 \end{bmatrix}$$

and from comparison of the suitable entries we obtain $k_1 = 0, k_2 = 2$. □

Let the matrices E, A and B of (7) satisfy the conditions (23) and (24).

If the system (E, A, B) is completely controllable then by Lemma 1 there exist orthogonal matrices P,Q such that

$$\widetilde{E} = PEQ = \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{12} & \cdots & \widetilde{E}_{1k} \\ 0 & \widetilde{E}_{22} & \cdots & \widetilde{E}_{2k} \\ \hdashline 0 & 0 & \cdots & \widetilde{E}_{kk} \end{bmatrix},$$

$$\widetilde{A} = PAQ = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \cdots & & \widetilde{A}_{1k} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \cdots & & \widetilde{A}_{2k} \\ 0 & \widetilde{A}_{32} & \cdots & & \widetilde{A}_{3k} \\ \hdashline 0 & 0 & \cdots & \widetilde{A}_{k,k-1} & \widetilde{A}_{kk} \end{bmatrix}, \widetilde{B} = PB = \begin{bmatrix} \widetilde{B}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(28)

with $B_{11} \in R^{\hat{n}_1 \times m}, A_{i,i-1} \in R^{\bar{n}_i \times \bar{n}_{i-1}}, i = 2,...,k$ of full row rank and $E_{11} \in \in R^{\bar{n}_i \times \bar{n}_i}, i = 2,...,k$ nonsingular.

Premultiplying the equation (7) with (8) by the matrix P, postmulplying by the matrix Q and using (28) we obtain

$$P[Es - A]Q + PBKQ = \widetilde{E}s - \widetilde{A} + \widetilde{B}\widetilde{K} = \widetilde{U}(s)$$

(29)

where $\widetilde{K} = KQ$ and $\widetilde{U}(s) = PU(s)Q$.

From the equality

$$P[Es - U(s)]Q = PDQ = \widetilde{D} = \widetilde{E}s - \widetilde{U}(s)$$

it follows that if D is a real matrix then $\widetilde{D}$ is also a real matrix.

Let $\widetilde{D} = \begin{bmatrix} \widetilde{D}_1 \\ \widetilde{D}_2 \end{bmatrix}$ and $\widetilde{A} = \begin{bmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{bmatrix}$,

where $\widetilde{D}_1, \widetilde{A}_1 \in R^{\bar{n}_1 \times n}, \widetilde{D}_2, \widetilde{A}_2 \in R^{(n-\bar{n}_1) \times n}$

Then from (28) and (29) we have

$$\begin{bmatrix} \widetilde{D}_1 \\ \widetilde{D}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{bmatrix} - \begin{bmatrix} \widetilde{B}_1 \\ 0 \end{bmatrix} \widetilde{K}$$

and

$$\widetilde{D}_1 = \widetilde{A}_1 - \widetilde{B}_1 \widetilde{K}, \widetilde{D}_2 = \widetilde{A}_2$$

(30)

Therefore we have the following theorem

Theorem 3. Let the matrices E,A,B satisfy the assumptions (9) and (26) and let the matrices be transformed to the forms (28). The equation (7) has a solution X,Y satisfying (8) if and only if

$$\widetilde{D}_2 = \widetilde{A}_2$$

(31)

Proof. The necessity of (31) follows immediately from (30). If the assumption (26) is satisfied then D is a real matrix and $\widetilde{D}$ is also a real matrix. The matrix $\widetilde{B}_1$ is nonsingular and from (30) we obtain

$$\widetilde{K} = \widetilde{B}_1^{-1}[\widetilde{A}_1 - \widetilde{D}_1]$$

and

$$X = K = \widetilde{K}Q^{-1} = \widetilde{B}_1^{-1}[\widetilde{A}_1 - \widetilde{D}_1]Q^{-1} \qquad \square$$

(32)

Remark 3. From comparison of the Theorems 2 and 3 and Example 2 it follows that the solvability conditions for Problem 2 are more restrictive than for the Problem 1.

Example 3. Find a solution (8) of the equation (7) with

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, U(s) = \begin{bmatrix} 1 & -1 & s \\ 0 & -\alpha & 0 \\ 0 & s-2 & -1 \end{bmatrix}$$

(33)

In this case the assumptions (9) and (26) are satisfied and the matrix

$$D = Es - U(s) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(34)

is real.

The orthogonal matrices $P, Q \in R^{3\times 3}$ transforming (33) to the forms (28) have the form

$$P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},Q=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

(35)

and

$$\widetilde{E}=PEQ=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$\widetilde{A}=PAQ=\begin{bmatrix}\widetilde{A}_1\\\widetilde{A}_2\end{bmatrix}=\begin{bmatrix}1&-1&2\\-1&0&1\\0&1&2\end{bmatrix},$$

$$\widetilde{B}=PB=\begin{bmatrix}\widetilde{B}_1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

(36)

Using (30), (34) and (35) we obtain

$$\widetilde{D}=PDQ=\begin{bmatrix}\widetilde{D}_1\\\widetilde{D}_2\end{bmatrix}=\begin{bmatrix}0&0&\alpha\\-1&0&1\\0&1&2\end{bmatrix}$$

(37)

From (36) and (37) it follows that the condition (31) is satisfied and the equation (7) with (33) has the solution X, Y satisfying (8).

Using (32), (36) and (37) we obtain

$$X=K=\widetilde{B}_1^{-1}\left[\widetilde{A}_1-\widetilde{D}_1\right]Q^{-1}=$$
$$[1,\ -1,\ 2-\alpha]\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=[1,\ 2-\alpha,\ -1]$$

(38)

It is easy to check that (38) and $Y=I_3$ satisfy the equation (7) with (33).

5. APPLICATIONS

Consider the singular system

$$E\dot{x}=Ax+Bu$$

(39a)

$$y=Cx \qquad (39b)$$

where $x\in R^n$, $u\in R^m$ and $y\in R^p$ are the semistate, input and output vectors, respectively and $E,A\in R^{n\times n},B\in R^{n\times m},C\in R^{p\times n}$ with $\det E=0$.

It is assumed that $rank\,C=p$ and (2) holds.

The singular system

$$E\dot{\hat{x}}=A\hat{x}-Bu-K(C\hat{x}-y),$$
$$\hat{x}(0)=\hat{x}_0,\ \hat{x}\in R^m, K\in R^{n\times p}$$

(40)

is called full order perfect observer of the system (39) if and only if $\hat{x}(t)=x(t)$ for $t>0$ and any initial conditions x_0 and $\hat{x}_0$ of (39) and (40).

It was shown [4] that there exists a full-order perfect observer (40) of the system (39) if the system is completely observable, i.e.

$$rank\begin{bmatrix}Es-A\\C\end{bmatrix}=n \qquad \text{for all finite } s\in\mathbf{C}$$

(41a)

and

$$rank\begin{bmatrix}E\\C\end{bmatrix}=n$$

(41b)

In this case there exists a matrix K such that

$$\det[Es-A+KC]=\alpha$$

(42)

(a nonzero scalar independent of s)

Note that by transposition of (42) we obtain (5). Therefore, the design problem of the
observer (40) for the system (39) has been reduced to the Problem 1.

The design problem of reduced-order perfect observers and of perfect functional
observers for the system (39) can be also reduced to the Problem 1. [4,5]

Consider the singular system (39) with the state-feedback (3). The transfer matrix of the closed-loop system described by (4) and (39b) is given by $T(s)=C[Es-A+BK]^{-1}B$. If $[Es-A+BK]=U(s)$ with $U(s)$ unimodular then the transfer matrix $T(s)=CU^{-1}(s)B$ is a polynomial matrix. Therefore, finding a solution (8) of (7) is equivalent to finding a state-feedback gain matrix K such that the closed-loop transfer matrix is polynomial.

6. CONCLUDING REMARKS.

Two associated problems: the problem of infinite eigenvalue assignment and the problem of solvability of polynomial matrix equations have been considered. Necessary and sufficient conditions for the existence of solutions to the problems have been established. The relationship between the problems has been discussed and some applications from the field of the perfect observers design for singular linear systems have been presented. The considerations have been illustrated by numerical examples. With slight modifications the considerations can be extended for singular discrete-time linear systems. An extension of the considerations for two-dimensional linear systems [6] is also possible.

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