Norm-Based Approach to Componentwise Asymptotic Stability

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***Abstract*-- In previous works on componentwise asymptotic stability (CWAS), the analysis of CWAS for a given linear system requested the investigation of an auxiliary system of difference (in the discrete-time case) or differential (in the continuous-time case) inequalities, built from the state equation of the studied system. Our paper shows that, by the adequate usage of the infinity norm, the analysis of CWAS can circumvent the construction of such inequalities and can apply the standard tools of asymptotic stability (ε - δ formalism, properties of the operator describing the system dynamics, Lyapunov functions) directly to the studied system. These novel results reveal the complete meaning of CWAS as a special type of asymptotic stability.**

***Index Terms*-- Stability analysis, Flow-invariant sets, Linear systems.**

I. INTRODUCTION

The concepts of *componentwise asymptotic stability* (CWAS) and *componentwise exponential asymptotic stability* (CWEAS) were introduced and characterized for continuous-time dynamical systems by Voicu, who explored the linear dynamics in [1], [2] and the nonlinear dynamics in [3]. Voicu's works relied on the theory of time-dependent flow-invariant sets [4] which allowed a refinement of the standard stability notions, by the individual monitoring of the state-space trajectories approaching an equilibrium point. Later on, CWAS and CWEAS were extended by Hmamed to continuous-time delay linear systems [5] and to 1-D and 2-D linear discrete systems [6]. Recently, Pastravanu and Voicu dealt with CWAS and CWEAS of interval matrix systems in both discrete-time and continuous-time cases [7], [8].

All the researches mentioned above focused on the characterization of CWAS / CWEAS via difference inequalities (in the discrete-time case) and differential inequalities (in the continuous-time case). Consequently, emphasis was placed on studying the properties of the operators defining such inequalities, which were different from the operators describing the system dynamics.

The purpose of the current paper is to point out the existence of direct links between the dynamics of the studied system and CWAS / CWEAS as a special type of asymptotic stability (AS). It is shown that such links are ensured by the usage of infinity norm and operate as particular forms of well-known results in the classical theory of stability. Thus, the analysis of CWAS / CWEAS can circumvent the construction of the inequalities mentioned above and can apply standard tools in stability theory directly to the investigated system.

During the last decade, the infinity norm has been used in several works devoted to the study of polyhedral invariant sets and their application in control - see, for instance, the remarkable survey paper [9] and the papers cited therein. For most of these researches, the polyhedral invariant sets do not depend on time, or if they do, the time-dependence is understood as a contraction of exponential type, operating uniformly on the constraints of the initial conditions (which is actually induced by the exponential-type decreasing of a nonquadratic Lyapunov function associated with linear systems). Therefore, such researches (focusing on the generality of the polyhedrons, but neglecting the generality of the time dependence) do not realize that the studied invariance is strongly related to a special type of asymptotic stability (actually meaning CWAS / CWEAS).

Besides the intrinsic value of the stability analysis tools developed by our paper, we are also able to bridge the gap between the research trend commented above and the CWAS / CWEAS framework. Thus, CWAS / CWEAS as special type of AS, reveal the complete meaning of the invariance for symmetrical rectangular sets, whose dependence of time is *a priori* known and explicitly defined.

II. BRIEF OVERVIEW OF CWAS AND CWEAS

This short presentation of the key concepts and results on CWAS and CWEAS is based on the initial formulation proposed for the continuous-time case in [1], [2] and, later on, unified for discrete-time and continuous-time cases in [7], [8].

Consider the linear system:

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),\ \mathbf{A} \in \mathbf{R}^{n\times n}, \tag{1}$$

where $t \in \mathbf{T}$ denotes the independent variable with discrete-time meaning $\mathbf{T} = \mathbf{Z}_+$, or continuous-time meaning $\mathbf{T} = \mathbf{R}_+$, and the action of the operator ( )' is defined by:

$$\mathbf{x}'(t) = \begin{cases} \mathbf{x}(t+1) \text{ for the discrete-time case } t \in \mathbf{T} = \mathbf{Z}_+; \\ \dot{\mathbf{x}}(t) \text{ for the continuous-time case } t \in \mathbf{T} = \mathbf{R}_+. \end{cases} \tag{2}$$

Definition 1. Given the vector function $\mathbf{h}(t) : \mathbf{T} \to \mathbf{R}^n$, which fulfils the following conditions:

(a) in the discrete-time case ($\mathbf{T} = \mathbf{Z}_+$), $\mathbf{h}$(t) has positive components $h_i(t) > 0$, $i = 1,\dots,n$, and $\lim_{t\to\infty} \mathbf{h}(t) = 0$,

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(b) in the continuous-time case ($\mathbf{T} = \mathbf{R}_+$), $\mathbf{h}(t)$ is differentiable, has positive components $h_i(t) > 0$, $i = 1, \dots, n$, and $\lim_{t\to\infty} \mathbf{h}(t) = 0$, system (1) is called componentwise asymptotically stable (CWAS) with respect to $\mathbf{h}(t)$ if

$$\begin{aligned} &\forall t_0, t \in \mathbf{T},\ t_0 \le t : |x_i(t_0)| \le h_i(t_0) \Rightarrow \\ &|x_i(t)| \le h_i(t),\ i = 1, \dots, n \end{aligned} \tag{3}$$

where $x_i(t), i = 1, \dots, n$ denote the state variables of system (1). □

CWAS allows the individual monitoring of each state variable and therefore it represents a refinement of the standard concept of asymptotic stability where the evolution is characterized in the global terms of a vector norm.

Theorem 1. All the functions $\mathbf{h}(t)$ that fulfil the conditions in Definition 1 are solutions of the difference inequality (in the discrete-time case) or differential inequality (in the continuous-time case):

$$\mathbf{h}'(t) \ge \overline{\mathbf{A}}\ \mathbf{h}(t), \tag{4}$$

where the matrix $\overline{\mathbf{A}} \in \mathbf{R}^{n\times n}$ is built from matrix $\mathbf{A}$ in equation (1), as follows:

(a) for the discrete-time case:

$$\overline{a}_{ij} = |a_{ij}|, \quad i, j = 1, \cdots, n; \tag{5a}$$

(b) for the continuous-time case:

$$\begin{aligned} &\overline{a}_{ii} = a_{ii}, i = 1, \cdots, n, \\ &\overline{a}_{ij} = |a_{ij}|,\ i \ne j, i, j = 1, \cdots, n \end{aligned} \tag{5b}$$

□

System (4) confers a consistent dynamical signification to the operator $\overline{\mathbf{A}}$, pointing out the origin of the CWAS concept in the theory of flow-invariant sets. Within this context, it is worth saying that system (4) might have solutions $\mathbf{h}(t)$ that do not fulfill the condition $\lim_{t\to\infty} \mathbf{h}(t) = 0$ in Definition 1, but such solutions are able to define time-dependent sets, which are flow-invariant with respect to system (1).

Theorem 2. System (1) is CWAS with respect to an arbitrary $\mathbf{h}(t)$ which fulfils the conditions in Definition 1, if and only if the matrix $\overline{\mathbf{A}}$ built according to (5a) or (5b) is stable in the Schur or Hurwitz sense, respectively. □

The usage of CWAS with respect to a particular vector function $\mathbf{h}(t)$ of exponential type yields:

Definition 2. (a) In the discrete-time case, system (1) is called componentwise exponential asymptotically stable (CWEAS) if there exist a vector $\mathbf{d} \in \mathbf{R}^n$, with positive components $d_i > 0, i = 1, \dots, n$, and a constant $0 < r < 1$ such that

$$\begin{aligned} &\forall t_0, t \in \mathbf{T} = \mathbf{Z}_+,\ t_0 \le t : |x_i(t_0)| \le d_i r^{t_0} \Rightarrow \\ &|x_i(t)| \le d_i r^t,\ i = 1, \dots, n \end{aligned} \tag{6a}$$

(b) In the continuous-time case, system (1) is called componentwise exponential asymptotically stable (CWEAS) if there exist a vector $\mathbf{d} \in \mathbf{R}^n$, with positive components $d_i > 0, i = 1, \dots, n$, and a constant $r < 0$ such that

$$\begin{aligned} &\forall t_0, t \in \mathbf{T} = \mathbf{R}_+,\ t_0 \le t : |x_i(t_0)| \le d_i e^{rt_0} \Rightarrow \\ &|x_i(t)| \le d_i e^{rt},\ i = 1, \dots, n \end{aligned} \tag{6b}$$

□

The linearity of the dynamics of system (1) guarantees the equivalence between CWAS and CWEAS.

Theorem 3. For both discrete-time and continuous-time cases, system (1) is CWAS with respect to an arbitrary $\mathbf{h}(t)$ which fulfils the conditions in Definition 1 if and only if system (1) is CWEAS. □

On the other hand, the exponential form of the vector function $\mathbf{h}(t)$ considered in Definition 2 results in an algebraic characterization of CWEAS, or, equivalently, CWAS.

Theorem 4. System (1) is CWAS (or equivalently CWEAS), if and only if the system of inequalities constructed with the matrix $\overline{\mathbf{A}}$ (5a) or (5b):

$$\overline{\mathbf{A}}\mathbf{d} \le r\mathbf{d},\ \mathbf{d} \in \mathbf{R}^n, d_i > 0,\ i = 1, \dots, n,\ r \in \mathbf{R} \tag{7}$$

has solutions $0 < r < 1$ in the discrete-time case, or $r < 0$ in the continuous-time case, respectively. □

The special structure of matrix $\overline{\mathbf{A}}$ built according to (5a) or (5b) induces a spectral property to $\overline{\mathbf{A}}$ of crucial importance for the compatibility of inequality (7):

Theorem 5. Denote by $\lambda_i(\overline{\mathbf{A}})$, $i = 1, \dots, n$, the eigenvalues of the matrix $\overline{\mathbf{A}}$.

i) (a) If $\overline{\mathbf{A}}$ is defined according to (5a), then $\overline{\mathbf{A}}$ has a real nonnegative eigenvalue (simple or multiple) denoted by $\lambda_{\max}(\overline{\mathbf{A}})$, meaning the spectral radius, which fulfills the dominance condition

$$|\lambda_i(\overline{\mathbf{A}})| \le \lambda_{\max}(\overline{\mathbf{A}}),\ i = 1, \dots, n. \tag{8a}$$

(b) If $\overline{\mathbf{A}}$ is defined according to (5b), then $\overline{\mathbf{A}}$ has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(\overline{\mathbf{A}})$, meaning the spectral abscissa, which fulfills the dominance condition

$$\mathrm{Re}[\lambda_i(\overline{\mathbf{A}})] \le \lambda_{\max}(\overline{\mathbf{A}}),\ i = 1, \dots, n. \tag{8b}$$

ii) The system of inequalities (7) is compatible if and only if

$$\lambda_{\max}(\overline{\mathbf{A}}) \le r. \tag{9}$$

□

III. CWAS / CWEAS) AND ε ~ δ FORMALISM

Although it was eminently clear that CWAS, or, equivalently, CWEAS represented a stronger concept than the standard asymptotic stability, no proof has been constructed yet for this statement in terms of norms (which actually provide the classical tools for defining asymptotic stability). Let us show that the exponential asymptotic stability incorporates the concept of CWEAS as a special case, by using the well known ε ~ δ language. Therefore consider the following general condition which ensures the *exponential asymptotic stability* for the equilibrium point {0} of linear system (1) (e.g. [10], pp. 107):

(a) for the discrete-time case:

$$\begin{aligned}&\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0, 0 < \omega < 1 : \| \mathbf{x}(t_0) \| \le \delta(\varepsilon) \Rightarrow \\ &\forall t \ge t_0 : \| \mathbf{x}(t) \| \le \varepsilon \omega^{(t-t_0)}\end{aligned} \tag{10a}$$

(b) for the continuous-time case:

$$\begin{aligned}&\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0,\ \omega < 0 : \| \mathbf{x}(t_0) \| \le \delta(\varepsilon) \Rightarrow \\ &\forall t \ge t_0 : \| \mathbf{x}(t) \| \le \varepsilon e^{\omega(t-t_0)}\end{aligned} \tag{10b}$$

where $\| \ \|$ denotes an arbitrary vector norm in $\mathbf{R}^n$.

On the other hand, define the vector norm:

$$\| \mathbf{x} \|_{\mathbf{D}\infty} = \| \mathbf{D}^{-1}\mathbf{x} \|_{\infty}, \tag{11}$$

where the diagonal matrix

$$\mathbf{D} = \mathrm{diag}\{d_1, \cdots, d_n\} \tag{12}$$

is built with the positive constants $d_i > 0, i = 1, \cdots, n$.

Theorem 6. System (1) is CWEAS if and only if condition (10) is met with $\delta(\varepsilon) = \varepsilon$, $\omega = r$ and for the vector norm $\| \ \|_{\mathbf{D}\infty}$ given by (11).

Proof: The inequality $\| \mathbf{x}(t_0) \|_{\mathbf{D}\infty} \le \varepsilon$ is equivalent to the componentwise inequality $| \mathbf{x}(t_0) | \le \varepsilon \mathbf{d}$ and

(a) for the discrete-time case, the inequality $\| \mathbf{x}(t) \|_{\mathbf{D}\infty} \le \varepsilon r^{(t-t_0)}$ is equivalent to the componentwise inequality $| \mathbf{x}(t) | \le \varepsilon \mathbf{d} r^{(t-t_0)}$ for $t \ge t_0$;

(b) for the continuous-time case, the inequality $\| \mathbf{x}(t) \|_{\mathbf{D}\infty} \le \varepsilon e^{r(t-t_0)}$ is equivalent to the componentwise inequality $| \mathbf{x}(t) | \le \varepsilon \mathbf{d} e^{r(t-t_0)}$ for $t \ge t_0$. □

Proving that the CWEAS property is obtainable from the general definition of the exponential asymptotic stability, this result motivates us to further explore the standard instruments used by the stability analysis of linear systems in order to characterize CWAS / CWEAS.

IV. CWAS / CWEAS and Properties of Operator A

Theorems 4 and 5 are extremely valuable in characterizing the CWAS (CWEAS) of system (1), because they permit a complete exploration of the link between the scalar *r*, vector **d** and matrix $\overline{\mathbf{A}}$ constructed according to (5). Nevertheless, they are unable to link *r* and **d** directly to matrix **A** used in system (1). One can overcome this disadvantage, by introducing the *matrix norm* subordinate to the vector norm $\| \ \|_{\mathbf{D}\infty}$ defined in (11) with (12):

$$\| \mathbf{M} \|_{\mathbf{D}\infty} = \| \mathbf{D}^{-1}\mathbf{M}\mathbf{D} \|_{\infty}, \quad \mathbf{M} \in \mathbf{R}^{n \times n}. \tag{13}$$

Theorem 7. Consider a square matrix **A** and the matrix $\overline{\mathbf{A}}$ built from it according to (5). A positive vector **d** and a constant r are a solution of inequality (7) if and only if

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) \le r, \tag{14}$$

where $\mu_{\mathbf{D}\infty}(\mathbf{A})$ denotes a matrix measure defined by:

(a) for $\overline{\mathbf{A}}$ built according to (5a):

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) = \| \mathbf{A} \|_{\mathbf{D}\infty}; \tag{15a}$$

(b) for $\overline{\mathbf{A}}$ built according to (5b):

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) = \lim_{\tau \to 0+} \frac{\| \mathbf{I} + \tau\mathbf{A} \|_{\mathbf{D}\infty} - 1}{\tau} \tag{15b}$$

Proof: Algebraic inequality (7) can be written as:

$$(1/d_i) \sum_{j=1}^{n} \overline{a}_{ij} d_j \le r, \ i = 1, \ldots, n, \tag{16}$$

or, equivalently:

$$\max_{i=1,\cdots,n} \left\{ (1/d_i) \sum_{j=1}^{n} \overline{a}_{ij} d_j \right\} \le r. \tag{17}$$

(a) For the discrete-time case, all the elements $\overline{a}_{ij}$ constructed in accordance with (5a) are nonnegative and, therefore, (17) is equivalent to:

$$\| \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \|_{\infty} \le r, \tag{18a}$$

which, taking into account (15a), means inequality (14).

(b) For the continuous-time case, in accordance with (5b) all the elements $\overline{a}_{ij}, i \ne j$ are nonnegative. If the same big positive constant $\frac{1}{\tau} \ge \| \mathbf{A} \|_2$, is added to both sides of each inequality (16), then all the elements $\overline{a}_{ii} + \frac{1}{\tau}$ become also nonnegative and, therefore, (16) is equivalent to:

$$\| \mathbf{D}^{-1}(\frac{1}{\tau}\mathbf{I} + \mathbf{A})\mathbf{D} \|_{\infty} \le r + \frac{1}{\tau}, \tag{18b}$$

which, taking into account (15b), means inequality (14). □

Remark 1. The matrix measure defined by (15b) for $\mathbf{D} = \mathbf{I}$ the identity matrix is frequently referred to as the "logarithmic norm" [11], although it does not meet all the properties of a norm. □

Remark 2. The *n* inequalities given by (16), which are equivalent to CWEAS, express the condition that the generalized Gershgorin disks of the matrix $\overline{\mathbf{A}}$ lay inside the unit circle or in the left half plane of the complex plane. In the continuous-time case these disks are identical to those of the matrix **A** (as pointed out in [2]), and in the discrete-time case, they can be identical to those of the matrix **A,** or symmetrical with respect to the imaginary axis of the complex plane. Therefore the usage, in the very recent paper [12], of condition (16) for the particular case $d_i = 1$, $i = 1, \cdots, n$, as a parametric definition for a property called "superstability" has no reason and yields particular forms of the CWEAS results available from [1], [2], [3], [7], [8]. □

Theorem 8. The dominant eigenvalue $\lambda_{\max}(\overline{\mathbf{A}})$ introduced in Theorem 4 fulfills the condition:

$$\lambda_{\max}(\overline{\mathbf{A}}) = \min_{\mathbf{D}=diag\{d_i\}} \mu_{\mathbf{D}\infty}(\mathbf{A}), \tag{19}$$

where $\mu_{\mathbf{D}\infty}(\mathbf{A})$ is defined by (15a) or (15b), in accordance with the procedure for building $\overline{\mathbf{A}}$ (5a) or (5b), respectively.

Proof: (a) In the discrete-time case, (19) results from the equality proven in (Theorem 2, [13]) for nonnegative matrices:

$$\lambda_{\max}(\overline{\mathbf{A}}) = \min_{\mathbf{D}=diag\{d_i\}} \| \mathbf{D}^{-1}\overline{\mathbf{A}}\mathbf{D} \|_{\infty}, \tag{20a}$$

together with:

$$\| \mathbf{D}^{-1}\overline{\mathbf{A}}\mathbf{D} \|_{\infty} = \| \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \|_{\infty} = \mu_{\mathbf{D}\infty}(\mathbf{A}). \tag{21a}$$

(b) In the continuous-time case, (19) results along the same lines, by taking into consideration the nonnegativeness of the matrix $\frac{1}{\tau}\mathbf{I}+\overline{\mathbf{A}}$, as well as the fact that for small $\tau > 0$ (i.e. $\tau \le 1/\|\overline{\mathbf{A}}\|_2$ satisfied) one can write:

$$\lambda_{\max}(\frac{1}{\tau}\mathbf{I}+\overline{\mathbf{A}}) = \min_{\mathbf{D}=diag\{d_i\}} \|\mathbf{D}^{-1}(\frac{1}{\tau}\mathbf{I}+\overline{\mathbf{A}})\mathbf{D}\|_\infty \quad (20b)$$

and

$$\begin{aligned}&\lim_{\tau\to 0+}\left(\|\mathbf{D}^{-1}(\frac{1}{\tau}\mathbf{I}+\overline{\mathbf{A}})\mathbf{D}\|_\infty - \frac{1}{\tau}\right) = \\ &\lim_{\tau\to 0+}\left(\|\mathbf{D}^{-1}(\frac{1}{\tau}\mathbf{I}+\mathbf{A})\mathbf{D}\|_\infty - \frac{1}{\tau}\right) = \mu_{\mathbf{D}\infty}(\mathbf{A})\end{aligned} \quad (21b)$$

□

Theorem 9. Linear system (1) is CWAS / CWEAS if and only if

(a) for the discrete-time case, there exists a vector with positive entries $\mathbf{d} \in \mathbf{R}^n$, such that

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) < 1, \quad (22a)$$

(b) for the continuous-time case, there exists a vector with positive entries $\mathbf{d} \in \mathbf{R}^n$, such that

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) < 0, \quad (22b)$$

where $\mu_{\mathbf{D}\infty}(\mathbf{A})$ is defined according to (15a) and (15b), respectively.

Proof: It results directly from Theorems 2 and 5 combined with Theorem 8. □

V. CWAS / CWEAS AND LYAPUNOV FUNCTIONS

The previous results fully motivate the idea of investigating CWAS by special Lyapunov functions, whose expressions contain precise information about the vector functions $\mathbf{h}(t)$ used in Definition 1.

Theorem 10. Consider a vector function $\mathbf{h}(t)$ that fulfills the conditions in Definition 1. System (1) is CWAS with respect to $\mathbf{h}(t)$, if and only if

$$\begin{aligned}&V(t,x(t)) = \|\mathbf{H}^{-1}(t)x(t)\|_\infty, \\ &\mathbf{H}(t) = \mathrm{diag}\{h_1(t),\ldots,h_n(t)\}\end{aligned} \quad (23)$$

is a weak Lyapunov function for system (1).

Proof: Given the properties of the vector function $\mathbf{h}(t)$, in both discrete-time and continuous-time cases $V(t,\mathbf{x}(t)) > 0$ for any t and $\mathbf{x}(t) \ne 0$.

(a) In the discrete-time case, $V(t,\mathbf{x}(t))$ is a weak Lyapunov function for system (1) means:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\frac{V(t+1,\mathbf{x}(t+1))}{V(t,\mathbf{x}(t))} \le 1\end{aligned} \quad (24a)$$

which can be also written as:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\frac{\|(\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty}{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} \le 1\end{aligned} \quad (25a)$$

If (25a) is true, then we have:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+: \\ &\max_{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty = 1} \|(\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty \le 1\end{aligned} \quad (26a)$$

that is equivalent to the boundedness of the operator norm:

$$\forall t \in \mathbf{T} = \mathbf{Z}_+: \|\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t)\|_\infty \le 1. \quad (27a)$$

Now, taking into account the equality:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+: \|\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t)\|_\infty = \\ &= \|\mathbf{H}^{-1}(t+1)\overline{\mathbf{A}}\mathbf{H}(t)\|_\infty,\end{aligned} \quad (28a)$$

relationship (27a) yields:

$$\forall t \in \mathbf{T} = \mathbf{Z}_+: \|\mathbf{H}^{-1}(t+1)\overline{\mathbf{A}}\mathbf{H}(t)\|_\infty \le 1, \quad (29a)$$

which means that inequality (4) is satisfied with $\mathbf{h}(t)$ meeting conditions in Definition 1, i.e. system (1) is CWAS with respect to $\mathbf{h}(t)$.

Conversely, if system (1) is CWAS with respect to $\mathbf{h}(t)$, then relationship (27a) holds and allows writing:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\frac{\|(\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty}{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} \\ &\le \frac{\|\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t)\|_\infty \|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty}{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} = \\ &= \|\mathbf{H}^{-1}(t+1)\mathbf{A}\mathbf{H}(t)\|_\infty \le 1\end{aligned} \quad (30a)$$

which shows that (25a) is true, i.e. $V(t,\mathbf{x}(t))$ defined by (23) is a weak Lyapunov function.

(b) In the continuous-time case, $V(t,\mathbf{x}(t))$ is a weak Lyapunov function for system (1) means:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\lim_{\tau\to 0+} \frac{V(t+\tau,\mathbf{x}(t+\tau)) - V(t,\mathbf{x}(t))}{\tau} \le 0\end{aligned} \quad (24b)$$

which, for small $\tau > 0$ can be also written as:

$$\begin{aligned}&\forall t \in \mathbf{T}=\mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\frac{\|(\mathbf{H}^{-1}(t+\tau)(I+\tau\mathbf{A})\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty}{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} \le 1\end{aligned} \quad (25b)$$

If (25b) is true, then, for small $\tau > 0$, we have:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \quad (26b) \\ &\max_{\|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty = 1} \|(\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty \le 1\end{aligned}$$

that is equivalent to the boundedness of the operator norm:

$$\forall t \in \mathbf{T} = \mathbf{R}_+: \|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t)\|_\infty \le 1. \quad (27b)$$

Now, taking into account the equality:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+: \|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t)\|_\infty = \\ &= \|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\overline{\mathbf{A}})\mathbf{H}(t)\|_\infty\end{aligned} \quad (28b)$$

valid for small $\tau > 0$, relationship (27b) yields:

$$\forall t \in \mathbf{T} = \mathbf{R}_+: \|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\overline{\mathbf{A}})\mathbf{H}(t)\|_\infty \le 1, \quad (29b)$$

which means that inequality (4) is satisfied with $\mathbf{h}$(t) meeting conditions in Definition 1, i.e. system (1) is CWAS with respect to $\mathbf{h}$(t).

Conversely, if system (1) is CWAS with respect to $\mathbf{h}(t)$, then relationship (27b) holds and allows writing:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\}: \\ &\frac{\|(\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t))(\mathbf{H}^{-1}(t)\mathbf{x}(t))\|_\infty}{\| \mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} \le \\ &\frac{\|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t)\|_\infty \|\mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty}{\| \mathbf{H}^{-1}(t)\mathbf{x}(t)\|_\infty} = \\ &= \|\mathbf{H}^{-1}(t+\tau)(\mathbf{I}+\tau\mathbf{A})\mathbf{H}(t)\|_\infty \le 1\end{aligned} \tag{30b}$$

which shows that (25b) is true, i.e. $V(t, \mathbf{x}(t))$ defined by (23) is a weak Lyapunov function.

□

For the particular case when testing CWEAS and the vector function $\mathbf{h}(t)$ considered in Definition 1 is of exponential type (see Definition 2), the explicit time-dependence of the Lyapunov function becomes redundant as shown below.

Theorem 11. System (1) is CWEAS with $d_i > 0, i = 1, \cdots, n$, if and only if

$$V(\mathbf{x}(t)) = \| \mathbf{x}(t) \|_{\mathbf{D}\infty} \tag{31}$$

is a strong Lyapunov function.

Proof: Given the particular form of matrix D used in (31), $V(\mathbf{x}(t)) > 0$ for any t and $\mathbf{x}(t) \neq 0$), in both discrete-time and continuous-time cases.

(a) In the discrete-time case, $V(\mathbf{x}(t))$ is a strong Lyapunov function for system (1) means:

$$\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \frac{V(\mathbf{x}(t+1))}{V(\mathbf{x}(t))} < 1 \tag{32a}$$

which can be also written as:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \\ &\forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \frac{\| (\mathbf{D}^{-1}\mathbf{A}\mathbf{D})(\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty}{\| \mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} < 1\end{aligned} \tag{33a}$$

If (33a) is true, then we have:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+ : \\ &\max_{\|\mathbf{D}^{-1}(t)\mathbf{x}(t)\|_\infty = 1} \| (\mathbf{D}^{-1}\mathbf{A}\mathbf{D})(\mathbf{D}^{-1}(t)\mathbf{x}(t)) \|_\infty < 1\end{aligned} \tag{34a}$$

that is equivalent to the boundedness of the operator norm:

$$\forall t \in \mathbf{T} = \mathbf{Z}_+ : \| \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \|_\infty < 1 . \tag{35a}$$

Thus, we have shown that

$$\mu_{\mathbf{D}\infty}(\mathbf{A}) = \| \mathbf{A} \|_{\mathbf{D}\infty} = \| \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \|_\infty < 1 \tag{36a}$$

which, in accordance with Theorem 9, ensures CWEAS of system (1) with $d_i > 0, i = 1, \cdots, n$.

Conversely, CWEAS of system (1) with $d_i > 0, i = 1, \cdots, n$, means CWAS with respect to $\mathbf{h}(t) = \mathbf{d}r^t$, $0 < r < 1$, which, according to Theorem 10, is equivalent to:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \\ &\frac{\| (r^{-(t+1)}\mathbf{D}^{-1}\mathbf{A}r^t\mathbf{D})(r^{-t}\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty}{\| r^{-t}\mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} \le 1\end{aligned} \tag{37a}$$

or, furthermore:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{Z}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \\ &\frac{\| (\mathbf{D}^{-1}\mathbf{A}\mathbf{D})(\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty}{\| \mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} \le r < 1.\end{aligned} \tag{38a}$$

Thus, we have proved the validity of (33a) and, consequently of (32a), i.e. $V(\mathbf{x}(t))$ is a strong Lyapunov function for system (1).

(b) In the continuous-time case, $V(\mathbf{x}(t))$ is a strong Lyapunov function for system (1) means:

$$\begin{aligned}&\forall t \in \mathbf{T} = R_+, \forall \mathbf{x}(t) \in \mathbf{R}^n : \\ &\lim_{\tau \to 0+} \frac{V(\mathbf{x}(t+\tau)) - V(\mathbf{x}(t))}{\tau} < 0\end{aligned} \tag{32b}$$

which, for small $\tau > 0$ can be also written as:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \\ &\frac{\| \mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})\mathbf{D}\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty}{\| \mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} < 1\end{aligned} \tag{33b}$$

If (33b) is true, then, for small $\tau > 0$, we have:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \\ &\max_{\|\mathbf{D}^{-1}\mathbf{x}(t)\|_\infty = 1} \| (\mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})\mathbf{D})(\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty < 1\end{aligned} \tag{34b}$$

that is equivalent to the boundedness of the operator norm:

$$\forall t \in \mathbf{T} = \mathbf{R}_+ : \| \mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})\mathbf{D} \|_\infty < 1 . \tag{35b}$$

Thus, we have shown that

$$\begin{aligned}\mu_{\mathbf{D}\infty}(\mathbf{A}) &= \lim_{\tau \to 0+} \frac{\| \mathbf{I}+\tau\mathbf{A} \|_{\mathbf{D}\infty} - 1}{\tau} = \\ &= \lim_{\tau \to 0+} \frac{\| \mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})\mathbf{D} \|_\infty - 1}{\tau} < 0\end{aligned} \tag{36b}$$

which, in accordance with Theorem 9, ensures CWEAS of system (1) with $d_i > 0,\ i = 1, \cdots, n$.

Conversely, CWEAS of system (1) with $d_i > 0, i = 1, \cdots, n$, means CWAS with respect to $\mathbf{h}(t) = \mathbf{d}e^{rt}$, $r < 0$, which, according to Theorem 10, is equivalent to:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \\ &\frac{\|(e^{-r(t+\tau)}\mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})e^{rt}\mathbf{D})(e^{-rt}\mathbf{D}^{-1}\mathbf{x}(t))\|_\infty}{\| e^{-rt}\mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} \le 1\end{aligned} \tag{37b}$$

or, furthermore:

$$\begin{aligned}&\forall t \in \mathbf{T} = \mathbf{R}_+, \forall \mathbf{x}(t) \in \mathbf{R}^n \setminus \{0\} : \\ &\frac{\| \mathbf{D}^{-1}(\mathbf{I}+\tau\mathbf{A})\mathbf{D}\mathbf{D}^{-1}\mathbf{x}(t)) \|_\infty}{\| \mathbf{D}^{-1}\mathbf{x}(t) \|_\infty} \le e^{r\tau} < 1.\end{aligned} \tag{38b}$$

Thus, we have proved the validity of (33b) and, consequently, of (32b), i.e. $V(x(t))$ is a strong Lyapunov function for system (1). □

Remark 3. In papers [14], [15], [16] the usage of Lyapunov function (31) is understood in the sense of standard AS, but pointing out the invariance of a time-independent polyhedral set. Papers [17], [18] notice that Lyapunov function (31) induces a time-dependence of exponential type for the invariant polyhedral sets; however the stability analysis is addressed within the classical framework, without any interpretation of the componentwise meaning. Moreover, the case of invariant polyhedral sets with arbitrary time-dependence (not only exponential) remains completely ignored by these two papers. □

VI. Illustrative Example

Consider system (1) in the continuous time case with the matrix **A** defined by:

$$\mathbf{A} = \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix}, \alpha > 0,\ \beta > 0. \tag{39}$$

For arbitrary $d_1 > 0, d_2 > 0$, we can simply write, according to (15b):

$$\mu_{D\infty}(\mathbf{A}) = \lim_{\tau \to 0+} \frac{1}{\tau} \left(\left\| \begin{bmatrix} 1 - \tau\,\alpha & \tau \dfrac{d_2}{d_1}\beta \\ -\tau \dfrac{d_1}{d_2}\beta & 1 - \tau\,\alpha \end{bmatrix} \right\|_\infty - 1 \right). \tag{40}$$

By applying Theorem 9, after some calculations it results that condition (22b) is equivalent to

$$\frac{\alpha}{\beta} > \max\left\{ \frac{d_1}{d_2}, \frac{d_2}{d_1} \right\}, \tag{41}$$

meaning that system (1) is CWAS / CWEAS if and only if $\alpha > \beta$.

The same conclusion is obtained if we use Theorem 11 and Lyapunov function (31) with the following concrete form:

$$V(\mathbf{x}(t)) = \left\| \begin{bmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\|_\infty . \tag{42}$$

For $x_1(t) \neq 0$ and using the notation $\mathrm{tg}\ \gamma(t) = x_2(t)/x_1(t)$, we can write:

$$V(\mathbf{x}(t)) = |x_1(t)| \left\| \begin{bmatrix} 1/d_1 \\ \mathrm{tg}\ \gamma(t)/d_2 \end{bmatrix} \right\|_\infty , \tag{43}$$

$$V(\mathbf{x}(t+\tau)) = \frac{|x_1(t)|\, e^{-\tau\alpha}}{\cos\gamma(t)} \left\| \begin{bmatrix} \cos(\gamma(t) - \beta\tau)/d_1 \\ \sin(\gamma(t) - \beta\tau)/d_2 \end{bmatrix} \right\|_\infty . \tag{44}$$

$V(\mathbf{x}(t))$ defined by (42) is a strong Lyapunov function for system (1) (i.e. inequality (32b) holds) if and only if condition (41) is met.

Remark 4. A class of continuous-time systems, that includes the system considered in this example, was extensively explored in [15]. The usage of a Lyapunov function built with the infinity norm refers to the standard AS, and the paper places its emphasis on the invariance of the polyhedral sets regarded as time-independent. □

VII. Conclusions

By using the infinity norm, well-known results from the classical theory of stability can be particularized so as to characterize CWAS / CWEAS as a special type of asymptotic stability. Thus, our approach allows developing connections between the dynamics of system (1) and CWAS / CWEAS, by circumventing the usage of auxiliary system (4) and applying standard tools in stability theory directly to system (1). The key results refer to the exploitation of the following instruments: ε - δ formalism (Theorem 6), properties of operator **A** (Theorem 9), time-dependent Lyapunov functions for testing CWAS with respect to an arbitrary vector function (Theorem 10) and time-independent Lyapunov functions for testing CWEAS (Theorem 11).

VIII. References

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