Design of Controllers for Unstable Time Delay Systems Using Polynomial Method

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***Abstract*-- The paper presents one methodology of the controller design for an unstable time delay system. The proposed method based on the polynomial approach and the pole assignment method yields a class of CT controllers ensuring setpoint tracking as well as load disturbance attenuation. The time delay term is approximated by the first order Padé approximation. The control configuration with two feedback controllers is considered. The resulting controllers obtained via polynomial Diophantine equations are stable ones. Three methods of the closed-loop pole assignment are used. Simulation results are presented to illustrate the proposed method.**

***Index terms*-- time delay system, time delay approximation, polynomial method, pole assignment**

I. Introduction

The existence of a time delay in input-output relations is a common property of many technological processes. Plants with a time delay can often not be controlled using usual controllers designed without a consideration for a presence of a dead-time. The control responses using such controllers then tends to destabilize the closed-loop system. A part of technological processes containing a time delay in input-output relations can be unstable. There exist various ways to the control of such systems. While some methods issue from several modifications of the Smith predictor [1, 2], other methods employ PI, PID [3, 4] or PI-PD [5] control strategies.

The paper presents one methodology of the controller design for unstable time delay systems. The method is proposed for an unstable first order time delay system (FOTDS) in conjunction with the first order Padé approximation of the time delay term. Preferable behaviour of this approximation in comparison with an application of the numerator and the denominator approximations has been demonstrated by authors of this paper in [7]. The control system configuration with two feedback controllers is used. An application of only single feedback controller (so called 1DOF configuration) is unsatisfactory for controlling unstable time delay systems, as shown in [7]. To derive of controllers, the polynomial approach, e.g. [6], and the pole assignment method are applied. The procedure is proposed for three choices of the characteristic polynomial of the closed-loop. One of these forms, based on the LQ control technique, was described in detail in [8]. The resulting controllers obtained via polynomial Diophantine equations are stable and proper. Even though any method based on a time delay approximation cannot guarantee the control system stability in general, the simulation results document a usability of the proposed method providing stable control responses of a good quality also for a higher ratio between the time delay and the unstable time constant of the controlled system.

II. Approximate Transfer Function

The transfer function of an unstable FOTDS has the form

$$G(s) = \frac{K}{\tau s - 1} e^{-\tau_d s} \tag{1}$$

where $K > 0$ is the gain, $\tau > 0$ is the time constant and $\tau_d > 0$ is the time delay. Using the 1/1 - order Padé approximation of the time delay term

$$e^{-\tau_d s} \approx \frac{2 - \tau_d\, s}{2 + \tau_d\, s} \tag{2}$$

in (1), the approximate transfer function takes the form

$$G_A(s) = \frac{K\,(2 - \tau_d\, s)}{(\tau\, s - 1)\,(2 + \tau_d\, s)} = \frac{b_0 - b_1 s}{s^2 + a_1 s - a_0} \tag{3}$$

where $b_0 = \frac{2K}{\tau \tau_d}$, $b_1 = \frac{K}{\tau}$, $a_0 = \frac{2}{\tau \tau_d}$, $a_1 = \frac{2\tau - \tau_d}{\tau \tau_d}$ and $\tau_d \neq 2\tau$.

Note that a higher order approximation leads to higher degrees of the numerator and the denominator in the

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approximate transfer function and, consequently, to more complex resulting controllers.

III. Control System Description

The control system configuration is depicted in Fig. 1. In the scheme, w is the reference signal, v is the load

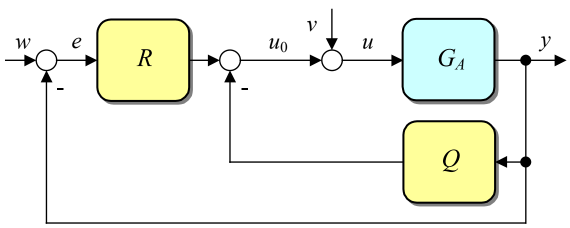

Fig. 1. Control system.

disturbance, e is the tracking error, u_0 is the controller output, y is the controlled output and u is the control input. Both w and v are considered to be step functions with transforms

$$W(s)=\frac{w_0}{s}\,,\quad V(s)=\frac{v_0}{s}\,. \tag{4}$$

The transfer function G_A represents the strictly proper approximate transfer function (3) in the general form

$$G_A(s)=\frac{b(s)}{a(s)}\,. \tag{5}$$

The transfer functions of controllers are

$$Q(s)=\frac{\tilde{q}(s)}{\tilde{p}(s)}\,,\quad R(s)=\frac{r(s)}{\tilde{p}(s)} \tag{6}$$

where $\tilde{q}, r$ and $\tilde{p}$ are polynomials in s.

IV. Application of the Polynomial Method

The controller design described in this section follows from the polynomial approach. The general conditions required to govern the control system properties are formulated as follows:

- Strong stability of the control system (in addition to the control system stability, also the stability of a controller is required).
- Internal properness of the control system.
- Asymptotic tracking of the reference.
- Load disturbance attenuation.

The procedure to derive admissible controllers can be carried out as follows:

Transforms of the controlled output and the tracking error take the form (for simplification, the argument s is in some equations omitted)

$$Y(s)=\frac{b}{d}\left[r\,W(s)+\tilde{p}\,V(s)\right] \tag{7}$$

$$E(s)=\frac{1}{d}\left[(a\tilde{p}+b\tilde{q})W(s)-b\tilde{p}\,V(s)\right]. \tag{8}$$

Here,

$$d(s)=a(s)\,\tilde{p}(s)+b(s)\big(r(s)+\tilde{q}(s)\big) \tag{9}$$

is the characteristic polynomial with roots as poles of the closed-loop.
Establishing the polynomial t as

$$t(s)=r(s)+\tilde{q}(s) \tag{10}$$

and substituting (10) into (9), the condition of the control system stability is ensured when polynomials $\tilde{p}$ and t are given by a solution of the polynomial Diophantine equation

$$a(s)\tilde{p}(s)+b(s)\,t(s)=d(s) \tag{11}$$

with a stable polynomial d on the right side.
With regard to transforms (4), an asymptotic tracking and load disturbance attenuation are provided by divisibility of both terms $a\tilde{p}+b\tilde{q}$ and $\tilde{p}$ in Eq. (8) by s. This condition is fulfilled for polynomials $\tilde{p}$ and $\tilde{q}$ in the form

$$\tilde{p}(s)=s\,p(s)\,,\quad \tilde{q}(s)=s\,q(s)\,. \tag{12}$$

Subsequently, the transfer functions of controllers take forms

$$Q(s)=\frac{q(s)}{p(s)}\,,\quad R(s)=\frac{r(s)}{s\,p(s)}\,. \tag{13}$$

A stable polynomial $p(s)$ in denominators of (13) ensures the stability of ontrollers.
The control system satisfies the condition of internal properness when the transfer functions of all its components are proper. Consequently, the degrees of polynomials q and r must fulfill inequalities

$$\deg q\le\deg p\,,\quad \deg r\le\deg p+1\,. \tag{14}$$

Now, the polynomial t can be rewritten to the form

$$t(s)=r(s)+s\,q(s)\,. \tag{15}$$

Taking into account a solvability of (11) and conditions (14), the degrees of polynomials in (11) and (15) can be easily derived as

$$\deg t=\deg r=\deg a\,,\ \deg q=\deg a-1\,,$$
$$\deg p\ge\deg a-1\,,\ \deg d\ge 2\deg a\,. \tag{16}$$

Denoting deg $a = n$, polynomials t, r and q have the form

$$t(s)=\sum_{i=0}^{n}t_i s^i\,,\quad r(s)=\sum_{i=0}^{n}r_i s^i\,,\quad q(s)=\sum_{i=1}^{n}q_i s^{i-1} \tag{17}$$

and among of their coefficients equalities

$$r_0=t_0\,,\quad r_i+q_i=t_i\ \text{ for }\ i=1,\ldots,n \tag{18}$$

hold. Since by a solution of the polynomial equation (11) only coefficients t_i can be calculated, unknown coefficients r_i and q_i can be obtained by a choice of selectable coefficients $\beta_i \in \langle 0,1 \rangle$ such that

$$r_i = \beta_i t_i \,, \quad q_i = (1-\beta_i)\, t_i \;\text{ for }\; i = 1, \ldots, n\,. \tag{19}$$

The coefficients β_i distribute a weight between numerators of transfer functions Q and R. With respect to the transform (7), it may be expected that higher values of β_i will speed up control responses to step references.

Remark: If $\beta_i = 1$ for all i, the control system in Fig. 1 demotes to the 1DOF control configuration. If $\beta_i = 0$ for all i and both reference and load disturbance are step functions, the control system corresponds to the 2DOF control configuration.

For the specific approximate transfer function (3) with $\deg a = 2$, the degrees of polynomials are

$$\deg t = \deg r = 2 \,, \deg q = 1 \,, \deg p \geq 1 \,, \deg d \geq 4 \tag{20}$$

and the transfer functions of controllers take the form

$$Q(s) = \frac{q_2\, s + q_1}{s + p_0} \,, \quad R(s) = \frac{r_2\, s^2 + r_1\, s + r_0}{s\,(s + p_0)} \tag{21}$$

for $\beta_1, \beta_2 \neq 0$.

The controller parameters then follow from solutions of the polynomial equation (11) and depend upon coefficients of polynomial d. The next problem here means to find a stable polynomial d that enables to obtain the acceptable stabilizing and stable controllers.

V. POLE ASSIGNMENT

A required control quality can be achieved by a suitable choice of the polynomial d on the right side of the polynomial equation (11). Generally, a stable polynomial have the form

$$d(s) = \prod_{i=1}^{\deg d} (s + s_i) \tag{22}$$

where $s_i = \alpha_i + j\,\omega_i$ and $\alpha_i > 0$ for all i. When $\omega_i = 0$ for all i, the control responses with aperiodic character will be obtained.

In this paper, three methods of the polynomial d determination are presented. In all cases, particular determination is realized for the specific approximate transfer function with $\deg a = 2$.

A: A quadruple pole of the closed-loop is chosen where the polynomial d takes the form

$$d(s) = (s + a)^4 \,, \;\; \alpha > 0. \tag{23}$$

B: The polynomial d is composed of two factors as

$$d(s) = n(s)\,(s+\alpha)^2 \,, \; \alpha > 0 \tag{24}$$

where n is a stable polynomial given by spectral factorization

$$n^*(s) n(s) = a^*(s) a(s) \tag{25}$$

where asterix denotes a conjugate polynomial.

Since $\deg n = \deg a = 2$, the degree of d is $\deg d = 4$.

C: The polynomial d is a product of two factors

$$d(s) = g(s)\, a^+(s) \tag{26}$$

where a^+ represents a stable part of the polynomial a and g is a stable polynomial obtained via spectral factorization

$$\left(s\, a(s)\right)^* \varphi\, s\, a(s) + b^*(s) b(s) = g^*(s) g(s)\,. \tag{27}$$

Here, $\deg a^+ = 1$, $\deg g = \deg a + 1 = 3$ and $\deg d = 4$.

Remark: Spectral factorization (27) is well known from the LQ control theory. There, the polynomial g is used to minimization of the quadratic cost function

$$J = \int_0^\infty \left\{ e^2(t) + \varphi \dot{u}^2(t) \right\} dt \tag{28}$$

where $e(t)$ is the tracking error, $\dot{u}(t)$ is the control input derivative and $\varphi > 0$ is the weighting coefficient.

Coefficients of the polynomial g arranged to the monic form

$$g(s) = s^3 + g_2\, s^2 + g_1\, s + g_0 \tag{29}$$

can be calculated from formulas

$$\begin{aligned} g_0 &= \frac{2K}{\tau\tau_d} \sqrt{\frac{1}{\varphi}} \\ g_1 &= \frac{1}{\tau\tau_d} \sqrt{4\left(K\tau\tau_d \frac{1}{\varphi} g_2 + 1\right) + K^2 \tau_d^2 \frac{1}{\varphi}} \\ g_2 &= \frac{1}{\tau\tau_d} \sqrt{2\tau^2\tau_d^2 \sqrt{\frac{1}{\varphi}}\, g_1 + 4\tau^2 + \tau_d^2} \end{aligned} \tag{30}$$

VI. CONTROLLER DESIGN

In all cases, the design procedure leads to controllers with transfer functions (13). The controller parameters were obtained by a solution of the polynomial equation (11) with right sides (23), (24) and (26). In behalf of shortness of the writing, important equations and derived formulas for considered cases are introduced in the form of the table in the following order:

- Form of $d(s)$.
- Forms of a^+ or n.
- Formulas for computation of the controller parameters.
- Condition of the resulting controller stability.

Table 1. Formulas for controller parameter computation.

A: $d(s)=(s+a)^4$
$p_0=\frac{1}{4\tau_d}\frac{\tau(\tau_d\alpha+2)^4-16(2\tau-\tau_d)}{2\tau-\tau_d}$
$t_0=\frac{\tau\tau_d}{2K}\alpha^4 \quad t_1=\frac{1}{K}\left[p_0+\tau\tau_d\alpha^3\left(2+\frac{\tau_d}{2}\alpha\right)\right]$
$t_2=\frac{\tau}{K}\left[p_0-4\alpha-\frac{2}{\tau_d}-\frac{1}{\tau}\right]$
$r_0=t_0,\ r_i=\beta_i t_i,\ q_i=(1-\beta_i)t_i$ for $i=1,2$
$p_0>0$ for $\tau_d<2\tau$ and $\alpha>\frac{2}{\tau_d}\left(\sqrt[4]{2-\frac{\tau_d}{\tau}}-1\right)$
B: $d(s)=n(s)(s+\alpha)^2$
$n(s)=\left(s+\frac{2}{\tau_d}\right)\left(s+\frac{1}{\tau}\right)$
$p_0=\frac{\alpha\left(\frac{\tau_d}{2}\alpha+2\right)(2\tau+\tau_d)+4}{2\tau-\tau_d}$
$t_0=\frac{1}{K}\alpha^2 \quad t_1=\frac{1}{K}\left[p_0+(\tau+\tau_d)\alpha^2+2\alpha\right]$
$t_2=\frac{1}{K}\left[\tau(p_0-2\alpha)-2\right]$
$r_0=t_0,\ r_i=\beta_i t_i,\ q_i=(1-\beta_i)t_i$ for $i=1,2$
$p_0>0$ for $\tau_d<2\tau$ and $\alpha>0$
$d(s)=g(s)a^+(s)$
$a^+(s)=s+\frac{2}{\tau_d}$
$p_0=\frac{\tau\left[2g_2+\tau_d\,(g_1+\frac{\tau_d}{2}g_0)\right]+2}{2\tau-\tau_d}$
$t_0=\frac{\tau}{K}g_0,\ t_1=\frac{1}{K}\left[p_0+\tau(g_1+\tau_d\,g_0)\right],$
$t_2=\frac{1}{K}\left[\tau(p_0-g_2)-1\right]$
$r_0=t_0,\ r_i=\beta_i t_i,\ q_i=(1-\beta_i)t_i$ for $i=1,2$
$p_0>0$ for $\tau_d<2\tau$ and $\alpha>0$

VII. Simulation Results

The aim in this section is to investigate the applicability of the proposed method to controlling the above considered unstable first order time delay system. The simulations were performed by MATLAB-Simulink tools. For all simulations, the unit step reference w was introduced at time $t = 0$ and the step load disturbance v at time t_v. The fixed parameters of the controlled time delay system were chosen as $K = 1$ and $\tau = 4$.

In most part of simulations, zero parameters β_1, β_2 in (19) have been chosen. Subsequently, the control system corresponds to the 2DOF configuration.

In this case, the controller parameters can be tuned by a single selactable parameter (α or φ).

A. *Simulation Results for* $d(s)=(s+a)^4$

The control responses obtained by a choice of the quadruple closed-loop pole are shown in Figs. 2 and 3. These responses exhibit the large settling time and expressive overshoots to the step reference for all α as well as an oscillatory character for its higher values. Hence, this form of d cannot be recommended for the control design.

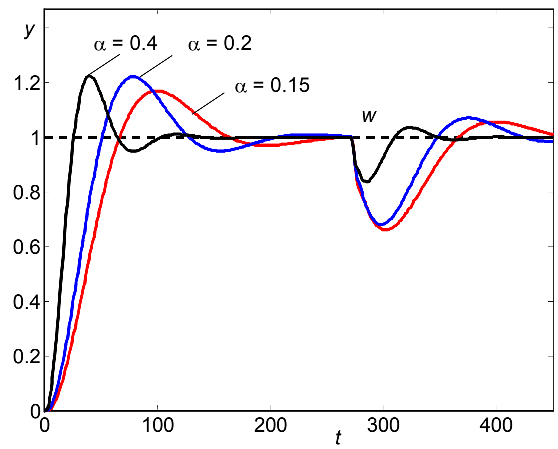

Fig. 2. Responses to step reference and load disturbance. ($\tau_d = 2$, $v = -0.1$, $t_v = 270$).

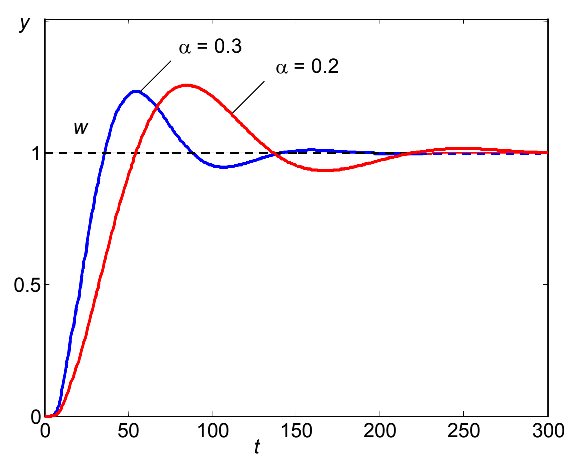

Fig. 3. Responses to step reference. ($\tau_d = 4$).

B. *Simulation Results for* $d(s) = n(s)\,(s+\alpha)^2$

The control responses for d in the above form are in Figs. 4 -6. The responses exhibit the significantly shorter settling time in comparison with the case *A*. By a selection of the suitable value of α, smooth responses without any overshoots and with the aperiodic character can be obtained. Clearly, for higher values of τ_d, smaller values of α must be used. The responses in Fig. 6 document the usability of the proposed method also for the ratio $\tau_d/\tau > 1$ (here, $\tau_d/\tau = 1.25$). Note that in many works this ratio is constrained as $\tau_d/\tau < 1$.

C. *Simulation Results for* $d(s) = g(s)\,a^{+}(s)$

Now, the control responses in Figs. 7 and 8 document the control of a similar quality as in the case *B* for the same values of τ_d. In this case, the controller parameters can be tuned by the single selectable parameter φ. Here, for higher

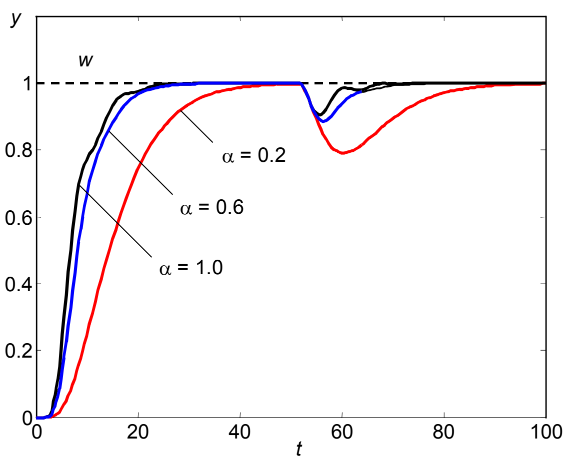

Fig. 4. Responses to step reference and load disturbance. ($\tau_d = 2$, $v = -0.1$, $t_v = 50$).

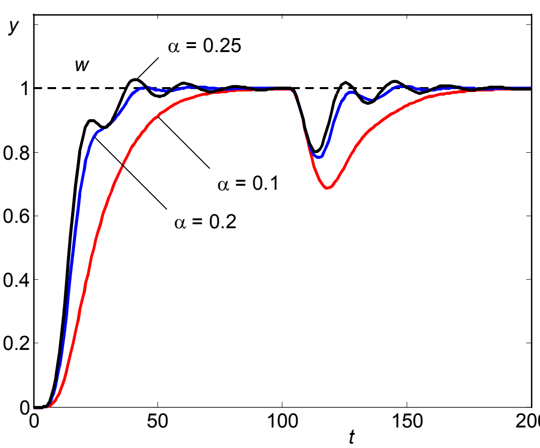

Fig. 5. Responses to step reference and load disturbance. ($\tau_d = 4$, $v = -0.05$, $t_v = 100$).

values of τ_d, greater values of φ must be chosen. A likeness of step reference responses obtained in the case *B* (for α = 0.08) and *C* (for φ = 400) is evident from Fig. 9. However, the responses to step load disturbance exhibit a smaller

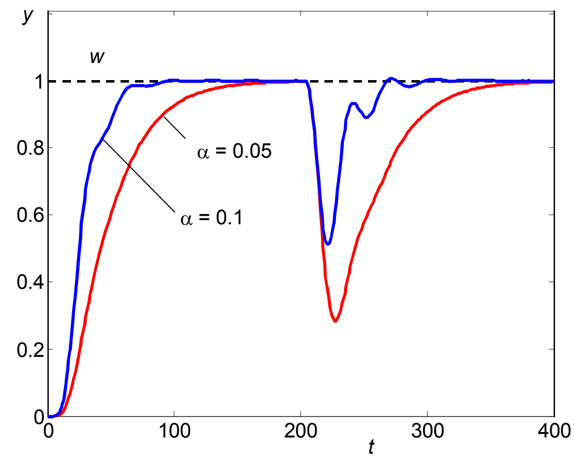

Fig. 6. Responses to step reference and load disturbance. ($\tau_d = 5$, $v = -0.05$, $t_v = 200$).

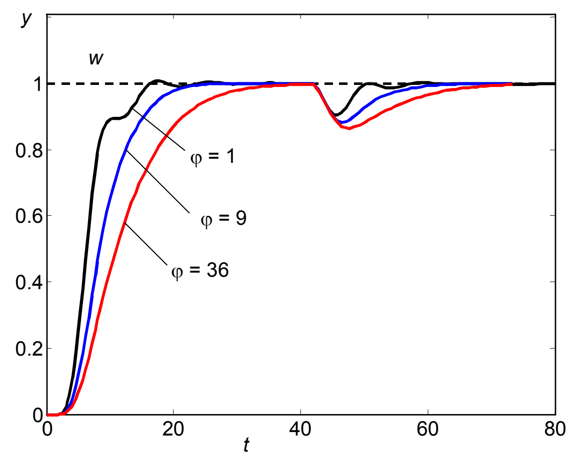

Fig. 7. Responses to step reference and load disturbance. ($\tau_d = 2$, $v = -0.1$, $t_v = 40$).

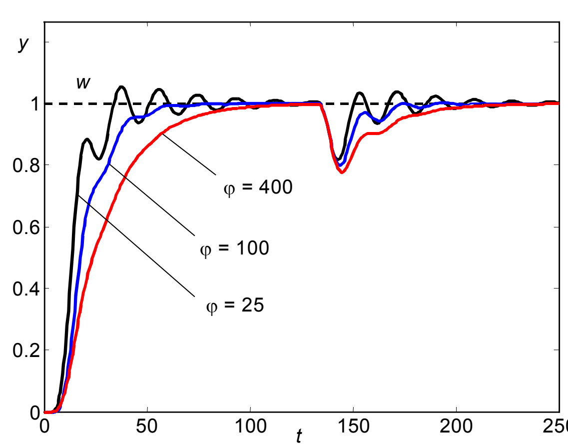

Fig. 8. Responses to step reference and load disturbance. ($\tau_d = 4$, $v = -0.05$, $t_v = 130$).

undershoot using the form *C*.

D. *Effect of Parameters β*

Simulation results shown in Fig. 10 demonstrate an influence of the parameter β_1 on the control responses. A small value of β_1 speeds up step reference responses but it does not affect load disturbance responses. Other simulations proved that a greater value of β_1 leads to overshoots and oscillations. The condition $\beta_2 = 0$ in (19) is necessary for the process here investigated. A nonzero value of β_2 leads to nonstrictly proper *R* and, consequently, to great overshoots in step reference responses. Note that establishing $\beta_2 = 0$ and $\beta_1 \neq 0$, the transfer functions of controllers take forms

$$Q(s) = \frac{q_2 s + q_1}{s + p_0}, \quad R(s) = \frac{r_1 s + r_0}{s(s + p_0)}$$

where

$$r_0 = t_0, \quad r_1 = \beta_1 t_1, \quad q_1 = (1 - \beta_1) t_1 .$$

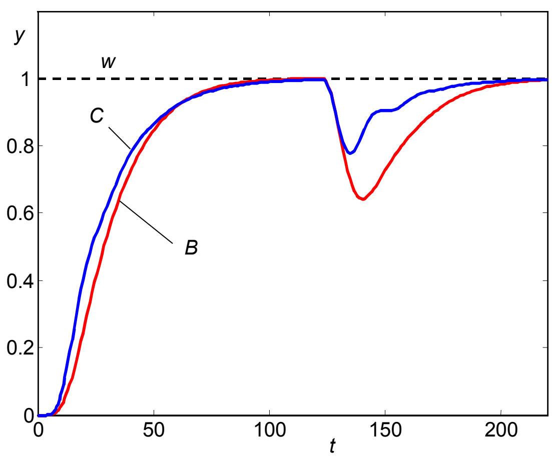

Fig. 9. Comparison of control responses.
(*B*: $\alpha = 0.08$, *C*: $\varphi = 400$, $v = -0.05$, $t_v = 120$).

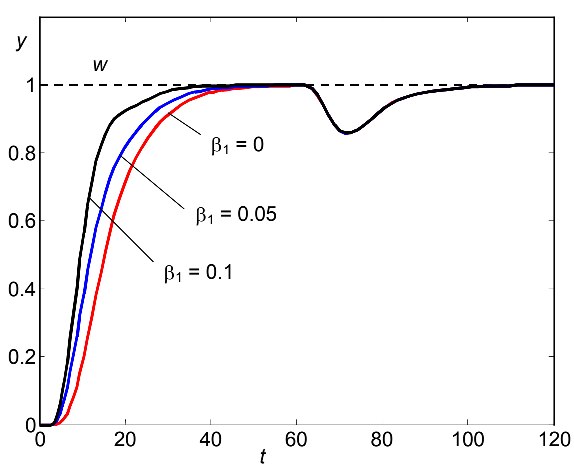

Fig. 10. Responses to step reference and load disturbance.
($\tau_d = 3$, $\alpha = 0.2$, $v = -0.1$, $t_v = 60$).

VIII. Conclusions

One method of control design for an unstable first order time delay system has been solved and analysed. The proposed method is based on the Padé time delay approximation. The controller design uses the polynomial synthesis and the controller setting employs the closed-loop pole assignment method. Three methods of a choice of the closed-loop characteristic polynomial were studied and compared. The presented results proved a usability of two methods which provide the control of a good quality also for relatively high ratio between the time delay and the controlled system time constant. The procedure makes possible a tuning of the controller parameters by most two selectable parameters. Using derived formulas, the controller parameters can be automatically computed. From this reason, the method could also be used for an adaptive control.

IX. Acknowledgments

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