THE REGULATOR PROBLEM FOR LINEAR SYSTEMS WITH CONSTRAINED CONTROL: AN LMI APPROACH

A. BENZAOUIA and S. EL FAIZ
Research Unit: Constrained and Robust Regulation, Department of Physics,
Faculty of Science Semlalia, P.B 2390, Marrakech, Morocco.
E-mail.: benzaouia@ucam.ac.ma

Abstract

A new methodology of the partial eigenstructure assignment by state feedback via Linear matrix inequality (LMI) is extended to obtain a solution of the constrained regulator problem for linear continuous-time and discrete-time systems by using the LMI formulation. **Key-words**: Linear systems, constrained control, positive invariance, LMI technique.

1 Notation

- For two vectors $x,\ y \in \mathbb{R}^n$, $x \preceq y$ if $x_i \preceq y_i$, $i = 1, \dots n$.
- A square matrix $Q > 0$ if Q is definite positive.
- $I\!I$ denotes the identity matrix.
- For a matrix H,

$$\tilde{H} = \begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix},$$

where

$$h_{ij}^+ = \mathrm{Sup}(h_{ij},\ 0),\ h_{ij}^- = \mathrm{Sup}(-h_{ij},\ 0),$$

$$\tilde{H}_c = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix},$$

where

$$H_1 = \begin{cases} h_{ij}^+ & \text{for} \quad i \neq j \\ h_{ii} & \text{otherwise} \end{cases},$$

$$H_2 = \begin{cases} h_{ij}^- & \text{for} \quad i \neq j \\ 0 & \text{otherwise} \end{cases}.$$

for $i,\ j = 1,\ \dots,\ n$, where h_{ij} denotes the matrix component $H(i,j)$.

2 Introduction

In this paper, we study the stability of linear systems with input saturation. This class of systems has obtained great interest during the last decade. It was shown by many authors that the global stabilization of linear systems with input saturation, can be achieved if and only if all the poles of the given system are stable, and, even then, in general one must use non linear control and only simple cases can be handled via linear control laws (see [16] and the references therein).

An efficient tool to develop local stabilizing controllers for linear systems with input saturation, when the given system is not stable, is the positive invariance approach [3]-[6], [8]. The stabilizing regulator F obtained with this approach is a solution to the non linear algebraic equation $FA + FBF = HF$, where matrix H satisfies the main condition of positive invariance of type $\tilde{H}q \preceq q$, vector q represents the constraints on the control. One can cite the work of [4] where the resolution of equation $XA + XBX = HX$ is presented as a technique of partial eigenstructure assignment. This resolution was also associated to the constrained regulator problem. For the same purpose, the following generalized Sylvester matrix equation $AX + BY = XJ$ is used in the literature [11], [13] [12] and the references therein. In [12], the partial eigenstructure assignment problem, in which both the open-loop and the closed-loop eigenvalues are allowed to possess arbitrary geometric and algebraic multiplicities, is addressed. Its solution is similar to the one obtained by [4].

The LMI theory has been successfully used in many areas of automatic control [9],[1], [2], [7] and the references therein. There exist many efficient algorithms to numerically solve a given LMI problem.

In this paper, we address the regulator problem for linear continuous-time and discrete-time systems with constrained control in terms of an LMI problem. This formulation is closely based on the work of [7] which allows one to find a non singular solution to the Sylvester equation with possible additional specifications on the eigenvectors of the closed-loop systems and without restrictive assumptions. Thus, a solution by using the LMI technique is provided for the partial stabilization problem, that is, find a state feedback F which ensures the asymptotic stability of the linear system with constrained control.

The rest of the paper is organized as follows: The background of the technique of the reduced order system together with the LMI of partial eigenstructure assignment are recalled in the second section . Section 3 presents the main result of this paper which consists in a LMI allowing a direct solution of the constrained regulator problem for linear systems. An algorithm and an example illustrating this new technique are also presented in this section.

3 Preliminary results

This paper is devoted to the study of linear systems described by equation (1)

$$\delta x(t) = Ax(t) + Bu(t), \qquad (1)$$

where the operator denoted here δ is defined as follows :

$$\delta x(t) = \begin{cases} \dot{x}(t), \\ x(t+1), \end{cases}$$

for continuous-time systems and discrete-time systems respectively, x is the state vector in $\mathbb{R}^n$, and u is the constrained control, satisfying

$$u \in \Omega \subset \mathbb{R}^m. \qquad (2)$$

Matrices A and B are constant, of appropriate size and satisfy the following assumption. We assume that:

- H1): The pair (A, B) is controllable.
- H2): The open-loop system has m undesirable or unstable eigenvalues.

Ω is the set of admissible controls defined as

$$\Omega = \{u \in \mathbb{R}^m / -q_2 \preceq u \preceq q_1;\ q_1,\ q_2 \in \mathbb{R}^m\}. \qquad (3)$$

This is a non symmetrical polyhedral set as is generally the case in practical situations.

Let us first consider the unconstrained case and assume that we construct a stabilizing controller for system (1) which consists in realizing a feedback law as :

$$u(t) = Fx(t),\ F \in \mathbb{R}^{mxn} \text{ with } rank(F) = m. \qquad (4)$$

In such a case, system (1) becomes :

$$\delta x(t) = (A + BF)x(t) = \Phi x(t), \qquad (5)$$

where F is generally chosen in such a way that an increase of system dynamics is obtained with the asymptotic stability of the closed loop system (5).

In the constrained case, we follow the approach proposed in [3], [5], [6]. Recall that this approach consists in giving conditions allowing the choice of a stabilizing controller (4) in such a way that model (5) remains valid every time. This is only possible if the state is constrained to evolve in a specified region defined by

$$\mathcal{D} = \{x \in \mathbb{R}^n / - q_2 \preceq Fx \preceq q_1\}; \qquad (6)$$

Note that these domains are convex and unbounded for $m < n$.

Definition 3.1 *A subset $\mathcal{S}$ of $\mathbb{R}^n$ is said to be positively invariant with respect to (w.r.t.) the motion of the system (5) if for every initial state $x_o \in \mathcal{S}$, the motion $x(x_o, \ t) \in \mathcal{S}$, for every t.*

The necessary and sufficient conditions of positive invariance of the sets $\mathcal{D}$ w.r.t the system (5) are well known and are given by [3], [5], [6] which is recalled by the following result.

Theorem 3.1 *Domain $\mathcal{D}$ is positively invariant w.r.t. the system (5), if and only if there exists a matrix $H \in \mathbb{R}^{mxm}$, solution to*

$$FA + FBF = HF \qquad (7)$$

and satisfying

$$\tilde{H}_\delta q \preceq 0, \qquad (8)$$

where $\tilde{H}_\delta$ is defined by $\tilde{H}_\delta =$

$$\begin{cases} \tilde{H} - \mathbb{I}, \text{for discrete} - \text{time systems}, \\ \tilde{H}_c, \text{for continuous} - \text{time systems}, \end{cases}$$

An efficient algorithm to built such controllers is given by the resolution of the algebraic equation $XA + XBX = HX$ [4] where matrix H is firstly given according to condition (8). Note that the obtained controller is stabilizing the system in the closed-loop while the control is admissible for all $x_0 \in \mathcal{D}$. This technique is so-called the inverse procedure. The resolution of this algebraic equation necessitates that matrix A admits at least $n - m$ stable eigenvalues as required by assumption H2. If not, one has to use the technique of augmentation [4] described below:

Rewrite the system (1) under the equivalent form:

$$\delta x(t) = Ax(t) + B_a w(t), \qquad (9)$$

with matrix B_a given by:

$$B_a = \left[\begin{array}{cc} B & \odot \end{array} \right],$$

where $\odot \in R^{n\mathrm{x}(n-m)}$ represents the null matrix. This augmentation technique leads to the introduction of $n-m$ fictitious entries together with their fictitious constraints given by: $-\varphi_2 \leq v \leq \varphi_1$. In this case, the control law is also modified and becomes

$$w(t) = \left[\begin{array}{c} u(t) \\ v(t) \end{array} \right],$$

$w(t) = Kx(t)$ and $v(t) = Ex(t)$. Note K and g as follows:

$$K = \left[\begin{array}{c} F \\ E \end{array} \right], g_1 = \left[\begin{array}{c} q_1 \\ \varphi_1 \end{array} \right],$$
$$g_2 = \left[\begin{array}{c} q_2 \\ \varphi_2 \end{array} \right], g = \left[\begin{array}{c} g_1 \\ g_2 \end{array} \right] \qquad (10)$$

where g represents the new vector constraint. Note that the system in closed-loop given with the augmented control $w(t)$ remains the same as (5) while the set of admissible constraints becomes with this augmentation,

$$\Omega_a = \{w \in \mathbb{R}^n / -g_2 \leq w \leq g_1\} \qquad (11)$$

It is worth noting that this technique does not modify the system, introduces new degree of freedom with φ, which are used to satisfy conditions (8), but in return, reduces the domain D which is transformed to the following bounded and convex set:

$$G = \{x \in \mathbb{R}^n / -g_2 \le Kx \le g_1\}; \quad (12)$$

Obviously, conditions (7) and (8) are to be written with matrices K and B_a, matrix H becomes of nxn size.
Let Λ_o: be the subset formed by the $(n-m)$ open-loop eigenvalues that belong to some desirable stable region.

Λ_1 : be the subset formed by the m eigenvalues that one wants to assign in closed-loop by using the state feedback (4).

The problem of partial pole assignment consists in computing matrix F such that $\Lambda = \sigma(A+BF) = \Lambda_o \cup \Lambda_1$.

This work is also based on the use of the technique of reduced-order system obtained by the projection of the original system trajectories in the subspace associated with the undesirable eigenvalues [10] . This can be achieved by a Schur decomposition of the system matrix in two blocks associated respectively with the desirable and undesirable open-loop eigenvalues. Thus, let us recall the main outlines of this technique detailed in [10].

Let us define a subspace S_o associated with the $(n-m)$ stable open-loop eigenvalues and consider S_m a complementary subspace to S_o, i.e $S_o \oplus S_m = \mathbb{R}^n$. Note that S_m can be associated with the unstable or undesirable eigenvalues.
In this way, consider the following change of basis in (1):

$$x = [Q_o \,| Q_m] \begin{bmatrix} z_o \\ z_m \end{bmatrix}; z_0 \in \mathbb{R}^{n-m}, z_m \in \mathbb{R}^m \quad (13)$$

where the matrix $Q \in \mathbb{R}^{n\times n}$ is orthonormal,

$$Q = [Q_o |\, Q_m]; Q_o \in \mathbb{R}^{n\times(n-m)}; Q_m \in \mathbb{R}^{n\times m};$$

$$Q^T Q = QQ^T = \mathbb{I}_n; Q_m^T Q_m = \mathbb{I}_m \quad (14)$$

such that,

$$\begin{cases} \text{the columns of } Q_o \text{ span } S_o \triangleq Ker(F) \\ \text{the columns of } Q_m \text{ span } S_m \text{ complementary to } S_o \end{cases}$$

Matrix Q can be obtained from a Schur decomposition of matrix A by reordering, if necessary, its Schur blocks [14].
In the orthonormal basis formed by the columns of matrix Q, the open-loop system (1) is represented by :

$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_m \end{bmatrix} = \begin{bmatrix} R_o & R_2 \\ 0 & R_m \end{bmatrix} \begin{bmatrix} z_o(t) \\ z_m(t) \end{bmatrix} + \begin{bmatrix} B_o \\ B_m \end{bmatrix} u(t) \quad (15)$$

where,

$$R = Q^T A Q = \begin{bmatrix} R_o & R_2 \\ 0_{(m\times(n-m))} & R_m \end{bmatrix};$$

$$B_Q = \begin{bmatrix} B_o \\ B_m \end{bmatrix} = \begin{bmatrix} Q_o^T \\ Q_m^T \end{bmatrix} B \quad (16)$$

$$\begin{cases} z_o \text{ is the projection of } x \text{ on } S_o \text{ along } S_m \\ z_m \text{ is the projection of } x \text{ on } S_m \text{ along } S_o \end{cases}$$

Note that the dynamic of z_m associated with the undesirable poles to be modified, is decoupled from z_o. Thus we can isolate the following open-loop reduced-order system:

$$\dot{z}_m = R_m z_m(t) + B_m u(t) \quad (17)$$

Recall that it is always possible to have $rank(B_m) = m$ and the reduced pair (R_m, B_m) completely controllable [10].
In the new basis, the feedback matrix F is represented by:

$$F_Q = F[Q_o \,|\, Q_m] = [0_{m\times(n-m)} |\, F_m], \quad (18)$$

with $rank(F_m) = m$. In this way, matrix F_m assigns the desired spectrum of the closed-loop reduced-order system :

$$\dot{z}_m = (R_m + B_m F_m)\, z_m(t) \quad (19)$$

The second result of this section concerns the problem of partial eigenstructure assignment which is related to a reduced-order Sylvester equation associated to a given matrix J by using the LMI technique.

Theorem 3.2 *[7] For a matrix $J \in \mathbb{R}^{m\times m}$ given such that $\sigma(J) = \Lambda_1$, and $X \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m\times m}$ solutions of the following LMI problem:*

$$\begin{cases} X^T + X > 0 \\ \text{u. c.: } R_m X + B_m Y - X J = 0 \end{cases} \tag{20}$$

the regulator of gain $F = F_m Q_m^T$; with $F_m = Y X^{-1}$ assigns the spectrum $\Lambda_o \cup \Lambda_1$ for the system (5) with assumptions H1- H2.

4 Main result

In this section, we present the main result of this paper which exprimes the asymptotic stability condition for linear systems with constrained control by means of LMI technique. This result is based on an equivalent formulation of the resolution of the algebraic equation (7) using the Sylvester equation of the reduced-order system.

Theorem 4.1 *A matrix F of full rank is the unique solution of the equation,*

$$FA + FBF = HF \tag{21}$$

where matrices A, B satisfy assumptions H1-H2 and $H \in \mathbb{R}^{m\times m}$ a given matrix; if and only if there exist non singular matrices $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{m\times m}$ solutions of the following system:

$$\begin{cases} R_m X + B_m Y - X J = 0, \\ HY - YJ = 0, \end{cases} \tag{22}$$

where the matrix J denotes the Jordan form of the matrix H. Moreover, the only solution of (21) is $F = Y\ X^{-1} Q_m^T$, matrix Q_m is given by (16).

Proof:

(if) Let X and Y be the non singular solutions of the system (22). According to the resolution of the Sylvester equation $R_m X + B_m Y - X J = 0$, [12], one can write,

$$(R_m + B_m F_m)X = X J, \tag{23}$$

which means that the columns of matrix X, which is non singular, represents the eigenvectors of the reduced matrix in closed-loop associated to the eigenvalues of matrix H. It follows that,

$$(R_m + B_m F_m) = X J X^{-1} \tag{24}$$
$$F_m = Y X^{-1}. \tag{25}$$

By multiplying in the left equation (24) by matrix X^{-1} and Y successively, one obtains,

$$Y X^{-1} R_m + Y X^{-1} B_m F_m = Y J X^{-1} \tag{26}$$

Taking account of the second equation $H Y = Y J$, one obtains,

$$F_m R_m + F_m B_m F_m = H F_m \tag{27}$$

By multiplying equation (27) in the right by Q_m^T, and using $Q_m^T Q_m = I\!I_m$ one should have,

$$F_m Q_m^T Q_m R_m Q_m^T + F_m Q_m^T Q_m B_m F_m Q_m^T = H F_m Q_m^T,$$

which leads to the algebraic equation (21), keeping the transformations (16) in mind. Finally, one can note that matrix F_m is also non singular, that is, matrix F is of full rank.

(only if) Let matrix F with full rank be the unique solution of the algebraic equation (21). Using the same transformations (16), matrix F_m is the non singular unique solution of the reduced algebraic equation (27). According to [4], this solution is given by $F_m = Y X^{-1}$ where X and Y satisfy:

$$F_m X = Y \tag{28}$$

and

$$HY = YJ \tag{29}$$

Multiply in the right equation (27) by matrix X, it follows,

$$F_m\,(R_m\,X + B_m F_m\,X) = H F_m\,X \tag{30}$$

Using (28) and the fact that matrix F_m is non singular, then,

$$R_m\,X + B_m\,Y = X\,Y^{-1} H\,Y, \tag{31}$$

Using (29), one obtains the Sylvester equation of (22). In conclusion, the unique solution of full rank of the algebraic equation (21) is given by $F = YX^{-1}\,Q_m^T$ with matrices X and Y are solutions of the system (22). $\nabla\nabla\nabla$

This result enables us to built a stabilizing regulator for system (5) despite the presence of the constraints on the control by using an LMI formulation.

Theorem 4.2 *For a given matrix $H \in \mathbb{R}^{m\times m}$ satisfying (8) and $\sigma(H) = \Lambda_1$, if there exist non singular matrices X and Y solutions of the following LMI problem:*

$$\left\{\begin{array}{c} X^T + X > 0 \\ \textit{u. c.:}\;\; R_m\,X + B_m Y - X\,J = 0 \\ \textit{and}\;\; HY - YJ = 0, \end{array}\right. , \tag{32}$$

where matrix J denotes the Jordan form of matrix H, then, the system in closed-loop (5) with matrix F given by $F = Y\,X^{-1}\,Q_m^T$ is asymptotically stable for every $x_0 \in \mathcal{D}$.

Proof:

According to Theorem 4.1, if there exist non singular matrices X and Y solutions of the LMI (32), then, matrix $F = Y\,X^{-1}\,Q_m^T$, which is of full rank, is the unique solution of the algebraic equation (7). Since matrix H is given according to condition (8), then, by virtue of Theorem 3.1, the set $\mathcal{D}$ is positively invariant w.r.t the system in closed-loop (5). Further, $\sigma(A + B\,F) = \Lambda_0 \bigcup \Lambda_1$ which guarantees the asymptotic stability of the system (5) for every $x_0 \in \mathcal{D}$. $\nabla\nabla\nabla$

The following algorithm summarizes the steps of calculations followed during the development of this new approach by introducing the LMI. It is worth to recall that the use of a Schur decomposition guarantees numerical robustness in the computation of the open-loop eigenvalues while it determines a new basis for the associated subspaces [14].

Algorithm:

- Step 1: Verify that matrix A possesses only m undesirable eigenvalues. Λ_0 is the set of the remainder $(n-m)$ eigenvalues of A.
- Step 2: Apply a Schur decomposition of matrix A by reordering, if necessary, its Schur blocks to have matrix $Q_m \in R^{n\times m}$ and the reduced-order system (19) associated with the undesirable eigenvalues of matrix A.
- Step 3: Give a matrix $H \in R^{m\times m}$ satisfying (8) and $\sigma(H) = \Lambda_1$. Compute its Jordan form J.
- Step 4: Compute the LMI (32); $F_m = YX^{-1}$
- Step 5: Compute the gain matrix $F = F_m Q_m^T$.
- Step 6: Verify that $\sigma(A + BF) = \Lambda_0 \bigcup \Lambda_1$

Example 4.1 *In order to illustrate the use of the proposed methodology, we consider* *the same continuous-time system treated by [12].*

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \; B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let the constraints on the control be as follows:

$$q_1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix}; \; q_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The open-loop eigenvalues of the system are given by:

$$\sigma(A) = \{-4.9711, \;\; -0.0973, \;\; 2.0684\}$$

Let the undesirable eigenvalues be $\{-0.0973, 2.0684\}$. *For that, the Schur decomposition of matrix A is given by:*

$$\begin{aligned} Q &= \begin{bmatrix} 0.9851 & 0.1427 & 0.0962 \\ 0.0284 & -0.6864 & 0.7267 \\ -0.1697 & 0.7131 & 0.6802 \end{bmatrix}; \\ Q_m &= \begin{bmatrix} 0.1427 & 0.0962 \\ -0.6864 & 0.7267 \\ 0.7131 & 0.6802 \end{bmatrix} \end{aligned}$$

The corresponding matrices R_m and B_m are given by,

$$\begin{aligned} R_m &= \begin{bmatrix} -0.09726 & 0.1413 \\ 0 & 2.0684 \end{bmatrix}; \\ B_m &= \begin{bmatrix} 0.713 & -0.6864 \\ 0.6802 & 0.7267 \end{bmatrix} \end{aligned}$$

Choose a matrix H satisfying (8) of spectrum $\sigma(H) = \{-3, -5\}$ as follows:

$$H = \begin{bmatrix} -3 & 1 \\ 0 & -5 \end{bmatrix}$$

The resolution of the LMI (32) yields the following non singular solutions X and Y,

$$\begin{aligned} X &= \begin{bmatrix} 813.2382 & -735.6532 \\ 456.3723 & 187.7618 \end{bmatrix}; \\ Y &= \begin{bmatrix} -3400.6862 & 1716.4146 \\ 0 & -3432.829 \end{bmatrix} \end{aligned}$$

From the Algorithm, we have the next feedback matrix that assigns the desired closed-loop spectrum,

$$F = Y X^{-1} Q_m^T = \begin{bmatrix} -0.6333 & 0.3527 & -3.616 \\ -0.09236 & -6.3552 & -1.6003 \end{bmatrix}$$

The assigned spectrum in closed-loop is then computed,

$$\begin{aligned} \sigma(A + BF) &= \{-4.9711, -3.0000, -5.0000\} \\ &= \Lambda_0 \cup \sigma(H) \end{aligned}$$

The algebraic equation (7) is also satisfied and can attain $\|FA + FBF - HF\| = 4.1e - 15$ *with a high precision of MATLAB.*

Example 4.2 *In order to illustrate the use of the augmented technique, consider the double integrator system in discrete-time,*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \; B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Let the constraints on the control be as follows:

$$q_1 = 5; \quad q_2 = 10$$

The open-loop eigenvalues of the system are given by:

$$\sigma(A) = \{1, \;\; 1\}$$

The open-loop system does not contain $n - m$ stable eigenvalues. For that, we use the augmentation technique.

$$B_a = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \end{bmatrix}$$

Choose a matrix H satisfying (8) with a given fictitious constraints $\varphi_1 = 2.5$ and $\varphi_2 = 5$ as follows:

$$H = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.4 \end{bmatrix}$$

The resolution of the Sylvester equation (32) yields the following non singular solution X and Y,

$$
\begin{aligned}
X &= \begin{bmatrix} 21950.499 & -16521.881 \\ -14633.666 & 14161.612 \end{bmatrix}; \\
Y &= \begin{bmatrix} 7316.833 & -8496.9673 \\ 0 & 849.6967 \end{bmatrix};
\end{aligned}
$$

$$K = YX^{-1} = \begin{bmatrix} -0.30 & -0.95 \\ 0.18 & 0.27 \end{bmatrix}$$

The effective gain matrix F is to be extracted from matrix K as follows,

$$F = \begin{bmatrix} -0.30 & -0.95 \end{bmatrix}$$

The assigned spectrum in closed-loop is then computed,

$$
\begin{aligned}
\sigma(A+BF) &= \{0.4, 0.5\} \\
&= \sigma(H)
\end{aligned}
$$

The algebraic equation (7) is also satisfied and can attain $\|FA+FBF-HF\| = 1e-16$ *with a high precision of MATLAB. Nevertheless, the continuous-time double integrator leads to a Sylvester equation which does not admit a non singular solution when the augmentation technique is used.*

5 Conclusion

In this paper, a new formulation for the constrained regulator problem is presented. This technique is based on the use of the reduced-order system and the LMI's to simplify the computations and to have a good numerical precision. The results are given for the continuous-time systems and the discrete-time systems. The paper presents a simple algorithm and two illustrative examples.

References

[1] L. El Ghaoui, F. Oustry and M. Ait Rami. "An LMI-based linearization algorithm for static output-feedback and related problems ". *IEEE Trans. Aut. Control*, **Vol. 42**, $N°8$, 1997.

[2] M. Ait Rami and X. Y. Zhou. "Linear Matrix Inequalities, Riccati Equations, and Indefinite Stochastic Linear Quadratic Controls". *IEEE Trans. Aut. Control*, **Vol. 45**, $N°8$, 2000.

[3] A. Benzaouia, C. Burgat. "Regulator problem for linear discrete-time systems with non symmetrical constrained control", *Int. J. Control*, **Vol.48**, No 6, 2441-2451, (1988).

[4] A. Benzaouia. "The resolution of equation XA + XBX = HX and the pole assignment problem". *IEEE Trans. Aut. Control*, **Vol. 39**, $N°10$, pp. 2091-2095, 1994.

[5] A. Benzaouia, A. Hmamed. "Regulator problem for linear continuous systems with non symmetrical constrained control". *IEEE Trans. Aut. Control*, **Vol. 38**, $N°10$, (1993).

[6] A. Benzaouia, " Further Results on the Saturated Controller Design for Linear Continuous-Time Systems" ". *Med. Conference, Portugal*, (2002).

[7] A. Benzaouia, S. Elfaiz. "Eigenstructure assignment problem for linear continuous systems: An LMI approach". *11th Mediterranean Conference on Control and Automation MED'03*, Submitted, (2002).

[8] F. Blanchini "Set invariance in control". *Automatica*, **Vol. 35**, pp. 1747-1767, (1999).

[9] Boyd. S. P, EL Ghaoui, E. Feron, and V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia, PA, 1994.

[10] E. B. Castelan, J. M. Gomes da Silva Jr., and José E. R. Cury ”A Reduced-order Framework Applied to Linear System with Constrained Control ”. IEEE Trans. on Aut. Control. **Vol. 41**, $N°.2$, 1996.

[11] E. De Souza and S. P. Bhattacharyya, ”Controllability, observability and the solution of $AX - XB = C$”. Lin. Alg. and Appl. **Vol.39**, pp. 167-188, 1981.

[12] G. R Duan, G. W. Irwin and G. P. Liu ” Partial Eigenstructure Assignment by State Feedback: A Complete Parametric Approach.” ECC, 1999.

[13] G. R Duan, ” Solution to matrix equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems”, IEEE Trans. on Automatic Control, **Vol.38**, $N°.2$, pp. 276-280, 1993.

[14] J. J Dongarra, S. Hammarling, and J. Wilkinson,” Numerical Considerations in Computing Invariant Subspaces ”, SIAM J. Matrix Anal. Appl., **Vol.13**, $N°.1$, pp. 145-161, 1992.

[15] J.M. Gomes da Silva Jr., E.B. Castelan and J.E.R. Cury, ”On Partial Eigenstructure Assignement”, in Portuguese, X CBA, VI CLACA, Rio de Janeiro, Sep. 1994.

[16] Z. Li, A. Saberi and A. A. Stoorvogel. “Semi-global Stabilization of Linear Discrete-time Systems Subject to Input Saturation via Linear Feedback. An ARE-Based Approach”. *Proceeding of the* 33*rd CDC, Lake Buena Vista, FL. December* (1994)