Numerical Accurate Computations for Ellipsoidal State Bounding

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***Abstract*— Ellipsoidal outer bounding is revisited from a numerical point of view. Numerically stable algorithms are discussed around square root factorization techniques. Geometrical interpretation is given through standard orthogonal projection arguments that shows directly some basic properties attached to these methods. Efficient numerical algorithm is given to test if two or more ellipsoids have non-empty intersection. Lastly, specific formulations are given to guarantee that the theoretical properties associated to trace or determinant criterion computations are numerically satisfied.**

***Index Terms*— Numerical stability, square root factorization, ellipsoidal bounding, determinant / trace criterion.**

I. Introduction

As in Kalman filtering, ellipsoidal state outer bounding is based upon two basic steps: prediction and correction [1]. Each of them involves two quite different kinds of computations, namely summations of ellipsoids and intersections of ellipsoids. Most of the published papers present algorithms directly issued from theoretical considerations. Unfortunately, these algorithms are basically unstable from a numerical point of view, which explains why they frequently fail when they are applied to real life problems. Actually, the arguments are, here, very similar to those invoked about standard Kalman filtering algorithms. Therefore, the way to construct numerically stable algorithms looks like the tricks used to stabilize Kalman's formulas via square root factorization updating. As it will be shown later, the proposed approach offers other mathematical and geometrical interpretations in a unified framework of all the classical families of enclosing ellipsoids. Specific algorithmic details are given to efficiently implement the two standard optimization trace and determinant criteria.

The paper is organized as follows. In section 2, classical definitions are reminded. Section 3 is devoted to the intersection problem while section 4 deals with the summation one. Section 5 is concerned with the optimization problem. Some numerical examples are given in section 6, while the last section gives some conclusion and future developments.

II. Definitions

Any ellipsoid $E(c,M)$ can be defined by:

$$\left\{x\in\mathbb{R}^n : (x-c)^T M(x-c)\le 1 \,, M=M^T\ge 0\right\}. \tag{1}$$

When M has no zero eigenvalue, the ellipsoid is bounded, otherwise it is unbounded. This is the case when parallel strips $S(y,d)$ are considered with:

$$S(y,d)=\left\{x\in\mathbb{R}^n : \left|y-d^T x\right|\le 1\right\}. \tag{2}$$

Clearly, (2) is equivalent to (1) with $M=dd^T$ and $y=d^T c$. Note that $S(y,d)$ is sometimes called the feasible parameter set [2], the parameter uncertainty set [3] or the likelihood set [4].

Another special feature appears when empty ellipsoid must be taken into account. Using (1) is not numerically attractive, because this situation means that M has at least one infinite eigenvalue.

A better alternative is to characterize the ellipsoid by the following equivalent formulation:

$$E(c,P)=\left\{x\in\mathbb{R}^n : (x-c)^T P^+(x-c)\le 1 \,, P=P^T\ge 0\right\} \tag{3}$$

When a non-empty bounded ellipsoid is considered, the Moore-Penrose pseudo-inverse P^+ becomes $P^{-1}=M$ and P has at least one zero eigenvalue for an empty ellipsoid.

This paper is now restricted to the intersection and summation problems of non-empty ellipsoids but not necessarily bounded. Two kinds of results will be presented, namely, the classical ones based upon standard mathematical computations and those dedicated to stable numerical implementations.

III. The Intersection Problem

A. Main results

Among many possibilities and technical presentations [5], it has been only kept here the solution families as developed by C. Durieu *et al.*, see for example [6]. Using and adapting their notations, let $\mathcal{A}$ be the set of $\alpha_i\ge 0$ such that $\sum_{i=1}^{K}\alpha_i=1$. If the matrix $\sum_{i=1}^{K}\alpha_i M_i$ is positive definite, then the ellipsoid:

$$E\left(c_\alpha,(1-\delta_\alpha)^{-1}M_\alpha\right) \tag{4}$$

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will contain the intersection of the K ellipsoids $E(c_i, M_i)$ with:

$$M_\alpha = \sum_{i=1}^{K} \alpha_i M_i, \quad c_\alpha = M_\alpha^{-1} \sum_{i=1}^{K} \alpha_i M_i c_i \tag{5}$$

$$\delta_\alpha = \sum_{i=1}^{K} \alpha_i c_i^T M_i c_i - c_\alpha^T M_\alpha c_\alpha \tag{6}$$

where $\alpha \in \mathbb{R}^K$ is the vector whose components are the α_i.

Equations (5)-(6) are reinterpreted from a new point of view. First, a quite different approach is used in order to show this result. Let us define:

$$f(x) = \sum_{i=1}^{K} \alpha_i f_i(x) \le 1$$

where $f_i(x) = (x - c_i)^T M_i (x - c_i) \le 1$.

By direct expansion, it follows that:

$$f(x) = x^T M_\alpha x - 2\sum_{i=1}^{K} \alpha_i c_i^T M_i x + \sum_{i=1}^{K} \alpha_i c_i^T M_i c_i \le 1 . \tag{7}$$

Afterward, a second order Taylor series expansion of $f(x)$ near its unique minimum $\hat{x}$ leads to:

$$f(x) = f(\hat{x}) + (x - \hat{x})^T M_\alpha (x - \hat{x}) \le 1 . \tag{8}$$

The comparison between equations (7) to (8) shows that:

$$\begin{aligned} &c_\alpha = \hat{x}, \\ &f(\hat{x}) = \sum_{i=1}^{K} \alpha_i c_i^T M_i c_i - c_\alpha^T M_\alpha c_\alpha = \delta_\alpha \ge 0. \end{aligned} \tag{9}$$

Furthermore, (8) gives a necessary and sufficient condition for a non-empty[1] intersection of the K ellipsoids $E(c_i, M_i)$:

$$\bigcap_{i=1}^{K} E(c_i, M_i) \ne \varnothing \Leftrightarrow \max_\alpha \delta(\alpha) = f(\hat{x}) < 1 \tag{10}$$

The following theorem presents an alternative to this preliminary result.

Theorem 1

The computation of c_α is equivalent to the determination of the unique solution $\hat{x}$ of the convex quadratic problem:

$$c_\alpha = \hat{x} = \arg\min_x \left\{ f(x) = \left\| \begin{bmatrix} \sqrt{\alpha_1} X_1 \\ \sqrt{\alpha_2} X_2 \\ \vdots \\ \sqrt{\alpha_K} X_K \end{bmatrix} x - \begin{bmatrix} \sqrt{\alpha_1} X_1 c_1 \\ \sqrt{\alpha_2} X_2 c_2 \\ \vdots \\ \sqrt{\alpha_K} X_K c_K \end{bmatrix} \right\|_2^2 \right. \tag{11}$$

$$f(\hat{x}) = \delta_\alpha,$$

with X_i the full rank factorization of M_i, i.e. $X_i^T X_i = M_i$, $i = 1,...,K$.

Proof 1

Let us rewrite the function in (11) as:

$$f(x) = \left\| [A]x - [b] \right\|_2^2 . \tag{12}$$

The minimum $\hat{x}$ is then given by the well-known normal equations $\hat{x} = \left(A^T A\right)^{-1} A^T b$. This immediately leads to:

$$A^T A = M_\alpha = \sum_{i=1}^{K} \alpha_i M_i ; \quad A^T b = \sum_{i=1}^{K} \alpha_i M_i c_i .$$

Consequently, $\hat{x} = c_\alpha$.

Lastly,

$$\begin{aligned} f(\hat{x}) &= \left\| [A]\hat{x} - [b] \right\|_2^2 = b^T \left(I - A\left(A^T A\right)^{-1} A^T \right) b \\ &= b^T b - b^T A \left(A^T A\right)^{-1} \left(A^T A\right) \left(A^T A\right)^{-1} A^T b \\ &= \sum_{i=1}^{K} \alpha_i c_i^T M_i c_i - c_\alpha^T M_\alpha c_\alpha \\ &= \delta_\alpha \end{aligned}$$

where I is the identity matrix. This result generalizes a previous one published in [8].

B. Numerical considerations

Afterwards, consider the computational aspects. The convex quadratic problem is solved using equation (12) by means of an orthogonal factorization [7] of the matrix $[A \quad b]$:

$$Q[A \mid b] = \left[\begin{array}{c|c} X_\alpha & v \\ 0 & \tau \\ \hline 0 & 0 \end{array} \right],$$

$Q \in \mathbb{R}^{Kn \times Kn}$, $X_\alpha \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^{n \times 1}$, $\tau \in \mathbb{R}$,

where X_α is a regular upper triangular matrix. Therefore, the function may be rewritten as:

$$f(x) = \left\| [A]x - [b] \right\|_2^2 = \left\| [X_\alpha]x - [v] \right\|_2^2 + \tau^2 .$$

Therefore, c_α is the solution of the triangular system:

$$X_\alpha c_\alpha = v , \tag{13}$$

and $\delta_\alpha = \tau^2$. Notice that $X_\alpha^T X_\alpha = M_\alpha$.

It is quite evident that the computed δ_α is now always numerically positive, and that the computed matrix M_α is symmetric positive definite, which is not guaranteed using equation (5). Furthermore, $X_\alpha^T X_\alpha = M_\alpha$ means that the condition numbers satisfy:

$$\xi(M_\alpha) = \left[\xi(X_\alpha)\right]^2 ,$$

so that the number of exact digits in c_α may be halved if (5) is used instead of (13).

A numerical example showing a negative δ_α using the standard computations (5) can be found in [8].

[1] non-empty means also "not reduced to a point".

IV. THE SUMMATION PROBLEM

The summation of K ellipsoids $E(c_i, P_i)$ is investigated. For this problem, the center computation is trivial:

$c_\alpha = c = \sum_{i=1}^{K} c_i$.

Consequently, this second problem only involves the computation of the shape and the size of an ellipsoid. In this context, the best representation is given by (3). According to [6]:

$P_\alpha = \sum_{i=1}^{K} P_i / \alpha_i \geq 0$

must be computed. Note that $\alpha_i \geq 0$ with $\sum_{i=1}^{K} \alpha_i = 1$.

Once again, the definiteness of the matrix P_α will be guaranteed using a square root factorization.

Theorem 2

The factorization Y_α of $P_\alpha = Y_\alpha^T Y_\alpha$ can be computed using an orthogonal factorization *via* Householder transformations of the composite matrix *Z*:

$$Z = \left[Y_1^T / \sqrt{\alpha_1}, \quad Y_2^T / \sqrt{\alpha_2}, \quad \cdots, \quad Y_K^T / \sqrt{\alpha_K} \right]^T \tag{14}$$

where Y_i is the full rank factorization of $P_i = Y_i^T Y_i$, $i = 1, ..., K$.

Proof 2

First, the orthogonal factorization of *Z* leads to:

$$Q[Z] = \left[\begin{array}{c} Y_\alpha \\ \hline 0 \end{array} \right], Q \in \mathbb{R}^{Kn \times Kn}, Y_\alpha \in \mathbb{R}^{n \times n} .$$

Then $Y_\alpha^T Y_\alpha = Z^T \left(Q^T Q \right) Z = Z^T Z = \sum_{i=1}^{K} P_i / \alpha_i = P_\alpha$.

Remark

Even if some of the matrices M_i or P_i are singular, usually M_α and P_α are positive definite. However computing one from the other may induce catastrophic numerical errors if their condition numbers ξ achieve very large values. Using square root formulation offers a drastic advantage due to the property:

$\xi(Y_\alpha) = \xi(X_\alpha) = \sqrt{\xi(M_\alpha)} = \sqrt{\xi(P_\alpha)}$.

V. THE OPTIMIZATION PROBLEM

A. Problem statement

The computation of the vector $\alpha = [\alpha_1 \quad \cdots \quad \alpha_K]^T$ is the solution of a constrained optimization problem. The criterion is usually based upon the trace or determinant of the matrix $P_\alpha = Y_\alpha^T Y_\alpha = \left(X_\alpha^T X_\alpha \right)^{-1}$ according to the context. This kind of problem will be referred to as OP1 problem. Note that other criterion may be used, as mentioned for instance in [5], but they are not investigated for comparison purpose in this paper.

Actually, a smaller ellipsoid is achieved if the matrix M_α or P_α is respectively replaced with $(1 - \delta_\alpha)^{-1} M_\alpha$ or $(1 - \delta_\alpha) P_\alpha$, and this second problem is referred to as OP2.

In practice, this harder situation is solved only with $K = 2$. This will be called OP3 problem in the sequel, and a particular case deals with one ellipsoid and one strip.

Generally, there is no explicit solution to those optimization problems. However, the trace criterion for the summation problem OP1 has a closed form solution [6]:

$$\alpha_i = \frac{tr(P_i)}{\sum_{i=1}^{K} tr(P_i)} = \frac{\|Y_i\|_F^2}{\sum_{i=1}^{K} \|Y_i\|_F^2}$$

where $\|*\|_F$ is the matrix Frobenius norm.

Although dealing sequentially with two ellipsoids leads to a sub-optimal solution, only the case $K = 2$ will be considered in the following.

B. Intersection of a strip and an ellipsoid

First, consider the problem OP3 with one strip $S(y, d)$ (which can be interpreted as the degenerated ellipsoid E_l) and one ellipsoid $E(c, P = M^{-1})$. The trace criterion leads to a particular root computation of a 3rd order polynomial equation [9] whose coefficients depend on terms like $v^T Q v$, where Q stands for $M, M^2, P, P^2, P^{-1}, \ldots$ Note that a particular case occurs when the ellipsoid $E(c, P = M^{-1})$ is entirely located in the strip. In that case, the measurement does not introduce information. Therefore, it can be discarded, leading to a "dead zone" for the algorithm [5]. As a result, the ellipsoid $E(c, P = M^{-1})$ is not updated. Another important situation appears when one of the two hyperplans defining $S(y, d)$ does not intersect $E(c, P = M^{-1})$. In that case, the strip is first reduced using a tangent hyperplan instead of the non-intersected one before any computation.

It is impossible to achieve theoretical properties associated with these expressions without working on their factorized counterpart. The following parameters have to be computed:

$$\begin{aligned} &\mu = tr\left(M^{-1}\right) = tr(P), \\ &g = d^T M^{-1} d = d^T P d, \\ &h = d^T M^{-2} d = d^T P^2 d, \end{aligned} \tag{15}$$

and the factorized forms yield:

$\mu = \|Y\|_F^2 = \|Z\|_F^2$, $Z : X^T Z = I$,

$g = \|Yd\|_2^2 = \|v\|_2^2$, $v = Zd$,

$h = \|Pd\|_2^2 = \|w\|_2^2$, $w = Z^T v$.

Note that for instance the computed value of $d^T Pd$ may have a wrong sign, whereas $\|v\|_2^2 \geq 0$ is true whatever the machine precision may be.

If the determinant criterion is involved, a particular root of a 2nd order polynomial equation has to be found. The only critical term to be evaluated is g (15) previously defined.

C. Intersection of 2 ellipsoids

The problem OP3 is now considered with two non-empty and bounded ellipsoids $E(c_1, M_1)$ and $E(c_2, M_2)$ (i.e. the matrices M_1, M_2 symmetric, definite positive).

Lemma

The intersection of two ellipsoids is equivalent to the following problems:

$$\begin{aligned} E(c_1, M_1) \cap E(c_2, M_2) &\Leftrightarrow E(0, I) \cap E(c, D) \\ &\Leftrightarrow E(c', I) \cap E(0, D) \end{aligned} \quad (16)$$

where D is a diagonal matrix, I is the identity matrix, 0 is the zero vector.

Remark

In other words, the intersection of two ellipsoids can always be reduced to the intersection of a unit ball with an ellipsoid whose axes are parallel to the coordinate system axes. In the sequel, the first equivalence in (16) will be proved in details, while some information will be given for the second one.

Proof 3

Using $M_1 = X^T X$, the first ellipsoid can be expressed as:

$$\begin{aligned} E(c_1, M_1) &= \left\{ x \in \mathbb{R}^n : (x - c_1)^T X^T X (x - c_1) \leq 1 \right\} \\ &= \left\{ y = X(x - c_1) : y^T y \leq 1 \right\}, \end{aligned} \quad (17)$$

so that $E(c_1, M_1) \Leftrightarrow E(0, I)$. A change in the variables leads to the second ellipsoid $E(c_2, M_2)$ rewriting:

$$\begin{aligned} E(c_2, M_2) = \{ & x \in \mathbb{R}^n, x = X^{-1} y + c_1 : \\ & \left(X^{-1} y + c_1 - c_2\right)^T M_2 \left(X^{-1} y + c_1 - c_2\right) \leq 1 \}. \end{aligned} \quad (18)$$

Let us define v_2 such that:

$$X^{-1} y + c_1 - c_2 = X^{-1}(y - v_2) \Leftrightarrow v_2 = X(c_2 - c_1).$$

Hence (18) becomes:

$$\begin{aligned} E(c_2, M_2) = \{ & x \in \mathbb{R}^n, x = X^{-1} y + c_1 : \\ & (y - v_2)^T M (y - v_2) \leq 1 \}, \end{aligned} \quad (19)$$

with $M = M^T = X^{-T} M_2 X^{-1} > 0$. The matrix M in (19) is then replaced by its eigenvalue decomposition:

$$M = V^T D V \quad (20)$$

where $D > 0$ is a diagonal matrix, and V is orthogonal. It follows that:

$$\begin{aligned} (y - v_2)^T M (y - v_2) &= \left[(y - v_2)^T V^T \right] D \left[V(y - v_2) \right] \\ &= \left[V(y - v_2) \right]^T D \left[V(y - v_2) \right] \end{aligned}$$

Define $z = Vy$. Because the matrix V is orthogonal, $y^T y \leq 1 \Leftrightarrow z^T z \leq 1$, and the second ellipsoid may be rewritten:

$$\begin{aligned} E(c_2, M_2) &= \left\{ x \in \mathbb{R}^n : (x - c_2)^T M_2 (x - c_2) \leq 1 \right\} \\ &= \left\{ z \in \mathbb{R}^n : (z - c)^T D (z - c) \leq 1 \right\} = E(c, D), \end{aligned} \quad (21)$$

with $c = V v_2$.

The second result in (16) follows in the same way with $z = V(y - v_2)$ and $(z - c')^T (z - c') \leq 1$.

D. Intersection Check for K=2

While the intersection of one ellipsoid $E(c, P = M^{-1})$ and one strip $S(y, d)$ can easily be checked, it is a more difficult problem when two ellipsoids are considered. But this preliminary test is an essential condition for computing the including ellipsoid (4). Actually, this test can be done using (6). Some simple but tedious calculations yield to the following useful identity:

$$\delta_\alpha = \alpha(1 - \alpha)(c_1 - c_2)^T \left(\alpha M_2^{-1} + (1 - \alpha) M_1^{-1} \right)^{-1} (c_1 - c_2) \quad (22)$$

Theorem 3

The function δ_α is concave and possesses a unique maximum $\hat{\alpha}$ such that $0 < \hat{\alpha} < 1$.

Proof 4

The ellipsoids $E(c_1 M_1)$ and $E(c_2, M_2)$ are transformed in their equivalent forms $E(c, I)$ and $E(0, D)$. Using (22), δ_α becomes:

$$\delta_\alpha = \alpha(1 - \alpha) c^T \left(\alpha D^{-1} + (1 - \alpha) I \right)^{-1} c.$$

Standard derivations lead to

$$\frac{\partial \delta_\alpha}{\partial \alpha} = -\sum_{i=1}^{n} \frac{\left(d_i^{-1} - 1\right)\alpha^2 - 2\alpha + 1}{\left[\alpha d_i^{-1} + (1 - \alpha)\right]^2} c_i^2, \quad (23)$$

where $D = diag(d_i), d_i > 0$, $c = [c_1 \quad c_2 \quad \cdots \quad c_n]^T$, and

$$\frac{\partial^2 \delta_\alpha}{\partial \alpha^2} = -2 \sum_{i=1}^{n} \frac{d_i^{-1} c_i^2}{\left[\alpha d_i^{-1} + (1 - \alpha)\right]^3} < 0 \quad (24)$$

Note that $\left[\alpha d_i^{-1} + (1 - \alpha)\right] > 0$ for all $\alpha \in]0,1[$. Furthermore, $\delta_{\alpha=0} = \delta_{\alpha=1} = 0$. δ_α being a concave function (24), and positive for $\alpha \in]0,1[$, then there is a unique maximum $\hat{\alpha} \in]0,1[$.

Remark

Any Newton-like optimization method will converge towards $\hat{\alpha}$ in a very few steps from any starting point $\alpha_0 \in \left]0,1\right[$.

E. Numerical implementation

The numerical computation of δ_α should not be done with (9) nor (22). The must efficient way is once again using an orthogonal factorization:

$$Q\begin{bmatrix} \sqrt{\alpha_1}X_1 & \sqrt{\alpha_1}X_1c_1 \\ \sqrt{\alpha_2}X_2 & \sqrt{\alpha_2}X_2c_2 \end{bmatrix} = \begin{bmatrix} X_\alpha & v \\ 0 & \tau \\ & 0 \end{bmatrix} \tag{25}$$

$$\delta_\alpha = \tau^2$$

Clearly, the knowledge of the enclosing ellipsoid does not appear in this computation, nor its center. Actually, c_α remains unknown until the best α is obtained.

VI. NUMERICAL EXAMPLE

First, an academic example will exhibit the powerful of the factorized form of the algorithm. It shows the computation of $\delta_\alpha < 0$ in one iteration for the intersection of a strip $S(y,d)$ and an ellipsoid $E\left(c, P = M^{-1}\right)$. Computations are done in the Matlab environment with double precision. The problem size is n = 8. M is the inverse Hilbert matrix (M = *invhilb(8))*. Vector d is constructed using the first n components of the last column of the inverse Hilbert matrix of dimension 9:

$$Y = invhilb(9)$$
$$d = Y(1:8,9)$$

The ellipsoid center is given by:

$$c = [1,1,...,1]^T \in \mathbb{R}^8$$

while y = 1 and $\alpha = 0.001$.

Standard computation leads to $\delta_\alpha(classical) = -68.23$ while the factorized form gives:

$$\delta_\alpha(factorized) = \mathbf{1.6997358414}37460\mathbf{e-002} \tag{26}$$

The theoretical value obtained using Maple is:
$\delta_\alpha(theoretic) = 1.69973584146841\text{e-}2$.

Bold face figures in (26) show the correct decimal places.

Now, consider an identification problem with unknown but bounded output error:

$$y_k = \begin{cases} u_{k-8} - 2u_{k-7} + 0.1u_{k-6} + 0.05u_{k-5} - 10u_{k-4} + \\ \qquad 0.01u_{k-3} + u_{k-2} - 0.001u_{k-1} + e_k \end{cases} \quad k \geq 9$$

where e_k is a zero mean uniformly distributed random noise with a 10% relative level. The input is given by $u_k = 1/k$. 200 samples have been simulated and the data have been reused five times. Data y_k are shown in Figure 1.

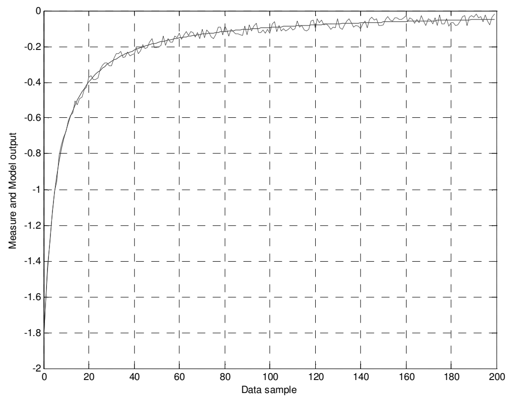

Figure 1: Measure and model output

The ellipsoid centre and minimal / maximal values of each parameter are given in Table 1 and Table 2. Only the significant digits are shown. As expected, very poor results are achieved using the standard formulation of the algorithm. The factorized form exhibits the best possible precision that can be achieved.

min	centre	max
-1.8032299e-002	9.81967701e-001	1.98196770e+000
-2.380407e-001	7.6195932e-001	1.7619593e+000
2.6856808e+001	2.7856808e+001	2.8856808e+001
-3.25567146e+002	-3.24567146e+002	-3.23567146e+002
1.52876100e+003	1.52976100e+003	1.53076100e+003
-3.37182701e+003	-3.37082701e+003	-3.36982701e+003
3.50771837e+003	3.50871837e+003	3.50971837e+003
-1.38851126e+003	-1.38751126e+003	-1.38651126e+003

Table 1: factorized algorithm

min	centre	max
-2.#e-002	9.8e-001	1.98e+000
#.#e+000	8.#e-001	2.##e+000
3.#e+001	3.#e+001	3.##e+001
-3.#e+002	-3.#e+002	-3.##e+002
1.5e+003	1.5e+003	1.5#e+003
-3.4e+003	-3.4e+003	-3.4#e+003
3.5e+003	3.5e+003	3.5#e+003
-1.4e+003	-1.4e+003	-1.4#e+003

Table 2: standard algorithm

Results displayed in Table 2 may seem strange. Actually, the symbol “#” stands for “not exact digit can be computed”. Therefore “#.#e+000” means that the numerical value cannot be computed with at least one exact digit using the standard algorithm and double precision computation

VII. CONCLUSION

This paper gives insight about the numerical difficulties that may appear when ellipsoidal outer-bounded approaches are used.

A unified technique based upon orthogonal factorization and square root representation of semi-definite matrices has been developed for the intersection and the summation problems. Real life problems have clearly exhibited the numerical superiority of such a methodology [8].

Another interesting property (even surprising at a first lecture) is that the intersection condition, $\delta_\alpha < 1$, can be numerically checked without computing any enclosing ellipsoid $E(c_\alpha, M_\alpha)$ or $E(c_\alpha, P_\alpha)$, because $f(\hat{x})$ can be obtained without the knowledge of $\hat{x}$, see (11).

For the intersection of two ellipsoids, the function δ_α has been shown to be concave, and there is a unique maximum $\hat{\alpha}$ over $]0,1[$.

Numerical examples show the expected superiority of the factorized form compared with the standard formulation.

Those algorithms have been successfully applied to industrial data (confidential). A model has been identified using a bounded but unknown error approach [10][11]. This model has been then used in a diagnosis context. The results will also be discussed during the lecture.

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