On the control of an uncertain rigid body dynamics

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***Abstract*— A new algebraic method, which is based on a recently developed algebraic identification approach proposed by the authors for the on-line identification of uncertain parameters in linear controlled dynamic systems, is here used for obtaining the solution of the fully actuated multi-variable feedback stabilization problem of the, unperturbed, classical Euler model of a rigid body with constant, but completely unknown, principal moments of inertia. The results can also be used in the under-actuated case.**

***Index Terms*— Algebraic identification, Euler equations**

I. INTRODUCTION

The feedback controlled stabilization of a rigid body (also referred to as the "de-tumbling" maneuver) is considered to be one of the classical problems in nonlinear multi-variable aerospace control systems. Regulation of the rigid body dynamics is the object of a vast literature from various viewpoints. The interested reader is referred to the tutorial review by Coppola and McClamroch appearing in the book by Levine [1]. We point out that a challenging problem associated to this particular subject is the case in which the control algorithm is to be derived, and implemented, with the rather realistic assumption of the principal moments of inertia being constant but, otherwise, completely unknown. A common approach for the solution of such an uncertain control problem is the *adaptive control* approach. However, this approach suffers from several difficulties arising from the non-linear nature of the system parametrization. Even if the over-parameterized description of the Euler equations can be induced to exhibit an asymptotically exponential convergence of the angular velocity state variables towards the equilibrium condition, the actual values of the unknown but constant moments of inertia parameters are never recovered and, moreover, angular accelerations are needed in the algorithm.

In this article, we treat the problem of a fully actuated controlled "de-tumbling" maneuver in a set of classical non-linear Euler equations describing the dynamics of a parameter uncertain rigid body. The problem of regulating, or stabilizing, such a nonlinear system with unknown inertia parameters does not fit the linear parametrization scheme, or even the over-parametrization alternative, commonly advocated in adaptive control based formulations. We find however, that the Euler system, with unknown moments of inertia, is indeed *linearly identifiable* in the sense made precise in a previous work by the authors (see Fliess-Sira[3]) to which the reader is referred for theoretical details and some illustrative examples. The key consideration to establish linear identifiability of the constant principal moments of inertia of a rigid body is related to the linear parametrization of the expression describing the *energy balance* of the system. From the unperturbed dynamics of a rigid body evolving in free outer space, the energy balance is simply associated with the rate of the total rotational kinetic energy written in terms of the external supplied power. By taking the same steps followed in the algebraic identification framework of a linear SISO, we arrive at an *explicit formula* for the unknown parameters, which requires a rather small computation time, is independent of the initial conditions, and which only requires the knowledge of the measured system inputs and outputs (i.e. it requires the input torques and the angular velocities), thus avoiding angular acceleration measurements. The computed value of the parameters can then be quickly substituted, in an on-line fashion, on the expressions of any stabilizing certainty equivalence controller.

Section 2 deals with the Euler equations of a rigid body dynamics. In this section we propose, under the assumption of perfect knowledge of the system parameters, a rather simple, and well known, stabilizing nonlinear feedback controller achieving a closed loop exact linearization of the angular velocities. This controller is regarded as a *certainty equivalence controller*. Section 3 demonstrates that the rigid body dynamics is indeed linearly identifiable and it presents an on-line formula for the fast (on-line) unknown parameter calculation. This section proposes then the combination of the certainty equivalence controller with the fast parameter identification scheme. The results of the proposed scheme are illustrated by means of computer simulations.

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II. THE FULLY ACTUATED RIGID BODY DYNAMICS

Consider the dynamic model, known as the *Euler equations*, of a rigid body fixed at its center of mass in free space (See Wertz, [5] and also Jurdjevic [4])

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + u_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3 \end{aligned} \tag{1}$$

where I_1, I_2 and I_3 are the moments of inertia, around the principal axes of the body, ω_1, ω_2 and ω_3 are the angular velocities, assumed to be measurable, around such axes and u_1, u_2 and u_3 represent the applied control input torques (these may be obtained, for instance, from the usual reaction jets symmetrically located on the body, as shown in Figure 1).

A. A certainty equivalence controller

System (1) is *flat* (see Fliess et al [2]) with the three flat outputs being the angular velocities, ω_1, ω_2 and ω_3. This means,

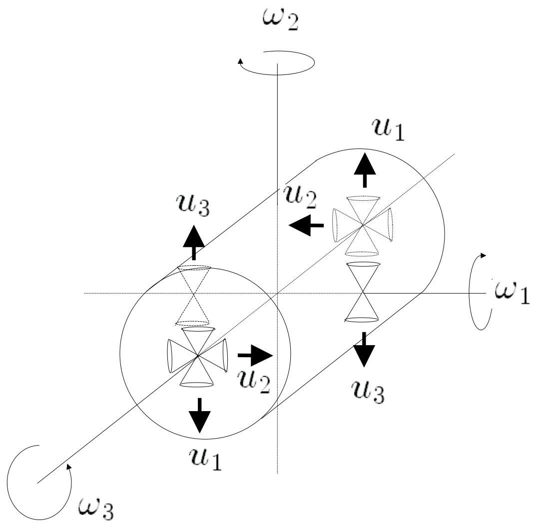

Fig. 1. Fully actuated rigid body

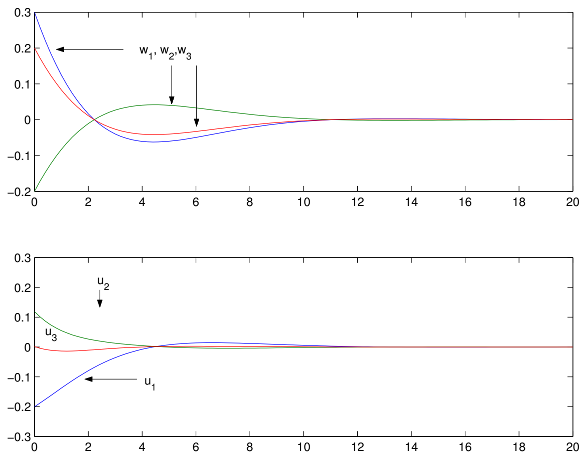

Fig. 2. Closed loop response of controlled rigid body with perfect knowledge of the system parameters

in this particular case, that the system is equivalent, under static state feedback and state coordinate transformations, to a set of three linear systems representing pure, decoupled, integration chains. Under the assumption of perfect knowledge of the moments of inertia, I_1, I_2 and I_3, a stabilizing, or de-tumbling, multi-variable feedback strategy is given by the following prescription of a control law, which includes integral compensation terms.

$$\begin{aligned} u_1 &= -(I_2 - I_3)\omega_2\omega_3 + I_1\left(-\lambda_{11}\omega_1 - \lambda_{01}\int_0^t \omega_1(\sigma)d\sigma\right) \\ u_2 &= -(I_3 - I_1)\omega_3\omega_1 + I_2\left(-\lambda_{12}\omega_1 - \lambda_{02}\int_0^t \omega_2(\sigma)d\sigma\right) \\ u_3 &= -(I_1 - I_2)\omega_1\omega_2 + I_3\left(-\lambda_{13}\omega_1 - \lambda_{03}\int_0^t \omega_3(\sigma)d\sigma\right) \end{aligned} \tag{2}$$

The closed loop system evolves in accordance with the following dynamics

$$\begin{aligned} \dot{\omega}_1 &= -\lambda_{11}\omega_1 - \lambda_{01}\int_0^t \omega_1(\sigma)d\sigma \\ \dot{\omega}_2 &= -\lambda_{12}\omega_1 - \lambda_{02}\int_0^t \omega_2(\sigma)d\sigma \\ \dot{\omega}_2 &= -\lambda_{13}\omega_1 - \lambda_{03}\int_0^t \omega_3(\sigma)d\sigma \end{aligned} \tag{3}$$

which can be made to have the origin as an asymptotically exponentially stable equilibrium point under suitable choice of the controller design parameters $\lambda_{1i}, \lambda_{0,i},\ i = 1,2,3$.

The performance of the proposed feedback controller (2), which is addressed as the *certainty equivalence controller*, is depicted in Figure 2

The numerical values, used in the simulations, for the moments of inertia, and for the design parameters were set to be:

$$I_1 = 1\ [\mathrm{N-m-s^2}],\quad I_2 = 0.5\ [\mathrm{N-m-s^2}],$$
$$I_3 = 0.2\ [\mathrm{N-m-s^2}]$$

$$\lambda_{1i} = 2\zeta_i\omega_{ni},\ \ \lambda_{0i} = \omega_{ni}^2\ \ \zeta_i = 0.707,\ \ \omega_{ni} = 0.5,\ \ i = 1,2,3$$

Remark 2.1: It is known that integral control actions, such as those specified in equation (2), are specifically aimed at counteracting possible unknown constant disturbance torque perturbation inputs. However, when such perturbation inputs are present in the dynamics of the rigid body, the combined controller-identification procedure, to be presented below, is valid provided the parameter identification part is carried out *before* the unknown constant disturbance moment vectors appear in the system dynamics. Otherwise, the proposed control scheme is not valid and it has to be substantially modified.

III. LINEAR IDENTIFIABILITY OF THE RIGID BODY DYNAMICS

The fundamental problem with the proposed feedback control law (2) is that the system parameters, represented by the moments of inertia, are not known, except for the fact that they are constant. We now concentrate our efforts in devising a parameter calculation scheme which is based on an algebraic identification approach proposed by the authors in previous works (see Fliess and Sira-Ramírez [3]). We first prove, in accordance with our established definitions, that the system is *linearly identifiable*.

The *energy balance* of the unperturbed system readily yields the following relationship, which is *linear* in the parameters representing the unknown principal moments of inertia

$$\begin{aligned} &\frac{1}{2}\left[I_1\frac{d}{dt}\left(\omega_1^2\right) + I_2\frac{d}{dt}\left(\omega_2^2\right) + I_3\frac{d}{dt}\left(\omega_3^2\right)\right] \\ &= \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \end{aligned} \tag{4}$$

We proceed as follows (See [3]): Multiply out both sides of the above equation by the time variable t. Integrate both sides from 0 to t and integrate by parts the terms containing the integral of the products of t with the time derivatives of the squared angular velocity. One obtains the following expression, which is clearly free of any initial condition at time $t = 0$,

$$
\begin{aligned}
& I_1\left[t\omega_1^2 - \int_0^t \omega_1^2(\sigma)d\sigma\right] + I_2\left[t\omega_2^2 - \int_0^t \omega_2^2(\sigma)d\sigma\right] \\
& + I_3\left[t\omega_3^2 - \int_0^t \omega_3^2(\sigma)d\sigma\right] \\
& = 2\int_0^t \sigma\left[\omega_1(\sigma)u_1(\sigma) + \omega_2(\sigma)u_2(\sigma) + \omega_3(\sigma)u_3(\sigma)\right]d\sigma
\end{aligned} \tag{5}
$$

From the obtained expression (5), we generate, by simple time integration, the following system of three equations in the three unknown constants

$$
\begin{bmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} = q(t) \tag{6}
$$

where, for $i = 1, 2, 3$,

$$
\begin{aligned}
p_{1i}(t) &= \left[t\omega_i^2 - \int_0^t \omega_i^2(\sigma)d\sigma\right], \\
p_{2i}(t) &= \int_0^t \left[\sigma\omega_i^2(\sigma) - \int_0^\sigma \omega_i^2(\sigma_1)d\sigma_1\right]d\sigma, \\
p_{3i}(t) &= \int_0^t \int_0^\sigma \left[\sigma_1\omega_i^2(\sigma_1) - \int_0^{\sigma_1} \omega_i^2(\sigma_2)d\sigma_2\right]d\sigma_1 d\sigma
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
q_1(t) &= 2\int_0^t \sigma\Big[\omega_1(\sigma)u_1(\sigma) + \omega_2(\sigma)u_2(\sigma) \\
&\quad + \omega_3(\sigma)u_3(\sigma)\Big]d\sigma \\
q_2(t) &= 2\int_0^t \int_0^\sigma \sigma_1\Big[\omega_1(\sigma_1)u_1(\sigma_1) + \omega_2(\sigma_1)u_2(\sigma_1) \\
&\quad + \omega_3(\sigma_1)u_3(\sigma_1)\Big]d\sigma_1 d\sigma \\
q_3(t) &= 2\int_0^t \int_0^\sigma \int_0^{\sigma_1} \sigma_2\Big[\omega_1(\sigma_2)u_1(\sigma_2) + \omega_2(\sigma_2)u_2(\sigma_2) \\
&\quad + \omega_3(\sigma_2)u_3(\sigma_2)\Big]d\sigma_2 d\sigma_1 d\sigma
\end{aligned} \tag{8}
$$

A. A feedback controller with fast identification

As in the linear system parameter identification case (see [3]), the matrix $P(t) = \big(p_{ij}(t)\big)$ is *not* invertible at time $t = 0$, but it is certainly invertible after an arbitrarily small time $t = \epsilon > 0$. In the proposed certainty equivalence controller (2), we utilize, during the calculation interval $[0, \epsilon]$, arbitrary numerical values for the moments of inertia, I_1, I_2, I_3. After time $t = \epsilon$, we proceed to substitute in the controller expression (2), the computed inertia parameters, as obtained from equation (6), which are now denoted by I_{1e}, I_{2e}, I_{3e}.

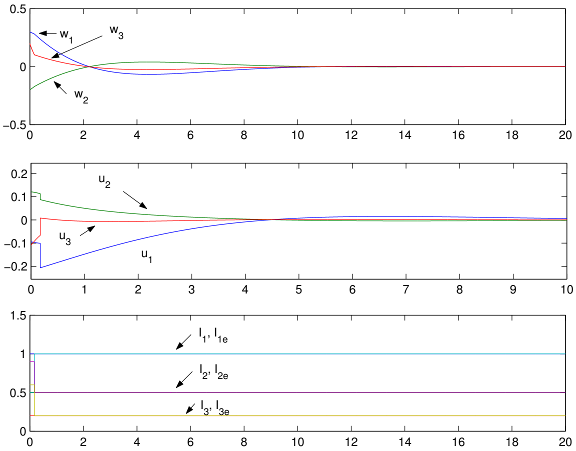

Fig. 3. Closed loop response of parameter uncertain rigid body to the proposed controller identifier scheme

We thus propose:

$$
\begin{aligned}
u_1 &= -(I_{2e} - I_{3e})\omega_2\omega_3 + I_{1e}\left(-\lambda_{11}\omega_1 - \lambda_{01}\int_0^t \omega_1(\sigma)d\sigma\right) \\
u_2 &= -(I_{3e} - I_{1e})\omega_3\omega_1 + I_{2e}\left(-\lambda_{12}\omega_1 - \lambda_{02}\int_0^t \omega_2(\sigma)d\sigma\right) \\
u_3 &= -(I_{1e} - I_{2e})\omega_1\omega_2 + I_{3e}\left(-\lambda_{13}\omega_1 - \lambda_{03}\int_0^t \omega_3(\sigma)d\sigma\right)
\end{aligned} \tag{9}
$$

with

$$
\begin{bmatrix} I_{1e} \\ I_{2e} \\ I_{3e} \end{bmatrix} = \begin{cases} \text{arbitrary} & \text{for } t \in [0, \epsilon] \\ P^{-1}(t)q(t) & \text{for } t > \epsilon \end{cases} \tag{10}
$$

Figure 3 depicts the performance of the feedback controller (9)-(10), where the arbitrary values of the moments of inertia were taken to be $I_{1e} = 0.5$ [N-m-s^2], $I_{2e} = 0.9$ [N-m-s^2] and $I_{3e} = 0.6$ [N-m-s^2] during a time interval $[0, \epsilon]$ where ϵ was determined by the violation of the condition: $|\det P(t)| \leq 10^{-16}$ (approximately 0.171 [s]). The computed values of the moments of inertia precisely coincide with the values used for the simulation of the system, i.e., $I_1 = 1.0[\mathrm{N-m-s^2}]$, $I_2 = 0.5$ $[\mathrm{N-m-s^2}]$, $I_3 = 0.2$ $[\mathrm{N-m-s^2}]$.

Note that the proposed linearizing controller with the fast identification scheme (9)-(10) is also robust with respect to sudden constant disturbance moments, provided these perturbations appear *after* the inertia parameters have been accurately computed. If the constant moments are present for all times (i.e. from the beginning), then a similar linear identification procedure can be still used by including the unknown perturbation moments as further constants to be identified in a linear fashion. Details of the solution of this possible variant of the problem will be presented elsewhere.

Figure 4 shows the performance of the proposed controller when the system is subject to sudden constant perturbation moments appearing after the system parameters have been identified. The perturbed model used in the simulations was

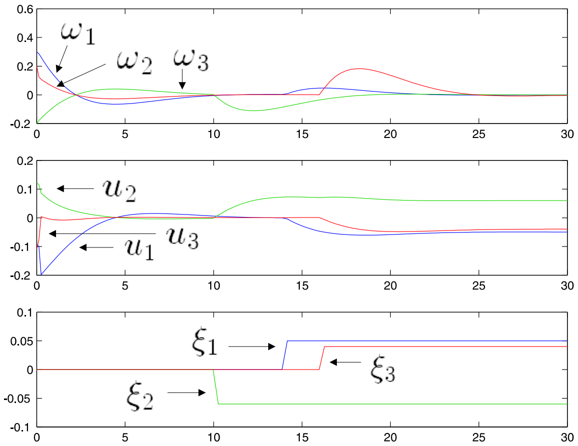

Fig. 4. Closed loop response of parameter uncertain rigid body subject to external constant perturbation moments

set to be,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 + \xi_1\mathbf{1}(t-\tau_1) \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + u_2 + \xi_2\mathbf{1}(t-\tau_2) \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3 + \xi_3\mathbf{1}(t-\tau_3) \end{aligned} \quad (11)$$

with $\xi_1 = 0.04$ [N-m], $\xi_2 = 0.05$ [N-m] and $\xi_3 = -0.06$ [N-m], with $\tau_1 = 16$ [s], $\tau_2 = 10$ [s] and $\tau_3 = 14$ [s].

In order to test the performance of our closed loop identification scheme with respect to stochastic inputs and stochastic measurement noises, we also run simulations on the following perturbed system including measurement noises.

$$\begin{aligned} I_1\dot{r}_1 &= (I_2 - I_3)r_2r_3 + u_1 + \xi_1 + \eta_1 \\ I_2\dot{r}_2 &= (I_3 - I_1)r_3r_1 + u_2 + \xi_2 + \eta_2 \\ I_3\dot{r}_3 &= (I_1 - I_2)r_1r_2 + u_3 + \xi_3 + \eta_3 \\ \omega_i &= r_i + \nu_i, \quad i = 1,2,3 \end{aligned} \quad (12)$$

with $\nu_i,\ i = 1,2,3$ representing zero mean stochastic measurement noises affecting the angular velocity measurements and $\xi_i,\ i = 1,2,3$ being the components of the constant perturbation moment vector. The additive perturbation signals η_1, η_2 and η_3 also represent zero mean stochastic input noises.

Figure 5 depicts the simulated performance of our controller identification scheme when subject to stochastic input and measurement noises. The computer generated noises used in the simulations are represented by normally distributed quasi random variables at each time instant. For the stochastic processes affecting the measurement we used amplitudes of the order 1×10^{-5} [rad/s]. The input perturbation noises were taken to be of amplitude 1×10^{-4} [N-m]. The constant moment disturbances representing the bias terms in the stochastic inputs were taken to be the same constant perturbations used in the previous simulation example.

IV. Conclusions

In this article we have proposed a direct, non-dynamic, identification scheme for the explicit and fast computation of the principal moments of inertia in a rigid body dynamics described by the classical Euler equations requiring only input

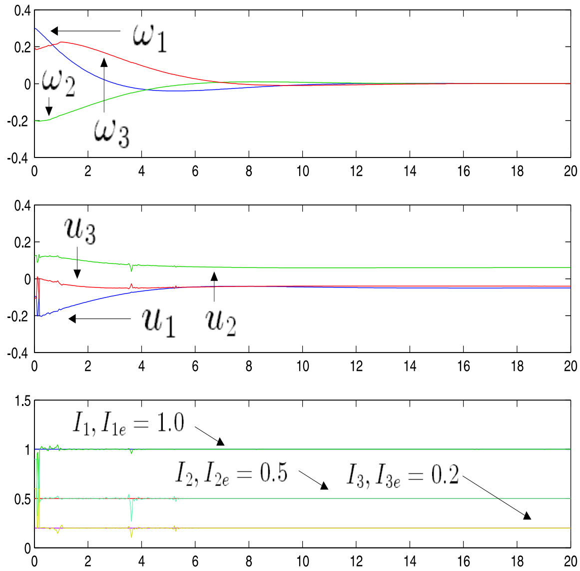

Fig. 5. Closed loop response of parameter uncertain rigid body subject to external stochastic perturbation moments with constant bias and stochastic measurement noises

torques and output angular velocities. The principal moments of inertia in the Euler equations were shown to be linearly identifiable from the rotational kinetic energy expression written in terms of the supplied energy. The identified principal moment of inertia parameters are then used on a certainty equivalence linearizing feedback controller, initialized with arbitrary system parameter values, which achieves exponential asymptotic stabilization (de-tumbling) of the angular velocities around the main body axes and counteracts constant but unknown torque disturbances.

When external unknown disturbance moment vectors are present, the proposed identification controller scheme still produces robust performance and equilibrium recovery provided these perturbations are all zero during the small time interval where the parameter computation is carried out. The lifting of this restriction is the subject of on-going research.

The proposed identification scheme is also applicable to the under-actuated rigid body case with only two torque inputs. Such system is also *differentially flat* (See [2]) and, hence, a certainty equivalence linearizing controller, which avoids singularities, is not difficult to devise.

References

[1] V. Coppola and H. McClamroch “Spacecraft attitude control” in W. S. Levine (Ed.), *Control Systems Applications*, Chapter 7, pp. 129-141, Boca Raton : CRC Press, 2000.

[2] M. Fliess, J. Levine, Ph. Martin and P. Rouchon “Flatness and defect of nonlinear systems: Introductory theory and examples” *International Journal of Control*, Vol. 61, No. 6, pp. 1327-1361, 1995.

[3] M. Fliess and H. Sira-Ramírez, “An algebraic framework for linear identification” *ESAIM: Control, Optimization and Calculus of Variations*, (accepted for publication, to appear in 2003)

[4] V. Jurdjevic, *Geometric Control Theory*, Cambridge studies in Advance Mathematics, Vol. 51. Cambridge: Cambridge Unversity Press, 1997.

[5] J. Wertz, *Spacecraft attitude determination and control*, Dordrecht: Kluwer Academic Publishers, 1990.