ROBUST STABILITY LIMIT AND ROBUSTNESS MEASURE FOR A *GTDOF* SYSTEM

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ABSTRACT

The very general robust stability condition is discussed and specialized for a generic two-degree of freedom control system. In case of a specific controller design procedures, it is shown how the time-delay mismatch influences the reachable robustness and performance measures.

Keywords: robustness, robust stability, performance, time-delay

1. INTRODUCTION

An important area of research in control theory is the design of feedback controllers for systems which have significant uncertainties in the plant and the explicit incorporation of model uncertainties in the design of high performance control systems. These uncertainties can result from a lack of precision in mathematical modeling of the plant and/or changes in the plant parameters with time. This leads to methods for designing robust stability and performance.

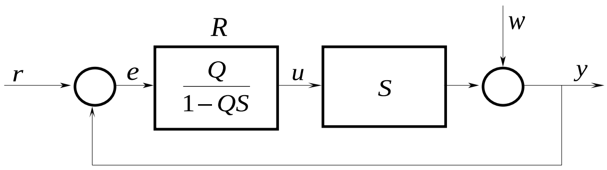

Fig. 1 Closed-loop with an *ARS* regulator

For open-loop stable *LTI* processes a good framework for designing all realizable stabilizing (*ARS*) control to use a regulator given by (see in Fig. 1)

$$R = \frac{Q}{1-QS} \tag{1}$$

where the "parameter" $Q \in \mathcal{S}$ ranges over all proper ($Q(\omega=\infty)$ is finite), stable transfer functions [1]. This is the Youla- (Y or Q)-parametrization of all stabilizing regulators and Q is the *Y*-parameter:

$$Q = \frac{R}{1+RS} \tag{2}$$

Q is anyway the transfer function from r to u and the closed-loop transfer function is

$$Z = \frac{RS}{1+RS} = QS \tag{3}$$

is linear (and hence convex) in Q.

2. ROBUST STABILITY OF *Y*-PARAMETRIZED CONTROL LOOPS

Be M the model of the process. Assume that the discrete-time process and its model are factorizable as

$$S = S_+\overline{S}_- = S_+S_-z^{-d} \quad \text{and} \quad M = M_+\overline{M}_- = M_+M_-z^{-d_{\mathrm{m}}} \tag{4}$$

where S_+ and M_+ mean the inverse stable (*IS*), S_- and M_- the inverse unstable (*IU*) factors, respectively. z^{-d} and $z^{-d_{\mathrm{m}}}$ correspond to discrete time delays, which are the integer multiple of the sampling time, usually $z^{-d} = z^{-d_{\mathrm{m}}}$ is assumed. (To get a unique factorization it is reasonable to ensure that S_- and M_- are monic, i.e., $S_-(1) = M_-(1) = 1$, having unity gain.) It is important that the inverse of the term z^{-d} is not realizable, because it would mean an ideal predictor z^d. These assumptions mean that $\overline{S}_- = S_-z^{-d}$ and $\overline{M}_- = M_-z^{-d_{\mathrm{m}}}$ are uncancelable invariant factors for any design procedure. Introduce the additive

$$\Delta = S - M \quad ; \quad \Delta_+ = S_+ - M_+ \quad ; \quad \Delta_- = \overline{S}_- - \overline{M}_- \tag{5}$$

and relative

$$\ell = \frac{\Delta}{M} = \frac{S-M}{M} \quad ; \quad \ell_+ = \frac{\Delta_+}{M_+} \quad ; \quad \ell_- = \frac{\Delta_-}{\overline{M}_-} \tag{6}$$

model errors. It is easy to show that the characteristic equation using the *ARS* regulator is (for $d = d_{\mathrm{m}} = 0$)

$$M_+M_- = 0 \tag{7}$$

if a $\hat{Q} = (M_+M_-)^{-1}\tilde{Q}$ parameter is applied, i.e., if someone tries to cancel both factors. This means that the zeros of the *IU* factor will appear in the characteristic equation and cause unstability. This is why these zeros (and the time delay itself) are called invariant uncancelable factors.

To investigate the robust stability of the closed-loop shown in Fig. 1 first the transfer function of an auxiliary closed-loop should be introduced similarly to (3) where the model M is used instead of S and a regulator $\hat{R}=\hat{R}(M)$ based on the model, i.e.

$$\hat{Z}=\frac{\hat{R}M}{1+\hat{R}M}=\hat{Q}M \tag{8}$$

where

$$\hat{Q}=\frac{\hat{R}}{1+\hat{R}M} \tag{9}$$

The well known robust stability condition [1] has several forms, here such form is shown, which is mostly used in iterative ID and control schemes. If ℓ is stable then $\hat{R}$ stabilizes S, if

$$\left|\hat{Z}\,\ell\right|<1 \quad \text{or} \quad \left|\hat{Z}\right|\le\frac{1}{|\ell|} \quad \text{or} \quad |\ell|\le\frac{1}{\left|\hat{Z}\right|} \quad \forall\omega \tag{10}$$

(If the system is discrete-time then $\forall\omega$ means $\omega\in[-\pi,\pi)$.) The inequality $\left|\hat{Z}\,\ell\right|<1$ is sometimes referred as

$$\left|\begin{matrix}\text{design}\\ \text{factor}\end{matrix}\right| \times \left|\begin{matrix}\text{modeling}\\ \text{factor}\end{matrix}\right|<1 \tag{11}$$

This condition has an interesting interpretation, because the first term is only influenced by the identification procedure and depends on the actual plant behavior, whereas the second one is influenced by the control design and the assumed plant model, but independent of the modeling error. To tell the truth this shaping condition does not mean too much: namely for small ω it is generally true that $\left|\hat{Z}\right|\cong 1$, therefore here $|\ell|<1$ is enough. For large ω, where $\left|\hat{Z}\right|<<1$, relatively bad model is also acceptable with $|\ell|>1$. However, for the medium frequency range, where $\left|\hat{Z}\right|$ has a maximum $|\ell|$ must be minimal.

The robust stability condition has a more serious form, when S and M have the same number of unstable poles and same unit-circle poles, then if $\left(\hat{R},M\right)$ is stable and either

$$\left|\hat{Z}\,\ell\right|<1 \qquad \omega\in[-\pi,\pi) \tag{12}$$

or

$$\left|Z\,\ell\right|<1 \qquad \omega\in[-\pi,\pi) \tag{13}$$

then $\left(\hat{R},S\right)$ is stable and vice versa. For the *ARS* regulator (10) gives $\left|\hat{Q}\,M\,\ell\right|<1$, i.e.,

$$\left|\hat{Q}\,M\right|<\frac{1}{|\ell|} \quad \text{or} \quad |\ell|<\frac{1}{\left|\hat{Q}\,M\right|} \quad \forall\omega \tag{14}$$

Thus the robust stability strongly depends on the model M and how the *Y-parameter* $\hat{Q}$ is selected.

3. A GENERIC TWO-DEGREE OF FREEDOM CONTROL SYSTEM

The *generic two-degree of freedom* (*GTDOF*) system [2] is used here which is based on the *Youla-parametrization* [1], providing *ARS* controller for open-loop stable plants and capable to handle the plant time-delay.

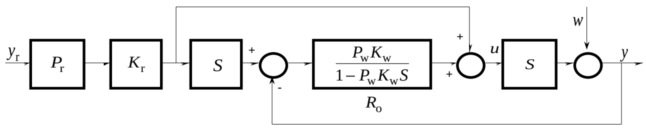

Fig. 2 The *generic TDOF* (*GTDOF*) control system

A *GTDOF* control system is shown in Fig. 2, where y_r, u, y and w are the reference, process input, output and disturbance signals, respectively. The optimal *ARS* regulator of the *GTDOF* scheme [3] is given by

$$R_o=\frac{P_w K_w}{1-P_w K_w S}=\frac{Q_o}{1-Q_o S}=\frac{P_w G_w S_+^{-1}}{1-P_w G_w S_- z^{-d}} \tag{15}$$

where

$$Q_o=Q_w=P_w K_w=P_w G_w S_+^{-1} \tag{16}$$

is the associated *Y-parameter* [4] furthermore

$$Q_r=P_r K_r=P_r G_r S_+^{-1}\ ;\ K_w=G_w S_+^{-1}\ ;\ K_r=G_r S_+^{-1} \tag{17}$$

assuming that the process is factorable by (4). Here P_r and P_w are assumed stable and proper transfer functions (reference models). An interesting result was [5] that the optimization of the *GTDOF* scheme can be performed in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm spaces by the proper selection of the serial G_r and G_w embedded filters.

It is interesting to note that the continuous-time equivalent form of the regulator (15) is

$$R_o=\frac{P_w K_w}{1-P_w K_w S}=\frac{Q_o}{1-Q_o S}=\frac{P_w G_w S_+^{-1}}{1-P_w G_w S_- e^{-s\tau}} \tag{18}$$

where τ is the time-delay of the plant and τ_m is the time-delay of the model and formally

$$S = S_+ \bar{S}_- = S_+ S_- e^{-s\tau} \quad \text{and} \quad M = M_+ \bar{M}_- = M_+ M_- e^{-s\tau_m} \quad (19)$$

4. ROBUST STABILITY OF *GTDOF* CONTROL SYSTEMS

The model based $\hat{R}_o$ and $\hat{Q}_o$ can be obtained from (15) and (16) by simply substituting S_+ and S_- with M_+ and M_-. It is also necessary to substitute z^{-d} by the model time delay z^{-d_m}. Analyze the basic robust stability condition (14) obtained for the *Y*-parametrized *ARS* regulators in case of the *generic scheme*, where the optimal regulator is given by (15) and $\hat{Q} = P_w G_w M_+^{-1}$ from (16). We get

$$\left|\hat{Q} M \ell\right| = \left|P_w G_w M_+^{-1} M \ell\right| = \left|P_w G_w M_- z^{-d_m} \ell\right| < 1 \quad \text{for} \quad \forall\omega \quad (20)$$

or in the other form (using that $\left|z^{-d_m}\right| = 1$)

$$|\ell| \le \frac{1}{|P_w|} \frac{1}{|G_w M_-|} \qquad \text{for} \qquad \forall\omega \qquad (21)$$

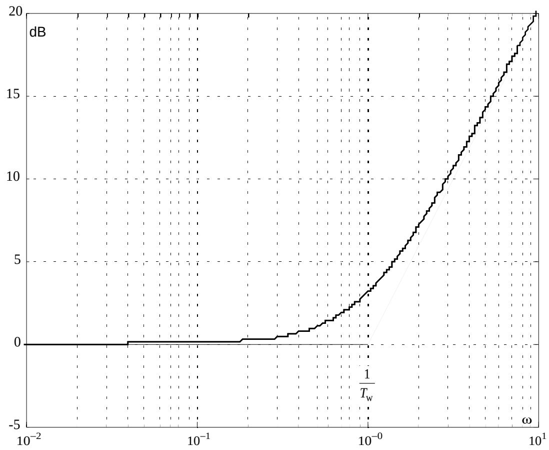

Fig. 3 The limiting condition for the relative model accuracy ℓ

In case of an *IS* plant with time-delay, when $M_- = 1$ and therefore $G_w = 1$, this robust stability condition is very simple

$$|\ell| \le \frac{1}{|P_w|} \qquad \text{for} \qquad \forall\omega \qquad (22)$$

i.e., the limiting right side depends only on the inverse of the P_w, which is the reference model for the regulatory property of the *GTDOF* system. So having known the design requirement P_w we got a direct inequality determining the necessary relative model accuracy to ensure robust stability. Assuming a simple first order reference model

$$P_w = \frac{1}{1 + sT_w} \qquad (23)$$

the limiting right side condition $1/|P_w|$ is shown in Fig. 3.

This limiting condition is very mild, because for low frequencies it is 100% and even increasing for high frequencies. It is important to note, if the relative model accuracy ℓ is known, then (22) gives a reachable limit for the design performance.

For *IU* plants the right side of (21) is determined by two factors. The first is $1/|P_w|$ which was discussed above. The second term is the inverse of the product $|G_w M_-|$. Here M_- is the model of the *IU* invariant factor S_-, which does not depend on us. The purpose of G_w is to attenuate the influence of the invariant factor M_- (and the true S_-) using an optimization in $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm space. So depending on the frequency characteristics of $|G_w M_-|$ the original $1/|P_w|$ limit could be slightly or drastically modified. The most unwanted case is when $|G_w M_-|$ gives a high gain at the cross-over frequency.

5. MODEL ERRORS FROM PLANT TIME-DELAY MISMATCH

Some controller design methodology, mostly for discrete-time systems, include the time-delay of the plant also into the parameters. Unfortunately relatively few papers [6], [7], [8]and [9] can be found dealing with the influence of the accuracy of the apriori knowledge or estimate of the time-delay, which is sometimes called the time-delay mismatch problem. In the sequel the influence of the time-delay mismatch on the robustness and performance will be discussed.

As a practical example let us compute the relative model error ℓ for an *IS* plant, where the model uncertainty comes only from a time-delay mismatch. The delay-free term is assumed to be known exactly, so $\bar{M}_- = 1$ and $M_+ = S_+$. In this case

$$\ell = \ell_d = \frac{\Delta}{M} = \frac{S - M}{M} = \frac{S_+ z^{-d} - S_+ z^{-d_m}}{S_+ z^{-d_m}} = z^{-(d - d_m)} - 1 \quad (24)$$

Assume an equivalent continuous time plant with time-delay τ and a model with time-delay τ_m. The analogous equivalence means

$$\ell = \ell_\tau = e^{-\Delta\tau s} - 1 \qquad (25)$$

where $\Delta\tau = \tau - \tau_m$. The robust stability condition (19) for the continuous time case is now

$$\sup_{\omega}\left|\ell_{\tau}\right| = \sup_{\omega}\left|e^{-j\Delta\tau\omega} - 1\right| \le 1/\left|P_{\mathrm{w}}(\omega)\right| \tag{26}$$

Assume a first order reference model

$$P_{\mathrm{w}} = \frac{1}{1 + s\mathrm{T}_{\mathrm{w}}} \tag{27}$$

and using the first order Taylor expansion of the exponential term one can get a good approximation of (26) for small deviations

$$\left|1 - \frac{\tau_{\mathrm{m}}}{\tau}\right| \le \frac{T_{\mathrm{w}}}{\tau} \tag{28}$$

The interpretation of (28) is very simple: for small T_{w}, which means high closed-loop performance, the model time delay τ_{m} must be close to the true delay τ. So it is obtained that the admissible time-delay mismatch is limited by the inverse of the performance. It could be furthermore very interesting how this limit influences the robustness of the loop.

Detailed investigation of this limiting behavior needs further numerical computations. Simple calculations give that the sensitivity function of the *GTDOF* system having time-delay mismatch for the discrete-time case is (assuming $G_{\mathrm{w}} = 1$)

$$E = \frac{1 - P_{\mathrm{w}} z^{-d_{\mathrm{m}}}}{1 + \ell P_{\mathrm{w}} z^{-d_{\mathrm{m}}}} = \frac{1 - P_{\mathrm{w}} z^{-d_{\mathrm{m}}}}{1 + \ell_{\mathrm{d}} P_{\mathrm{w}} z^{-d_{\mathrm{m}}}} \tag{29}$$

and the continuous time equivalent follows as

$$E = \frac{1 - P_{\mathrm{w}} e^{-s\tau_{\mathrm{m}}}}{1 + \ell P_{\mathrm{w}} e^{-s\tau_{\mathrm{m}}}} = \frac{1 - P_{\mathrm{w}} e^{-s\tau_{\mathrm{m}}}}{1 + \ell_{\tau} P_{\mathrm{w}} e^{-s\tau_{\mathrm{m}}}} \tag{30}$$

For P_{w} given by (27) the sensitivity function (30) becomes

$$E = \frac{1 + s\mathrm{T}_{\mathrm{w}} - e^{-s\tau_{\mathrm{m}}}}{1 + s\mathrm{T}_{\mathrm{w}} + P_{\mathrm{w}}\left(e^{-s\tau} - e^{-s\tau_{\mathrm{m}}}\right)} \tag{31}$$

The well-known Nyquist stability margin (the simplest robustness measure) is defined by

$$\rho_{\mathrm{m}} = \rho_{\min}(R) = \min_{\omega}\left|\rho(\omega, R)\right| = \min_{\omega}\left|1 + RS\right| = \min_{\omega}\left|1 + Y(j\omega)\right| = \frac{1}{\|E\|_{\infty}} \tag{32}$$

which is the distance between the point $(-1 + 0j)$ and the closest point of the open-loop transfer function $Y(j\omega)$. The reciprocal value of the norm is $\|E\|_{\infty}$. Unfortunately there is no simple analytical solution to obtain how the robustness of the closed-loop depends on the time-delay mismatch and on the performance. It is, however, possible to compute the graphical plot of a complex functional relationship $\rho_{\mathrm{m}} = \rho_{\min}(\tau_{\mathrm{m}}/\tau, T_{\mathrm{w}}/\tau)$ with the help of MATLAB.

As a result Fig. 4 shows the function $\rho_{\min}(T_{\mathrm{w}}/\tau)$ for $\tau_{\mathrm{m}} = 0.5\tau, \tau, 2\tau$. For the ideal $\tau_{\mathrm{m}} = \tau$ (no mismatch) case $\rho_{\min}$ depends only on our design goal (T_{w}) and on the plant time-delay (τ), more exactly on their relative value T_{w}/τ. The best robustness measure is $\rho_{\min}(0) = 0.5$ for cases when the reference model P_{w} requires a very fast transient response from the time-delay process and the measure is $\rho_{\min}(\infty) = 1$, if τ is negligible comparing to the time lag of P_{w}. It can be well seen that either under- or over-estimation of the time-delay causes considerable decrease of the robustness. Virtually $\rho_{\min}$ is more sensitive for over-estimation. (The left ends of the plots correspond to the robust stability limit.) While the no mismatch case provides an all stabilizing property for any performance requirement, in case of a non zero time-delay mismatch one can always expect the violation of the robustness stability limit for higher performance design.

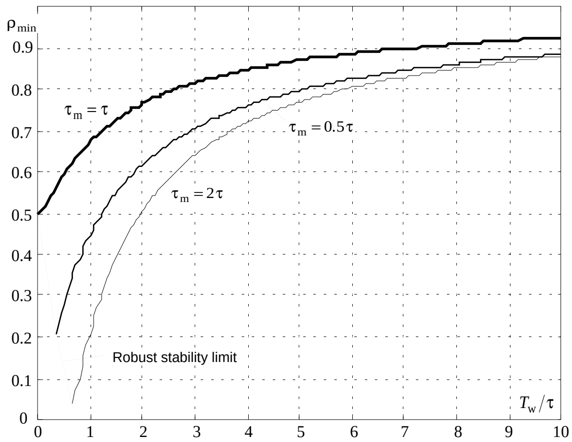

Fig. 4 The function $\rho_{\min}(T_{\mathrm{w}}/\tau)$ for $\tau_{\mathrm{m}} = 0.5\tau, \tau, 2\tau$

complex functional relationshipIt may be more reasonable to plot the function $\rho_{\min}(\tau_{\mathrm{m}}/\tau)$ parametrized by T_{w}/τ as Fig. 5 shows. One can see how extremely the robustness is sensitive for high performance requirement, when T_{w}/τ is small and how this sensitivity decreases when T_{w}/τ is large for low performance design. It is also interesting to observe, that for small mismatch the over-estimation of the delay gives higher $\rho_{\min}$, however, for $\rho_{\min}$ is somewhat more sensitive for large mismatch, as it was shown in Fig. 4.

In a relatively wide range of T_{w}/τ, the over-estimation of the time-delay by τ^{*}/τ improves (i.e. increases) the $\rho_{\min}$ to $\rho^{*}_{\min}$ according to the maxima of the curves observable in Fig. 5. The over-estimation is less than 25% and the improvement is marginal, less than 5% as Fig. 5 shows.

If we assume that the time-delay mismatch is less than 20% in a practical case, the robustness degradation is always less than 10% for $T_w/\tau \geq 0.5$, which can be well seen in Fig. 5. So if we want to speed up the open-loop process to a time constant, considerable less than the delay, then it can be done only using a quite accurate knowledge of the time-delay. Contrary, if someone can expect a considerable variation in the time delay then only a less demanding (slower) design is much more reliable and robust.

(The jags of both figures origin from the relative accuracy of the numerical computations. Do not forget that the Nyquist plot of a time-delay process has infinite number of winds around the origin and sometimes even the radius of the external wind is quite small. So it is not easy to find such frequency scaling which provides to determine both $\rho_{\min}$ (i.e. $\|E\|_\infty$) and the robust stability limit at the same time within a proper accuracy.)

The above results strengthen the conservative practical design experience that the time-delay is practically equivalent to an *IU* zero, i.e. invariant.

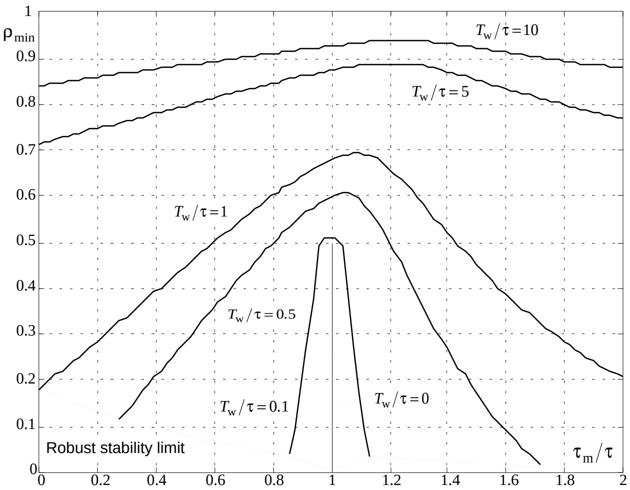

Fig. 5 The function $\rho_{\min}(\tau_m/\tau)$ parametrized by T_w/τ

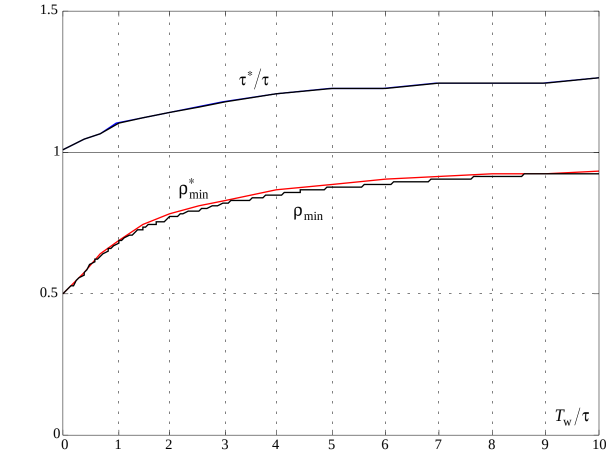

Fig. 6 Influence of the time-delay over-estimation

6. CONCLUSIONS

The well-known robust stability condition is a very general requirement, which is difficult to use for practical design problems. In case of some specific controller design procedures, however, it is possible to introduce such forms, which are very useful in the evaluation of the applicability and performance of the desired closed-loop. These specifications are introduced in this paper for a *GTDOF* control system. Then these results are applied to investigate the influence of time-delay mismatch on the robust stability.

Most of the widely applied identification and adaptive control methods assume an apriori known time-delay. It is not easy (although possible) to incorporate the iterative or adaptive estimation of the delay into the recursive methods. Therefore one can always assume a time-delay uncertainty or mismatch at all practical applications. It was discussed here how this mismatch influences the robustness degradation and the reachable closed-loop performance.

The investigations show that bandwidth higher than the bandwidth of the delay term ($T_w < \tau$) can be reached only for a considerable lower robustness and at the same time a much more accurate knowledge of the time-delay is necessary. This corresponds to the practical design experience that the corner frequency of a delay term corresponds to an unstable zero, i.e., similarly invariant. So the acceptable performance domain means $T_w \geq \tau$.

We found that a certain slight over-estimation of the time-delay improves the robustness, however, a higher over-estimation causes considerable robustness degradation again. This observation can be used for model predictive algorithms, too.

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