Induced $\mathcal{L}_2$-gain Domain for LPV-Gain Scheduled Control Systems

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Abstract— **Recent methods for gain scheduling controller design based on linear parameter-varying (LPV) systems offer a systematic way to obtain a nonlinear controller that covers different operating conditions. However, despite that the LPV synthesis part of the process of obtaining a gain scheduled controller is theoretically sound, the nonlinear closed loop system may not exhibit the expected induced $\mathcal{L}_2$ norm for the operating conditions considered in the design. In this paper, this property is illustrated by a simple second order nonlinear system, the well-known Van der Pol equation. Furthermore, an estimate of the domain of validity of the $\mathcal{L}_2$ gain is given, based on the LPV analysis for the nonlinear system.**

Keywords— **Linear parameter-varying systems, Gain scheduled control systems, Induced $\mathcal{L}_2$-norm, Lyapunov functions, Linear matrix inequalities.**

I. Introduction

One of the most popular controller design methods in practical problems is gain scheduling. This method uses a quasi-stationary heuristic approach to the design of nonlinear controllers. The nonlinear control law is formed by a divide and conquer strategy, leading to a synthesis problem for different operating settings together with a mapping of these to cover a wide range of settings. Due to the heuristics, the method has until the last decade or so received little attention in the academic world, see [1], [2].

One decade ago, Linear Parameter-Varying (LPV) systems, see [3], were introduced in the context of gain scheduling. Such systems enable a systematic way of obtaining the controller. The synthesis can incorporate the operating conditions in the scheduling parameter of the system resulting in a controller that is directly parameter dependent, eliminating the explicit mapping of linear controllers.

In parallel to the above mentioned development of LPV system theory, the use of Linear Matrix Inequalities (LMI) in control theory has been developed, see e.g. [4] and references therein. In particular, robust $\mathcal{H}_2$, $\mathcal{H}_\infty$ and μ methods fit into this framework of LMI constraints, see e.g. [5]. The well known Riccati equation for $\mathcal{H}_\infty$ has a corresponding LMI formulation. In the case of full order or state feedback controller synthesis, the problem is convex, see [5], and can be solved readily with available numerical LMI software. The combination of the LMI based synthesis methods and the use of LPV systems led to a systematic way of obtaining a gain scheduled controller in a numerically appealing way.

Using LPV synthesis methods means that a nonlinear system has to be formulated as an LPV system. The LPV system description is conservative in the sense that the nonlinearities of the system are captured by the (scheduling) parameter vector, which usually is allowed to take values within a bounded box, and sometimes there are also constraints on the rate of change of the parameter vector. This means that the LPV system not only describes the original nonlinear system, but also all nonlinear systems obtained when changing the parameter vector arbitrarily, as long as its value stays in the bounding box. The goal of the synthesis is to maintain stability and performance ($\mathcal{L}_2$-norm) for all parameter values in the bounding box, and hence the obtained LPV controller is valid also for the nonlinear system.

The controller synthesis of LPV systems has drawn much attention in the literature. Given an LPV system, the method of obtaining a controller is fairly straight forward. However, the problem how to end up in an LPV description of the nonlinear system is far from straight forward. A standard anzats to this problem is an approximation of the nonlinear system by mapping Taylor linearizations for different operating conditions. It is clear that such LPV models can deviate much from the nonlinear model, and the LPV design may perform badly or even result in an unstable closed loop system of the original nonlinear system, at least for some operating conditions. This procedure is however motivated under the assumption of slowly varying parameters. In this paper, only nonlinear systems that can be exactly included by LPV systems will be considered.

The properties of the nonlinear system will be studied in this paper. In particular, local $\mathcal{L}_2$ gain properties of the nonlinear system is investigated in the context of LPV systems that satisfy the bounded real lemma.

The notation in the paper is standard. It is made difference between local properties of the nonlinear system and the corresponding local LPV properties. In the later, e.g. $\mathcal{L}_2$ gain is only considered for parameter trajectories that stay inside the bounding box and without the connection to the (possible) underlying nonlinear system. This means that there is no distinction of whether the parameter is time-varying, resulting in a linear time-varying system, or depend on the state vector, implying a nonlinear system. This is the common approach to LPV gain scheduling in the literature.

This paper is organized as follows. In the following section, preliminaries is given about LPV systems, $\mathcal{L}_2$ gain etc. This is followed by a section of a motivating example showing that the $\mathcal{L}_2$ gain is a local property and that the LPV analysis does not guarantee the $\mathcal{L}_2$ gain even in the region of the analysis. The next section gives a domain for which the $\mathcal{L}_2$ gain of the LPV system implies the same $\mathcal{L}_2$ gain for the underlying nonlinear system. Also, connected to this, the set of inputs that keeps the system in this domain is given. This section is succeeded by a section that illustrates how these domains can be computed, based on the LPV analysis. Finally some concluding remarks are given.

II. Preliminaries

In this paper nonlinear systems,

$$\dot{x} = f(x,w),\ x \in D \subseteq \mathbb{R}^n,\ w \in \mathbb{R}^k$$

and their local properties will be studied. It will be assumed that all conditions concerning existence and uniqueness of the solution are satisfied. Also, it is assumed that the origin is an isolated stationary point for the unforced system. Nonlinear system relations to Linear Parameter-Varying (LPV) systems are in particular studied. An LPV system is described as follows,

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))w(t) \\ z(t) &= C(\rho(t))x(t) + G(\rho(t))w(t) \end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}^m$ is the output, $w \in \mathbb{R}^k$ is the (disturbance) input and $\rho \in \mathbb{R}^p$ is the (scheduling) parameter vector. The parameter vector may be an exogenous measurable input or contain endogenous (state) variables. In the later, the system is most often referred to as a quasi-LPV system to point out that it is a nonlinear system. The fact that some nonlinear systems can be written (or approximated) as (quasi-) LPV systems is useful when searching for controllers.

The well known *Bounded Real Lemma* (BRL) for LTI systems has an LPV system extension, see [6] and [7]. To be able to solve the BRL for an LPV system as a Linear Matrix Inequality (LMI) problem, bounds on the parameter vector are introduced restricting ρ to be in the (bounded and connected) set Ω, and the exact relationship between the parameter and the internal variables is neglected. Often it is meaningful to introduce bounds on the rates of change of the parameter as well, to express the fact that the parameter vector cannot change arbitrarily fast. This means that $\dot{\rho}$ is restricted to the set $\tilde{\Omega}$.

Let the $\mathcal{L}_2$ gain be defined by,

$$\sup_{w \in \mathcal{L}_2} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}$$

where $\mathcal{L}_2$ denotes the space of Lebesgue square integrable (vector) functions with the corresponding norm,

$$\|f\|_{\mathcal{L}_2} = \sqrt{\int f^T(\tau) f(\tau) d\tau}$$

The bounded real lemma states that the LPV system (1) has an induced $\mathcal{L}_2$ gain bounded by γ for all $\rho \in \Omega$ and $\dot{\rho} \in \tilde{\Omega}$ if there exists a positive definite matrix function $P(\rho) : \Omega \to \mathbb{R}^{n \times n}$ satisfying,

$$\begin{bmatrix} A^T(\rho)P(\rho) + P(\rho)A(\rho) + \dot{P}(\rho) & \star & \star \\ B^T(\rho)P(\rho) & -\gamma I & \star \\ C(\rho) & G(\rho) & -\gamma I \end{bmatrix} < 0 \qquad (2)$$

for all $\rho \in \Omega$ and $\dot{\rho} \in \tilde{\Omega}$. In this formulation, $\star$ denotes the transpose of the corresponding block matrix.

Since the bounds of the parameter vector implicitly defines a set of validity of the BRL condition (2), the $\mathcal{L}_2$ gain of a system is a local property.

It should be pointed out that the BRL condition (2) is conservative in the sense that an LPV system like (1) might have a finite $\mathcal{L}_2$ gain even if the conditions fails to be satisfied. This is not true in the LTI case where the system matrices are constant.

III. Motivating example

It is easy to believe that a closed loop LPV system that satisfies the bounded real lemma (2) for all parameters varying in the specified bounding set Ω (and possible $\tilde{\Omega}$) implies that the $\mathcal{L}_2$ gain is satisfied for the underlying nonlinear system. In this section, a simple example is given illustrating that this, in general, is not true.

Consider the well known *Van der Pol* equation (with reversed vector field and the input w added),

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - 0.3(1 - x_1^2)x_2 + w \\ z &= x_2 \end{aligned} \qquad (3)$$

This equation (3) (without the input w) is a special case of Liénard's equation, see e.g. [8], and it is well known that

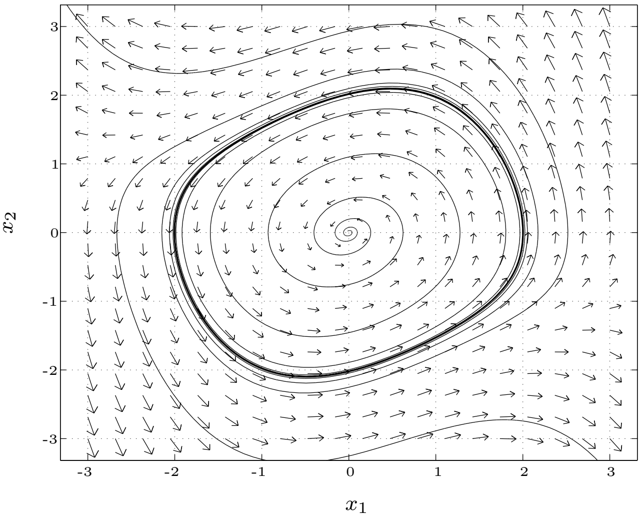

Fig. 1. Phase-portrait of the autonomous Van der Pol equation with reversed vector field.

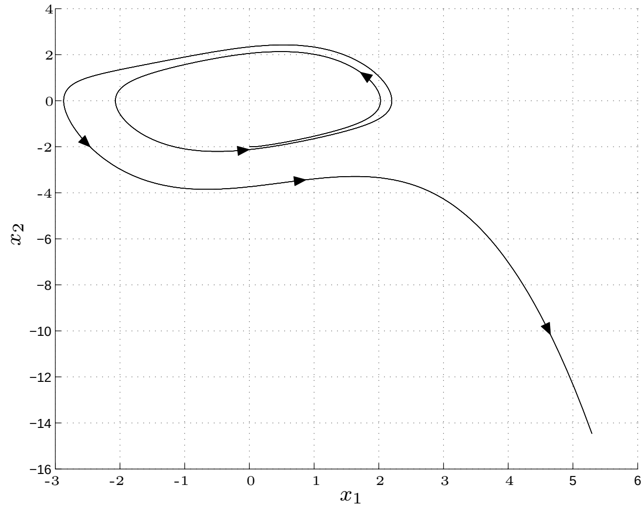

Fig. 2. Trajectory of the Van der Pol system with initial condition $x(0) = [0, -2]$ and with a disturbance of -0.5 during the first 0.06 seconds.

a limit cycle exists for such systems. This reversed vector field version has the property that for the unforced system, all trajectories starting outside this limit cycle diverges and all trajectories starting inside converges to the origin, see figure 1. An intuitive LPV description of the Van der Pol equation is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -0.3 + 0.3\rho \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \qquad (4)$$

where

$$\rho = x_1^2. \qquad (5)$$

The only nonlinear term of the right hand side of (3) is hidden in the parameter ρ. Observe that (4) is an exact description of (3) in the sense that the trajectories of the nonlinear system (3) has the same trajectories as the LPV system (using the relation (5)).

Using a constant P matrix in the BRL condition (2) and optimizing over the upper bound of the $\mathcal{L}_2$ gain γ using the LMI software package SeDuMi, see [9], with the SeDuMi interface, see [10], for Matlab results in $\gamma = 66.83$ if the parameter is allowed to take values in the set $\Omega = \{\rho \in \mathbb{R} \mid 0 \leq \rho \leq 0.9\}$. By using a constant P matrix in the BRL condition, no bounds on the rates of change are needed to be introduced. The use of a constant P matrix is conservative compared to the use of a parameter dependent matrix $P(\rho)$. However, the point here is to illustrate that the $\mathcal{L}_2$ gain is a local property and this LPV analysis is sufficient for this purpose.

Observe that if $\rho \geq 1$ then the eigenvalues of the A matrix would be in the right half plane. Since one possibility of the parameter is a time constant value $\rho(t) = \rho_0$, there can not exist a solution to the BRL condition (2). This is in contrast to linear time varying systems which do can have right half plane eigenvalues for periods of time and still be stable and have a finite $\mathcal{L}_2$ gain.

It is easy to believe that the computed upper bound of the $\mathcal{L}_2$ gain is valid for at least the set that corresponds to parameter values in Ω, that is,

$$\{x_1 \in \mathbb{R} \mid -\sqrt{0.9} \leq x_1 \leq \sqrt{0.9}\}. \qquad (6)$$

This is not true. In fact, not even stability is guaranteed for all values in the set (6). For example the solution diverges in the case with an initial condition $x(0) = [0, -2]$ (in the set Ω), and a disturbance w of -0.5 during 0.06 seconds ($w \in \mathcal{L}_2$), see figure 2.

IV. Domain of $\mathcal{L}_2$ Gain

As the example in the forgoing section illustrates, the domain where the system has an $\mathcal{L}_2$ gain does not coincide with the part of the state space implicitly defined by the bounding set Ω of the parameter vector. However, as long as the stationary point x_0 of the nonlinear system implies that $\rho(x_0) \in \Omega$ (a natural restriction) it is possible to find such domain for which the computed upper bound of the gain for the LPV system also holds for the underlying nonlinear system.

In this section such a domain is given, as well as the set of input signals that keeps the system in this domain where the upper bound of the gain is guaranteed. This is given in the following theorem.

Theorem IV.1: Consider the nonlinear system,

$$\begin{aligned} \dot{x} &= f(x, w), \; x \in D \\ z &= h(x, w) \end{aligned} \qquad (7)$$

exactly included by the LPV description,

$$\begin{aligned} \dot{x} &= A(\rho)x + B(\rho)w \\ z &= C(\rho)x + G(\rho)w \end{aligned} \qquad (8)$$

where $\rho = \phi(x)$. Assume that the LPV system (8) satisfies the BRL condition (2) for all parameters satisfying $\rho \in \Omega$ and $\dot{\rho} \in \tilde{\Omega}$. Define the following sets,

$$\begin{aligned} \mathcal{X} &= \{x \in D \mid \phi(x) \in \Omega\} & (9) \\ \tilde{\mathcal{X}} &= \{x \in D \mid \dot{\phi}(x) \in \tilde{\Omega}\} & (10) \\ \Gamma_\beta &= \{x \in D \mid V(x) \leq \beta\} & (11) \end{aligned}$$

where $V = x^T P(\phi(x))x$. If $\Gamma_\beta \subseteq (\mathcal{X} \cap \tilde{\mathcal{X}})$ then the system (7) is asymptotically stable for all initial values $x(t_0) \in \Gamma_\beta$ and (for $x(t_0) = 0$),

$$\sup_{w \in \mathcal{W}} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}} \leq \gamma$$

for the set of inputs defined by,

$$\mathcal{W} = \{w \in \mathcal{L}_2 \mid \frac{\partial V}{\partial x}\Big(A(\phi(x))x + B(\phi(x))w\Big) \leq 0, \quad \forall x \in \partial\Gamma_\beta\} \quad (12)$$

Proof: According to the BRL condition, the LPV system (8) has an induced $\mathcal{L}_2$ gain bounded by γ for all $\rho \in \Omega$ and $\dot{\rho} \in \tilde{\Omega}$, and in particular for $\rho = \phi(x)$. The LPV system with the relation $\rho = \phi(x)$ is exactly the nonlinear system (7) which implies that the nonlinear system has an induced $\mathcal{L}_2$ gain bounded by γ as long as the state x remains in the region $\mathcal{X} \cap \tilde{\mathcal{X}}$.

In general, there are states $x \in (\mathcal{X} \cap \tilde{\mathcal{X}})$ for which the trajectories of the nonlinear system leaves the region $\mathcal{X} \cap \tilde{X}$, c.f. figure 2. However, using the function $x^T P(\phi(x))x$, which serves as a Lyapunov function for the unforced system ($w = 0$), implies that there are regions in $\mathcal{X} \cap \tilde{\mathcal{X}}$ for which the trajectories remains inside, as long as there are restrictions on w. An estimate of the region of attraction using this Lyapunov function is Γ_β, see for example [8]. Hence, it can be concluded that Γ_β is a subset of $(\mathcal{X} \cap \tilde{\mathcal{X}})$ that will keep the unforced system inside Γ_β.

Consider now the forced system ($w \neq 0$). Taking the derivative of V along the trajectories of the LPV system results in,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}(A(\rho)) + B(\rho))w).$$

Hence, if (12) is satisfied for all $x \in \partial\Gamma_\beta$ the trajectories inside Γ_β can never leave Γ_β, which can be concluded by contradiction. This is as well true for the nonlinear system (7) since it is obtained particularly for the LPV system with $\rho = \phi(x)$. ■

A graphical interpretation of the set $\mathcal{W}$ is that the input (w) to the system must be of the nature that the resulting vector field is directed toward decreasing level surfaces of the Lyapunov function V at the boundary, due to the scalar product between the field and the level surface,

$$-\frac{\partial V}{\partial x} f(x,w) = \|\frac{\partial V}{\partial x}\| \|f(x,w)\| \cos(\theta) \geq 0$$

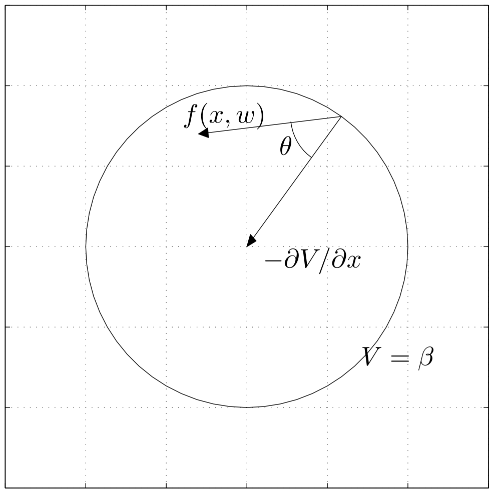

Fig. 3. Illustration of the effects of the input set $\mathcal{W}$.

see figure 3.

In the set Γ_β, the Lyapunov function $V = x^T P(\rho(x))x$ is decreasing for the unforced system, which implies that

$$\frac{\partial V}{\partial x}(A(\phi(x))x < 0$$

for all $x \in \Gamma_\beta$. Hence,

$$\frac{\partial V}{\partial x}(A(\phi(x))x + B(\phi(x))w) \leq \frac{\partial V}{\partial x} B(\rho(x))w,$$

on the border of Γ_β, which means that the requirement,

$$\frac{\partial V}{\partial x} B(\phi(x)w \leq 0 \;\forall x \in \partial\Gamma_\beta. \quad (13)$$

implies that (12) is fulfiled. The condition (13) results in a input restriction that is cheaper to compute than the one in the set $\mathcal{W}$ in (12).

It should be noted that the BRL condition for the LPV system does guarantee local $\mathcal{L}_2$ gain for the nonlinear system near the stationary point. This is due to the positiveness of the Lyapunov function, $V = x^T P(\rho(x))x$, and as long as the sets $\mathcal{X}$ and $\tilde{X}$ contains the origin. That the sets $\mathcal{X}$ and $\tilde{X}$ contain the origin is rather natural, since the origin is the stationary point of the nonlinear system and hence must be included in $\mathcal{X}$ and if $\dot{\rho} = 0$ is included in $\tilde{\Omega}$ then the origin is included in $\tilde{X}$ according to

$$\dot{\rho} = \frac{\partial \rho}{\partial x} f(0) = 0.$$

Hence, there will always exist a non-empty Γ_β for which the upper bound of $\mathcal{L}_2$ gain of the LPV system is valid for the nonlinear system, for reasonable $\mathcal{X}$ and $\tilde{\mathcal{X}}$.

V. Illustration

Recall the Van der Pol equation (3) from the motivating example section. Using the LPV description (4) can only result in a domain of $-1 \leq x_1 \leq 1$ where the bounded real lemma can be satisfied, due to that the system becomes unstable for larger values of x_1. To increase the domain consider the following LPV description of the Van der Pol equation (3)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1+0.3\rho & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

where

$$\rho = x_1 x_2. \tag{15}$$

Observe that this system matrix has negative eigenvalues as long as $|\rho| < \frac{10}{3}$, though the possible domain of $\mathcal{L}_2$ gain is larger than in the case of (4). To additionally increase the domain, a parameter dependent P matrix in the Bounded Real Lemma is used. To be able to solve this problem using linear matrix inequalities, the parameter dependence must be chosen in advance. A common initial anzats to the parameter dependence is to mimic the parameter dependence of the system, which in this case is affine. In addition, a strategy is introduced to relax the infinite dimension of the problem by either using e.g. multi-convexity, see [11], or gridding of the parameter space and solve the problem for this grid points.

Repeated optimization using the LMI software package SeDuMi, [9], with the SeDuMi interface, [10], on a grid in the parameter space resulted in the following matrix function, (using the most significant lower order terms in ρ)

$$\begin{aligned} P = & \begin{bmatrix} 1.0204 & -0.1813 \\ -0.1813 & 1.0788 \end{bmatrix} + \\ & + \begin{bmatrix} 0.0984 & 0.0166 \\ 0.0166 & -0.0879 \end{bmatrix} \rho + \\ & + \begin{bmatrix} -0.6069 & -1.5704 \\ -1.5704 & 5.4487 \end{bmatrix} 10^{-3}\rho^3 + \\ & + \begin{bmatrix} -1.9442 & 0.0173 \\ 0.0173 & 0.6815 \end{bmatrix} 10^{-3}\rho^5 \end{aligned} \tag{16}$$

and an upper bound gain estimate of $\gamma = 141.09$, by allowing the parameter to vary in,

$$|\rho| = |x_1 x_2| \leq 1.69. \tag{17}$$

and its rate of change,

$$|\dot{\rho}| = |-x_2^2 + x_1(x_1 - 0.3(1 - x_1^2)x_2)| \leq 3. \tag{18}$$

Since the gridding does not give a guarantee that the original parameterized LMIs are satisfied, a post analysis on a denser grid of the result was performed to ensure correctness of the solution.

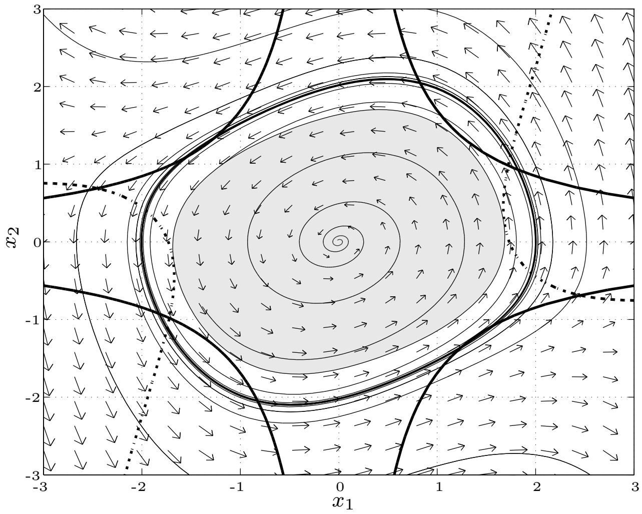

Fig. 4. Phase-portrait of the autonomous Van der Pol with the estimate of the region of attraction (shaded area) inside the sets $\mathcal{X}$ (bold lines) and $\tilde{\mathcal{X}}$ (dashed bold lines).

Since the $\mathcal{L}_2$ gain is defined as sup over all inputs w, one particular choice of input gives a lower bound on the $\mathcal{L}_2$ gain. First, consider inputs that keep the trajectory inside the limit cycle. For example, introduce an initial pulse in w that brings the trajectory of the system (3) close to the unstable limit cycle. To maintain an orbit close to the limit cycle, small pulses in w can be added. This results in a periodic system. The energy of the initial pulse is neglectable when computing the gain of this particular input, since initial transient does not affect the value of the ratio of the input and output energies. The closer the trajectory is to the limit cycle the slower it leaves it, see figure 1, and consequently the smaller energy in the input is needed to maintain the orbit. This indicates that the $\mathcal{L}_2$ gain of the system (3) can be arbitrarily large.

The estimate of the gain based on the BRL condition is obviously not valid for the true region of attraction of the nonlinear system (3). However, according to theorem IV.1 the domain for which this estimate is valid is the largest level curve (Γ_β) of the Lyapunov function $x^T P x$ that is enclosed by intersection of the set defined by the inequalities (17) and (18), c.f. the shaded area in figure 4.

The same strategy to obtain a simulated estimate (lower bound) of the gain was used, as previously mentioned, but keeping the trajectory inside the domain Γ_β for which the computed upper bound is valid. One choice of input and the resulting output is shown in figure 5. The gain for this particular choice of input is 13.6. Hence, the upper bound estimate using the LPV analysis is at the most of the order of 10 greater than the true gain of the system within Γ_β.

To illustrate that the input set $\mathcal{W}$ does keep the trajectory inside Γ_β, a simulation was performed. By using white noise with power 1 as input with the restriction that if the

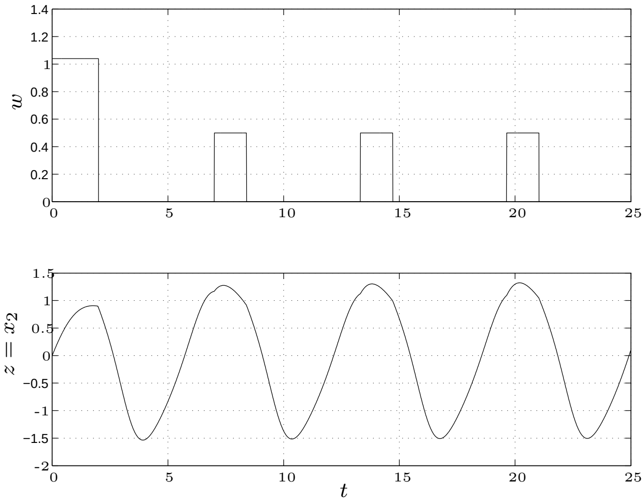

Fig. 5. Input and output of the system.

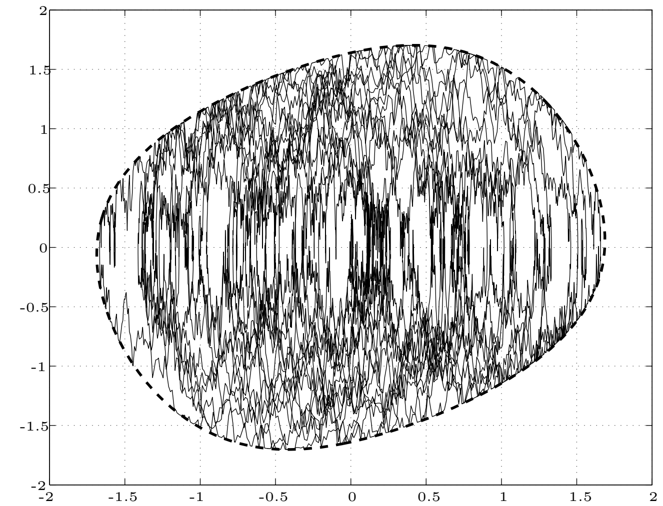

Fig. 6. White noise, which belongs to $\mathcal{W}$, of power 1 as input to (3) and the level surface of the Lyapunov function (16) (dashed line).

trajectory was on the border of Γ_β and if the direction of Bw was pointing outward of Γ_β the input changed sign. The result is shown in figure 6. As one can observe, the set Γ_β is invariant to the system with an input $w \in \mathcal{W}$.

VI. Conclusion

In LPV based gain scheduling controller synthesis, it is easy to believe that local properties like $\mathcal{L}_2$ gain for the closed loop system implies the same $\mathcal{L}_2$ gain for the nonlinear closed loop system in the domain for which the synthesis is made. This is not true, in general. This indicates that the LPV based gain scheduling controller synthesis should be carried out with great caution.

However, a domain for which the LPV $\mathcal{L}_2$ gain guarantee the same $\mathcal{L}_2$ gain for the underlying nonlinear system can be computed based on the LPV analysis (synthesis). Restrictions on the input can be made explicit, that keeps the system in this domain. These results are formalized in this paper by a theorem.

As this paper illustrates, nonlinear systems can be described by many different LPV systems, by different choices of the (scheduling) parameter. It is also illustrated that this particular choice of parameter plays an important roll for the domain of $\mathcal{L}_2$ gain based on the LPV analysis (synthesis) condition. This choice affects as well the solvability of the LPV analysis condition.

The domain of validity of the $\mathcal{L}_2$ gain is illustrated by a simple second order nonlinear system, the well known Van der Pol equation, as well as the input restriction that keeps the nonlinear system in this domain.

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