Output Feedback Nonlinear $\mathcal{H}_\infty$ Control of Underactuated Manipulators

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Abstract— **This paper presents a control system for underactuated manipulators based on output feedback nonlinear $\mathcal{H}_\infty$ control, where only the position is available. Experimental results are also presented.**

I. Introduction

Parametric uncertainties and exogenous disturbances increase the difficulty of reference tracking control for robotic manipulators. Additionally, actuator fault can suddenly occur during the manipulator control and, if the robot is working in hazardous or unstructured environment, the movement must be completed according to the manipulator fault configuration. Among the actuator fault types, the free torque fault, where the torque supply in the motor of each joint breaks down suddenly, can turn the system uncontrolled with the possibility of damage for the manipulator components. When a free torque fault occurs the fully actuated manipulator changes to a underactuated configuration. This kind of mechanical system, with less actuators than degrees of freedom, is of interest of many researchers [1], [2], [3], [4]. A control strategy for underactuated manipulator was first proposed in [1]. Following that strategy, firstly all the passive joints (without actuators) are controlled to the desired final position and then, with the passive joints braked, the active ones (with actuators) are controlled. In [2], three possibilities of selecting the joints to be controlled in each phase are proposed. In the above references, the control strategies were based on state feedback control, positions and velocities. In this paper, we apply the output feedback nonlinear H_∞ control technique for underactuated manipulators, using only position measurements.

Nonlinear $\mathcal{H}_\infty$ control consists in guarantee that the $\mathcal{L}_2$ gain between the disturbance and the output be bounded by an attenuation level γ. The development of the linear parameter varying (LPV) technique, considered in this paper, provides a systematic way to design controllers that schedule on varying parameters of the system and satisfy the $\mathcal{L}_2$ gain condition [6]. The nonlinear dynamics can be represented as LPV system with the parameters as function of the state, namely, quasi-LPV representation.

In most cases, when only the position is measured directly, the velocity signal is obtained via position derivative, and controllers based on state feedback are used. Also, in most cases, it is need to use filters in this velocity to avoid noises, that can generate delays. An output feedback control system can be used in order to avoid this problem since one can use only the available states in the synthesis of the controller without affecting its performance. [5] and [6] show methodologies to obtain such controllers for LPV systems.

This paper is organized as follows: the output feedback $\mathcal{H}_\infty$ control via a quasi-LPV representation is presented in Section II; in Section III, the quasi-LPV representation of the underactuated manipulator is showed; and results using this technique, and obtained from the experimental manipulator UArm II, are presented in Section IV. We denote $\mathcal{C}^1(\Re^{\mathrm{m}}, \Re^{\mathrm{n}})$ as the set of continuously differentiable functions that map $\Re^m$ to $\Re^n$. The Euclidean norm of a vector is denoted by $\|.\|$, i.e., $\|z\|^2 = z^T z$ for $z \in \Re^k$. The notation $\mathcal{L}_2$ will be used for bounded energy signals, i.e., $\mathcal{L}_2(0,\mathrm{T}) = \left\{ \mathrm{w} : \int_0^{\mathrm{T}} \|\mathrm{w(t)}\|^2 \, \mathrm{dt} < \infty \right\}$.

II. Output feedback nonlinear $\mathcal{H}_\infty$ control via quasi-LPV representation

In this section, the output feedback $\mathcal{H}_\infty$ control problem for LPV systems is presented. A parameter dependent controller, that stabilizes the closed loop and guaranties that the $\mathcal{L}_2$ gain between the disturbance and the output be bounded by an attenuation level γ, is considered. This problem can also be defined for special LPV systems where the parameters are some of the states of the system, named, quasi-LPV systems. Consider the LPV system:

$$\begin{aligned}
\dot{x} &= A(\rho(t))x + B_{11}(\rho(t))d_1 + B_{12}(\rho(t))d_2 + B_2(\rho(t))u \\
e_1 &= C_{11}(\rho(t))x + D_{1111}(\rho(t))d_1 + D_{1112}(\rho(t))d_2 \\
e_2 &= C_{12}(\rho(t))x + D_{1121}(\rho(t))d_1 + D_{1122}(\rho(t))d_2 + u \\
y &= C_2(\rho(t))x + d_2
\end{aligned} \tag{1}$$

where $d_1 \in \Re^{n_{d1}}$, $d_2 \in \Re^{n_{d2}}$, $e_1 \in \Re^{n_{e1}}$ and $e_2 \in \Re^{n_{e2}}$.

Assume that the underlying parameter ρ varies in the allowable set

$$F_P^\nu = \left\{ \rho \in \mathcal{C}^1(\Re^+, \Re^m) : \rho(\mathrm{t}) \in P, |\dot{\rho}_i| \leq \nu_i, i = 1, \ldots, m \right\}$$

where $P \subset \Re^m$ is a compact set and $\{\nu_i\}_{i=1}^m$ are nonnegative numbers.

The system (1) has $\mathcal{L}_2 - \mathrm{gain} \leq \gamma$ if

$$\int_0^T \|e(t)\|^2\, dt \leq \gamma^2 \int_0^T \|d(t)\|^2\, dt$$

for all $T \geq 0$ and all $u \in \mathcal{L}_2(0,\mathrm{T})$ with the system starting from $x(0) = 0$.

Suppose a m-dimensional parameter dependente controller K_P:

$$\begin{aligned} \dot{x}_k &= A_K(\rho(t),\dot{\rho}(t))x_k + B_K(\rho(t),\dot{\rho}(t))y \\ u &= C_K(\rho(t),\dot{\rho}(t))x_k + D_K(\rho(t),\dot{\rho}(t))y \end{aligned} \tag{2}$$

where $\rho \in F_P^\nu$ and $x_k(t) \in \Re^m$ is the controller state. Note that the controller depends on ρ and $\dot{\rho}$.

Defining $x_{clp}^T(t) = [x^T(t) \;\; x_k^T(t)]$, $e^T(t) = [e_1^T(t) \;\; e_2^T(t)]$ and $d^T = [d_1^T(t) \;\; d_2^T(t)]$, then the closed loop LPV system is given by:

$$\begin{aligned} \dot{x}_{clp} &= A_{clp}(\rho(t),\dot{\rho}(t))x_{clp} + B_{clp}(\rho(t),\dot{\rho}(t))d \\ e &= C_{clp}(\rho(t),\dot{\rho}(t))x_{clp} + D_{clp}(\rho(t),\dot{\rho}(t))d \end{aligned}$$

where

$$\begin{aligned} A_{clp} &= \begin{bmatrix} A(\rho) + B_2(\rho)D_K(\rho,\dot{\rho})C_2(\rho) & B_2(\rho)C_K(\rho,\dot{\rho}) \\ B_K(\rho,\dot{\rho})C_2(\rho) & A_K(\rho,\dot{\rho}) \end{bmatrix} \\ B_{clp} &= \begin{bmatrix} B_{11}(\rho) & B_{12}(\rho) + B_2(\rho)D_K(\rho,\dot{\rho}) \\ 0 & B_K(\rho,\dot{\rho}) \end{bmatrix} \\ C_{clp} &= \begin{bmatrix} C_{11}(\rho) & 0 \\ C_{12}(\rho) + D_K(\rho,\dot{\rho})C_2(\rho) & C_K(\rho,\dot{\rho}) \end{bmatrix} \\ D_{clp} &= \begin{bmatrix} D_{1111}(\rho) & D_{1112}(\rho) \\ D_{1121}(\rho) & D_{1122}(\rho) + D_K(\rho,\dot{\rho}) \end{bmatrix} \end{aligned}$$

Lemma II.1: [5] Given the open loop LPV system, Equation 1 and the performance level $\gamma > 0$. If there exist an integer $m \geq 0$, a function $W \in \mathcal{C}^\infty$ and continuous matrix functions $(A_K,\; B_K,\; C_K,\; D_K)$ such that $W(\rho) > 0$ and

$$\begin{bmatrix} E(\rho) & W(\rho)B_{clp}(\rho,\beta) & \gamma^{-1}C_{clp}^T(\rho,\beta) \\ B_{clp}^T(\rho,\beta)W(\rho) & -I_{n_d} & \gamma^{-1}D_{clp}^T(\rho,\beta) \\ \gamma^{-1}C_{clp}(\rho,\beta) & \gamma^{-1}D_{clp}(\rho,\beta) & -I_{n_e} \end{bmatrix} < 0 \tag{3}$$

where

$$E(\rho) = A_{clp}^T(\rho,\beta)W(\rho) + W(\rho)A_{clp}(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial W}{\partial \rho_i}\right)$$

for all $\rho \in P$ and $|\beta_i| \leq \nu_i$, $i = 1, \cdots, s$, then the closed loop LPV system with the controller K_P defined in (2) is stable and has $\mathcal{L}_2 - \mathrm{gain} \leq \gamma$.

$\square$

To simplify the notation, denote:

$$\begin{bmatrix} D_{111.}(\rho) \\ D_{112.}(\rho) \end{bmatrix} = \begin{bmatrix} D_{1111}(\rho) & D_{1112}(\rho) \\ D_{1121}(\rho) & D_{1122}(\rho) \end{bmatrix}$$

$$\begin{bmatrix} D_{11.1}(\rho) & D_{11.2}(\rho) \end{bmatrix} = \begin{bmatrix} D_{1111}(\rho) & D_{1112}(\rho) \\ D_{1121}(\rho) & D_{1122}(\rho) \end{bmatrix}$$

Theorem II.1 ([5]) Given the LPV system, Equation (1), and the compact set P. A controller K_P will be found if and only if there exist matrix functions $X \in \mathcal{C}^\infty$ and $Y \in \mathcal{C}^\infty$, such that for all $\rho \in P$, $X(\rho) > 0$, $Y(\rho) > 0$, and

$$\begin{bmatrix} \hat{E}(\rho) & X(\rho)C_{11}^T(\rho) & \gamma^{-1}\hat{B}(\rho) \\ C_{11}(\rho)X(\rho) & -I_{n_{e1}} & \gamma^{-1}D_{111.}(\rho) \\ \gamma^{-1}\hat{B}^T(\rho) & \gamma^{-1}D_{111.}^T(\rho) & -I_{n_d} \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} \tilde{E}(\rho) & Y(\rho)B_{11}^T(\rho) & \gamma^{-1}\tilde{C}^T(\rho) \\ B_{11}^T(\rho)Y(\rho) & -I_{n_{d1}} & \gamma^{-1}D_{11.1}^T(\rho) \\ \gamma^{-1}\tilde{C}(\rho) & \gamma^{-1}D_{11.1}(\rho) & -I_{n_e} \end{bmatrix} < 0, \tag{5}$$

$$\begin{bmatrix} X(\rho) & \gamma^{-1}I_n \\ \gamma^{-1}I_n & Y(\rho) \end{bmatrix} \geq 0 \tag{6}$$

where

$$\begin{aligned} \hat{E}(\rho) &= -\sum_{i=1}^{m} \pm\left(\nu_i \frac{\partial X}{\partial \rho_i}\right) + \hat{A}(\rho)X(\rho) + \\ &\quad X(\rho)\hat{A}(\rho)^T - B_2(\rho)B_2^T(\rho), \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{E}(\rho) &= \sum_{i=1}^{m} \pm\left(\nu_i \frac{\partial Y}{\partial \rho_i}\right) + \tilde{A}^T(\rho)Y(\rho) + \\ &\quad Y(\rho)\tilde{A}(\rho)^T - C_2^T(\rho)C_2(\rho) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \hat{A}(\rho) &= A(\rho) - B_2(\rho)C_{12}(\rho) \\ \hat{B}(\rho) &= B_1(\rho) - B_2(\rho)D_{112.}(\rho) \\ \tilde{A}(\rho) &= A(\rho) - B_{12}(\rho)C_2(\rho) \\ \tilde{C}(\rho) &= C_1(\rho) - D_{11.2}(\rho)B_2(\rho) \end{aligned}$$

If the conditions are satisfied, then by continuity and compactness, it is possible to perturb $X(\rho)$ such that the two LMIs (4 and 5) still hold and $Q(\rho) = Y(\rho) - \gamma^{-2}X^{-1}(\rho) > 0$ uniformly on P. Define:

$$\begin{aligned} \Omega(\rho) &= -D_{1122}(\rho) - D_{1121}(\rho)[\gamma^2 I_{n_{d1}} - \\ &\quad D_{1111}^T(\rho)D_{1111}]^{-1}D_{1111}^T(\rho)D_{1112}, \\ \bar{A}(\rho) &= A(\rho) + B_2(\rho)\Omega(\rho)C_2(\rho), \\ \bar{B}_1(\rho) &= B_1(\rho) + B_2(\rho)\Omega(\rho)D_{21}, \\ \bar{C}_1(\rho) &= C_1(\rho) + D_{12}\Omega(\rho)C_2(\rho), \\ \bar{D}_{11}(\rho) &= D_{11}(\rho) + D_{12}\Omega(\rho)C_2(\rho), \\ \bar{D}_h(\rho) &= [I_{n_e} - \gamma^{-2}\bar{D}_{11}(\rho)\bar{D}_{11}^T(\rho)]^{-1}, \\ \bar{D}_t(\rho) &= [I_{n_d} - \gamma^{-2}\bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)]^{-1} \end{aligned}$$

and

$$\begin{aligned} F(\rho) &= -(D_{12}^T D_h(\rho)D_{12})^{-1} \star [(B_2(\rho) \\ &\quad +\gamma^{-2}\bar{B}_1(\rho)\bar{D}_{11}^T(\rho)D_h(\rho)D_{12})^T X^{-1}(\rho) \\ &\quad +D_{12}^T D_h(\rho)\bar{C}_1(\rho)], \end{aligned}$$

$$\begin{aligned} L(\rho) &= -[Y^{-1}(\rho)(C_2(\rho) + \gamma^{-2}D_{21}D_t(\rho)\bar{D}_{11}^T(\rho)\bar{C}_1(\rho))^T \\ &\quad +\bar{B}_1(\rho)D_t(\rho)D_{21}^T] \star (D_{21}D_t(\rho)D_{21}^T)^{-1}, \end{aligned}$$

$$\begin{aligned} H(\rho,\dot{\rho}) &= -[X^{-1}(\rho)A_F(\rho) + A_F^T(\rho)X^{-1}(\rho) \\ &\quad + \textstyle\sum_{i=1}^{s}\dot{\rho}_i \frac{\partial X^{-1}}{\partial \rho_i} + C_F^T(\rho)C_F(\rho) \\ &\quad +(X^{-1}(\rho)\bar{B}_1(\rho) + C_F^T(\rho)\bar{D}_{11}(\rho)) \\ &\quad \star(\gamma^{-2}I - \bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho))^{-1}(\bar{B}_1^T(\rho)X^{-1}(\rho) \\ &\quad +\bar{D}_{11}^T(\rho)C_F(\rho))], \end{aligned}$$

with $A_F(\rho) = \bar{A}(\rho) + B_2(\rho)F(\rho)$ and $C_F(\rho) = \bar{C}_1(\rho) + D_{12}(\rho)F(\rho)$. Let:

$$\begin{aligned} M(\rho,\dot{\rho}) \;=\; & H(\rho,\dot{\rho}) + F^T(\rho)[B_2^T(\rho)X^{-1}(\rho) + D_{12}^T(\bar{C}_1(\rho)+ \\ & D_{12}(\rho)F(\rho))][\gamma^2 Q(\rho)(-Q^{-1}(\rho)Y(\rho)L(\rho)D_{21} \\ & -\bar{B}_1(\rho)) + F^T(\rho)D_{12}^T\bar{D}_{11}(\rho)] \\ & \star[\gamma^{-2}I - \bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)]^{-1}[\bar{B}_1^T(\rho)X^{-1}(\rho)+ \\ & \bar{D}_{11}^T(\rho)(\bar{C}_1(\rho) + D_{12}(\rho)F(\rho))]. \end{aligned}$$

A m-dimensional proper controller K_P that solve the output feedback problem is given by:

$$\begin{aligned} A_K(\rho,\dot{\rho}) &= \bar{A}(\rho) + B_2(\rho)F(\rho) + Q^{-1}(\rho)Y(\rho)L(\rho)C_2(\rho) \\ & \quad -\gamma^{-2}Q^{-1}(\rho)M(\rho,\dot{\rho}), \\ B_K(\rho) &= -Q^{-1}(\rho)Y(\rho)L(\rho), \\ C_K(\rho) &= F(\rho), \\ D_K(\rho) &= \Omega(\rho). \end{aligned}$$

□

The proposed controller can be applied to a nonlinear system represented in the quasi-LPV form:

$$\begin{aligned} \dot{x} &= A(\rho(x))x + B_{11}(\rho(x))d_1 + B_{12}(\rho(x))d_2 + B_2(\rho(x))u \\ e_1 &= C_{11}(\rho(x))x + D_{1111}(\rho(x))d_1 + D_{1112}(\rho(x))d_2 \\ e_2 &= C_{12}(\rho(x))x + D_{1121}(\rho(x))d_1 + D_{1122}(\rho(x))d_2 + u \\ y &= C_2(\rho(x))x + d_2 \end{aligned} \tag{9}$$

where the parameter ρ contains some of the state variables.

A practical scheme using basis functions for $X(\rho)$ and $Y(\rho)$, and gridding the parameter set P was developed to solve the constraints (4), (5) and (6). First choose a set of $\mathcal{C}^1$ functions $\{f_i(\rho)\}_{i=1}^M$ as a basis for $X(\rho)$

$$X(\rho) = \sum_{i=1}^{M} f_i(\rho)X_i$$

where $X_i \in S^{n\times n}$ is the coefficient matrix for $f_i(\rho)$, and a set of $\mathcal{C}^1$ functions $\{g_i(\rho)\}_{i=1}^M$ as a basis for $Y(\rho)$

$$Y(\rho) = \sum_{i=1}^{M} g_i(\rho)Y_i$$

where $Y_i \in S^{n\times n}$ is the coefficient matrix for $g_i(\rho)$.

If the matrices $X(\rho)$ and $Y(\rho)$, as given above, are substituted in (4), (5) and (6), the constraint turn to a set of LMI in terms of the matrix variables $\{X_i\}_{i=1}^M$ and $\{Y_i\}_{i=1}^M$ when the parameter ρ is fixed. To solve this infinite dimensional optimization problem, we can grid the parameter set P in L points $\{\rho_k\}_{k=1}^L$ in each dimension. Since (4), (5) and (6) consists of $2^{m+1}+1$ constraints, a total of $(2^{m+1}+1)L^m$ matrix inequalities in terms of the matrices X_i and Y_i have to be solved.

There is no method known to find the basis functions for the matrices $X(\rho)$ and $Y(\rho)$. The basis functions used in Section IV where obtained by experimenting several functions and choosing one that fits a solution.

The parameter set P was obtained using a developed MatLab program that ask the boundary values and the number of grids one wishes to use. This program is also used to find the solution for (4), (5) and (6).

III. Quasi-LPV representation of a underactuated manipulator

The dynamic equations of a robot manipulator can be found by the Lagrange theory as

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) \tag{10}$$

where $q \in \Re^n$ are the joint positions, $M(q) \in \Re^{n\times n}$ is the symmetric positive definite inertia matrix, $C(q,\dot{q}) \in \Re^{n\times n}$ is the Coriolis and centripetal matrix, $F(\dot{q}) \in \Re^n$ are the frictional torques, $G(q) \in \Re^n$ are the gravitational torques and $\tau \in \Re^n$ are the applied torques. The parametric uncertainty can be introduced dividing the parameter matrices $M(q)$, $C(q,\dot{q})$, $F(\dot{q})$ and $G(q)$ into a nominal and a perturbed part:

$$\begin{aligned} M(q) &= M_0(q) + \Delta M(q) \\ C(q,\dot{q}) &= C_0(q,\dot{q}) + \Delta C(q,\dot{q}) \\ F(\dot{q}) &= F_0(\dot{q}) + \Delta F(\dot{q}) \\ G(q) &= G_0(q) + \Delta G(q) \end{aligned}$$

where $M_0(q)$, $C_0(q,\dot{q})$, $F_0(\dot{q})$ and $G_0(q)$ are the nominal matrices, and $\Delta M(q)$, $\Delta C(q,\dot{q})$, $\Delta F(\dot{q})$ and $\Delta G(q)$ are the parametric uncertainties. A finite energy exogenous disturbance, τ_d, can also be introduced. After these considerations the Equation (10) becomes:

$$\tau + \delta = M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + F_0(\dot{q}) + G_0(q) \tag{11}$$

with

$$\delta = -(\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta F(\dot{q}) + \Delta G(q) - \tau_d).$$

Underactuated robot manipulators are mechanical systems with less actuators than degrees of freedom. For this reason, the control of the passive joints (joint without actuator) is made considering the dynamic coupling between them and the active joints (with actuator). Here, we consider that the passive joints have brakes. The strategy is control all the passive joints to reach the desired position, applying torques in the active ones, and then turn on the brakes. After that, all the active joints are controlled by themselves.

Consider a manipulator with n joints, of which n_p are passive and n_a are active joints. It is known [1] that, using breaks, no more than n_a joints of the manipulator can be controlled at every instant. Using this, we group the n_a joints being controlled in the vector $q_c \in \Re^{n_a}$. The remaining joints are grouped in the vector $q_r \in \Re^{n-n_a}$. There are three possibilities of forming the vector q_c [2]:

1. q_c contains only passive joints: when $n_p \geq n_a$ and all other passive joints, if any, are kept locked.
2. q_c contains passive and active joints: all other passive joints, if any, are kept locked.
3. q_c contains active joints.

With these possibilities in mind we can define the control strategy: first, choose the vector q_c as the possibilities 1 or 2 (according to n_p), until all passive joints have reached the desired position; second, choose q_c as the possibility 3 and control the active joints to the desired position.

The dynamic equation (11) can now be partitioned as:

$$\begin{bmatrix} \tau_a \\ 0 \end{bmatrix} + \begin{bmatrix} \delta_a \\ \delta_u \end{bmatrix} = \begin{bmatrix} M_{ar} & M_{ac} \\ M_{ur} & M_{uc} \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_c \end{bmatrix} +$$

$$\begin{bmatrix} C_{ar} & C_{ac} \\ C_{ur} & C_{uc} \end{bmatrix} \begin{bmatrix} \dot{q}_r \\ \dot{q}_c \end{bmatrix} + \begin{bmatrix} F_a \\ F_u \end{bmatrix} + \begin{bmatrix} G_a \\ G_u \end{bmatrix} \tag{12}$$

where the indices a and u represent the active and free (breaks not actioned) passive joints, respectively. Isolating the vetor $\ddot{q}_r$ in the second line of (12) and substituting in the first one, we have:

$$\tau_a + \overline{\delta} = \overline{M}_0\ddot{q}_c + \overline{C}_0\dot{q}_c + \overline{D}_0\dot{q}_r + \overline{F}_0 + \overline{G}_0$$

with

$$\begin{aligned}
\overline{M}_0(q) &= M_{ac} - M_{ar}M_{ur}^{-1}M_{uc} \\
\overline{C}_0(q,\dot{q}) &= C_{ac} - M_{ar}M_{ur}^{-1}C_{uc} \\
\overline{D}_0(q,\dot{q}) &= C_{ar} - M_{ar}M_{ur}^{-1}C_{ur} \\
\overline{F}_0(\dot{q}) &= F_a - M_{ar}M_{ur}^{-1}F_u \\
\overline{G}_0(q) &= G_a - M_{ar}M_{ur}^{-1}G_u \\
\overline{\delta}(q) &= \delta_a - M_{ar}M_{ur}^{-1}\delta_u.
\end{aligned}$$

The state is defined as:

$$x_c = \begin{bmatrix} \dot{q}_c \\ q_c \end{bmatrix}. \tag{13}$$

Hence, a quasi-LPV representation of the underactuated manipulator can be defined as follows

$$\dot{x}_c = A(q,\dot{q})x_c + B(q)u + B(q)\overline{\delta} \tag{14}$$

with

$$\begin{aligned}
A(q,\dot{q}) &= \begin{bmatrix} -\overline{M}_0^{-1}(q)\left(\overline{C}_0(q,\dot{q}) + \overline{F}_0(\dot{q})\right) & 0 \\ I_{n_a} & 0 \end{bmatrix} \\
B(q) &= \begin{bmatrix} I_{n_a} \\ 0 \end{bmatrix} \\
u &= \tau_a - \overline{D}_0(q,\dot{q})(\dot{q}_r - \overline{G}_0(q)).
\end{aligned}$$

To apply the technique described in Section II, the robot manipulator has to be represented according to the Equation (9). Consider as system disturbances, the desired position, q_c^d, and the combined torque disturbance, δ, that is: $d_1 = q_c^d$ and $d_2 = \overline{\delta}$. The system outputs, e_1 and e_2, are the position error, $[q_c^d - q_c]$, and control input, u, respectively. The control output is the position error, $y = [q_c^d - q_c]$, since we only have the position measure directly. Using these definitions in (9), we avoid using the velocities of the joints as a parameter, once we cannot measure them directly using the encoders available on the manipulator. Hence, the robot system can be described by Equation (9) with:

$$\begin{aligned}
A(\rho(x_c)) &= A(q) \\
B_{11}(\rho(x_c)) &= B(q) \\
B_{12}(\rho(x_c)) &= 0 \\
B_2(\rho(x_c)) &= B(q) \\
C_{11}(\rho(x_c)) &= [0 \quad -I] \\
C_{12}(\rho(x_c)) &= 0 \\
C_2(\rho(x_c)) &= [0 \quad -I] \\
D_{22}(\rho(x_c)) &= 0 \\
D_{1111}(\rho(x_c)) &= 0 \\
D_{1112}(\rho(x_c)) &= I \\
D_{1121}(\rho(x_c)) &= 0 \\
D_{1122}(\rho(x_c)) &= 0 \\
D_{12}(\rho(x_c)) &= [0 \quad -I]^T \\
D_{21}(\rho(x_c)) &= [0 \quad -I]
\end{aligned}$$

where the matrices $A(q)$ and $B(q)$ are obtained by Equation (14), with $\dot{q} = 0$.

IV. Experimental Results

The proposed $\mathcal{H}_\infty$ control was applied to our experimental underactuated manipulator UArm II (Underactuated Arm II), designed and built by H. Ben Brown, Jr. of Pittsburgh, PA, USA (Fig. 1). This 3-link manipulator has special-purpose joints containing each an actuator and a brake, so that they can act as active or passive joints. The manipulator configuration can be changed enabling or not the DC motor of each joint. All possible configurations, according to the active (A) and passive (P) joints location in the arm, are accept: AAA, AAP, APA, PAA, APP and PAP. For example, the configuration AAP means that the joints 1 and 2 are active and 3 is passive.

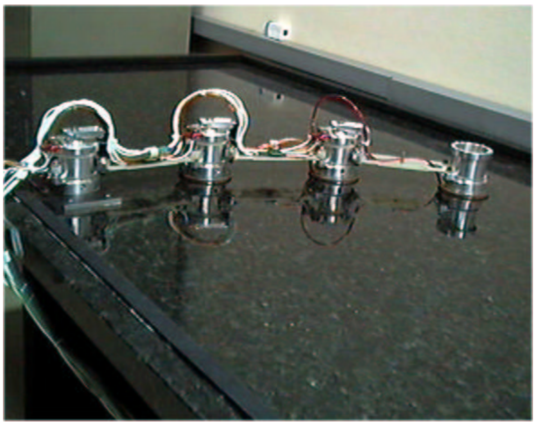

Fig. 1. Underactuated Arm II.

The manipulator's kinematic and dynamic nominal parameters, which are used to calculate the nominal matrices $M_0(q)$, $C_0(q,\dot{q})$ (see Appendix) and $F_0(\dot{q})$, are shown in Table I, where F_i is the coefficient of a velocity-dependent frictional term $F_0(\dot{q})$.

The underactuated configuration used to validate the proposed methodology was the APA configuration, i.e., the joint 2 is passive and the joints 1 and 3 are actives. For this configuration, two phases of control are necessary to

TABLE I
ROBOT PARAMETERS.

i	m_i	I_i	L_i	lc_i	F_i
	(kg)	(kgm^2)	(m)	(m)	(kgm^2/s)
1	0.850	0.0075	0.203	0.096	0.28
2	0.850	0.0075	0.203	0.096	0.18
3	0.625	0.0060	0.203	0.077	0.10

control all joints to the set-point, as showed in Section III. In the first phase, $q_c = [q_2]^T$, and, in the second one, $q_c = [q_1\ q_3]^T$.

For the experiment, the initial position and the desired final position were, respectively, $q(0) = [0°\ 0°\ 0°]^T$ and $q(T^1, T^2) = [20°\ 20°\ \ -20°]^T$, where $T^1 = [1.0]\ s$ and $T^2 = [5.0\ 5.0]\ s$ are the trajectory duration time for the phases 1 and 2, respectively.

For the first phase, the parameters ρ chosen are the state representing the position of the joint 2, i.e.:

$$\rho(x_c) = \left[\ q_2\ \right]^T.$$

The compact set , P, is defined as $\rho \in [0,\ 20]°$. The parameter variation rate is bounded by $|\dot{\rho}| \leq [60°/s]$. Since the value $\dot{\rho}$ cannot be measured directly, and $\mathrm{H}(\rho, \dot{\rho})$ has a term that depends on it, the matrix $\mathrm{X}(\rho)$ will be defined as a constant, leading the $\dot{\rho}$ dependency term to zero:

$$\textstyle\sum_{i=1}^{s} \dot{\rho}_i \frac{\partial X^{-1}}{\partial \rho_i} \quad = \quad 0.$$

The basis functions chosen are then:

$$f_1(\rho(\tilde{x}_c)) \quad = \quad 1.$$

and

$$\begin{aligned} g_1(\rho(\tilde{x}_c)) &= 1 \\ g_2(\rho(\tilde{x}_c)) &= cos(q_2) \\ g_3(\rho(\tilde{x}_c)) &= sin(q_2). \end{aligned}$$

The parameter space was divided in $L = 9$. The best level of attenuation found was $\gamma = 1.699$.

For the second phase, the chosen parameters, that are part of the state vector, were:

$$\rho(x_c) = \left[\ q_1 \quad q_3\ \right]^T$$

The compact set , P, is defined as $\rho \in [0,\ 20]° \times [0,\ 20]°/s$. The parameter variation rate is bounded by $|\dot{\rho}| \leq [60°/s\ 60°/s]$. The basis functions chosen are:

$$f_1(\rho(\tilde{x}_c)) \quad = \quad 1.$$

and

$$\begin{aligned} g_1(\rho(\tilde{x}_c)) &= 1 \\ g_2(\rho(\tilde{x}_c)) &= sin(q_1)\ +\ cos(q_1) \\ g_3(\rho(\tilde{x}_c)) &= sin(q_3)\ +\ cos(q_3). \end{aligned}$$

The parameter space was divided in $L = 5$ in each dimension. The best level of attenuation found was $\gamma = 2.052$.

Experimental results of joint positions and applied torques are showed in the Fig. 2 and 3, respectively. The joint positions are represented by:

- Solid line - joint 1;
- Dashed line - joint 2;
- Dot line - joint 3.

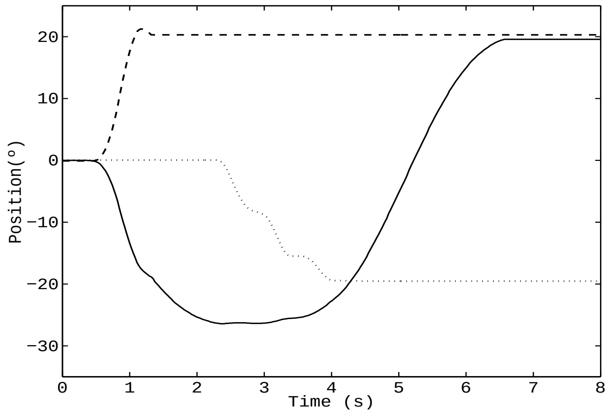

Fig. 2. Joint position, quasi-LPV control, underactuated configuration.

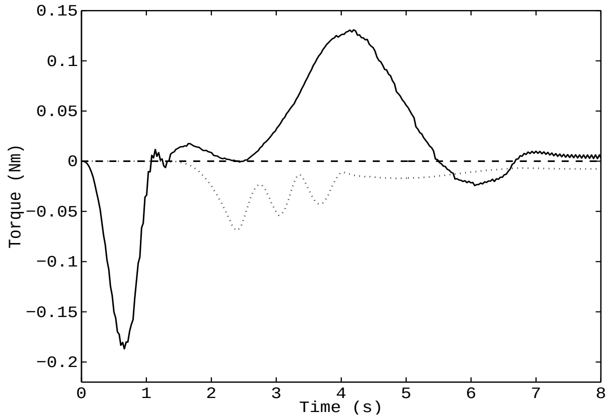

Fig. 3. Applied torque, quasi-LPV control, underactuated configuration.

V. Conclusion

An output feedback nonlinear $\mathcal{H}_\infty$ control methodology is applied to an experimental underactuated manipulator. Experimental results have showed that the proposed technique is efficient in the position control of the robot system even though the velocity is not available for the controller.

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Appendix

The matrices M and C for this kind of manipulator are given by:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\begin{aligned}
M_{11} &= m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2} cos(q_2)) + \\
&\quad m_3(l_1^2 + l_2^2 + l_{c_3}^2 + 2l_1 l_2 cos(q_2) + 2l_2 l_{c_3} cos(q_3)) + \\
&\quad 2m_3 l_1 l_{c_3} cos(q_2 + q_3) + I_1 + I_2 + I_3 \\
M_{12} &= m_2(l_{c_2}^2 + 2l_1 l_{c_2} cos(q_2)) + m_3(l_2^2 + l_{c_3}^2 + l_1 l_2 cos(q_2)) \\
&\quad m_3(l_1 l_{c_3} cos(q_2 + q_3) + 2l_2 l_{c_3} cos(q_3)) + I_2 + I_3 \\
M_{13} &= I_3 + m_3(l_{c_3}^2 + l_1 l_{c_3} cos(q_2 + q_3) + l_2 l_{c_3} cos(q_3)) \\
M_{21} &= M_{12} \\
M_{22} &= I_2 + I_3 + m_2(l_{c_2}^2) + m_3(l_2^2 + l_{c_3}^2 + 2l_2 l_{c_3} cos(q_3)) \\
M_{23} &= I_3 + m_3(l_{c_3}^2 + l_2 l_{c_3} cos(q_3)) \\
M_{31} &= M_{13} \\
M_{32} &= M_{23} \\
M_{33} &= I_3 + m_3(l_{c_3}^2)
\end{aligned}$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\begin{aligned}
C_{11} &= -[(m_2 l_1 l_{c_2} sin(q_2) + m_3 l_1 l_2 sin(q_2) + \\
&\quad m_3 l_1 l_{c_3} sin(q_2 + q_3))\dot{q_2} + (m_3 l_1 l_{c_3} sin(q_2 + q_3) + \\
&\quad m_3 l_2 l_{c_3} sin(q_3))\dot{q_3}] \\
C_{12} &= -[(m_2 l_1 l_{c_2} sin(q_2) + m_3 l_1 l_2 sin(q_2) + \\
&\quad m_3 l_1 l_{c_3} sin(q_2 + q_3))(\dot{q_1} + \dot{q_2}) + \\
&\quad (m_3 l_1 l_{c_3} sin(q_2 + q_3) + m_3 l_2 l_{c_3} sin(q_3))\dot{q_3}] \\
C_{13} &= -[(m_3 l_1 l_{c_3} sin(q_2 + \theta_3) + m_3 l_2 l_{c_3} sin(\theta_3)) \\
&\quad (\dot{q_1} + \dot{q_2} + \dot{q_3})] \\
C_{21} &= (m_2 l_1 l_{c_2} sin(q_2) + m_3 l_1 l_2 sin(q_2) + \\
&\quad m_3 l_1 l_{c_3} sin(q_2 + q_3))\dot{q_1} - m_3 l_2 l_{c_3} sin(q_3)\dot{q_3} \\
C_{22} &= -m_3 l_2 l_{c_3} sin(q_3)\dot{q_3} \\
C_{23} &= -m_3 l_2 l_{c_3} sin(q_3)(\dot{q_1} + \dot{q_2} + \dot{q_3}) \\
C_{31} &= (m_3 l_1 l_{c_3} sin(q_2 + q_3) + m_3 l_2 l_{c_3} sin(q_3))\dot{q_1} + \\
&\quad m_3 l_2 l_{c_3} sin(q_3)\dot{q_3} \\
C_{32} &= m_3 l_2 l_{c_3} sin(q_3)(\dot{q_1} + \dot{q_2}) \\
C_{33} &= 0,
\end{aligned}$$

where m_i, l_i, l_{c_i}, I_i, q_i and $\dot{q}_i$, is the mass, the length, the center of mass, the inertia momentum, the angular position and the angular velocity of the i-link, respectively.