A note on the relationships between high-gain state feedback and relay systems

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Abstract— **In this note we study some relationships existing between two widely applied control techniques, namely relay feedback control and high-gain saturated feedback control.**

I. Introduction

In the last thirty years, many different approaches have been employed to study high-gain control systems schemes (see for instance [4], [5], [6], [7]). As is known, high-gain feedback is commonly used in several situations to reduce the effects of disturbances and nonlinearities on the system performance, in terms of both stability and properties of the controlled variables. However, when hard bounds on the control signal magnitude are present, i.e. $|u(t)| \leq u_{max}$, as it happens in many realistic control problems, the effectiveness of the high-gain control approach is reduced, since such a technique usually requires very large control actions. As noted in [8] and [9], there exist even open-loop stable systems where a high-gain feedback law with saturation does not make the origin a global attractor.

High-gain control also plays an important role in sliding mode control systems [2], [3]. As is known (see the classic textbook [1]), when relay control is adopted, the controlled system exhibits two modes of operation, namely the *bang-bang* mode and the *sliding* mode, where the chattering phenomenon may occur, due to the presence of unmodeled dynamics, switching time delays, and other parasitic effects. The most commonly cited approach to reduce the effects of chattering has been that of approximating the switching element with a linear feedback gain in a boundary layer of the sliding manifold [2], [7]. In order for the system behavior to be close to that of the ideal sliding mode, particularly when an unknown disturbance is to be rejected, sufficiently high gain in the linear term is needed.

Referring to Fig.1, we can consider an "ideal" relay as the limit of a saturated feedback device ("real" version of a relay) as the feedback gain of the linear term tends to infinity. Then, a question that naturally arises is the following: Is it possible to derive some of the structural properties of an ideal relay control system by a limiting approach starting from a feedback saturated system? In this note, we show that the sliding manifolds and the regions where the devices behave linearly get arbitrarily close as the gain in the "real" relay becomes increasingly large, as well as the linear dynamics on the sliding manifolds. The note is organized as follows. In Section II some basic facts on relay systems are briefly reviewed. In Section III, the properties of the high-gain feedback systems are discussed and in Section IV their behavior as the linear gain tends to infinity are analysed. A numerical example is reported in Section V, and some concluding remarks are presented in Section VI.

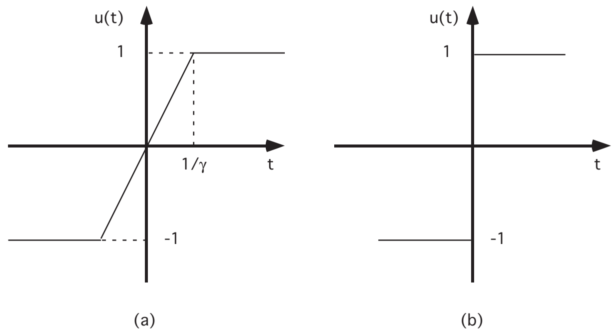

Fig. 1. (a) "Real" and (b) "ideal" relay.

II. Preliminaries on relay systems

In this Section we recall some basic facts and results on relay systems. Let us consider the single input system with state feedback control and an ideal relay at the input

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + bu(t) \\ u(t) & = & \operatorname{sign}(Kx(t)) \end{array} \right. . \qquad (1)$$

Assume that (1) is completely controllable and $Kb \neq 0$ (by the properties of the *sign* function, with no loss of generality we take $Kb = \pm 1$). Let the region R be defined as

$$R := \{x \in \mathbb{R}^n : Kx = 0 \text{ and } |KAx| < 1\} .$$

It is well known that if $Kb = -1$ trajectories corresponding to initial conditions $x(0)$ sufficiently "near" to R are attracted to R, whereas if $Kb = 1$ the state trajectories "avoid" R. If $x(t)$ enters the region R at a (finite) time instant t_0, and remains in R in a time interval $[t_0\ t_1]$, with $t_1 > t_0$ (possibly, $t_1 = +\infty$), then the value of $u(t)$ is univocally determined (simply use the condition on $x(t)$ to belong to R for $t \in [t_0\ t_1]$) from the following formula

$$u(t) = KAx + \gamma Kx = K(A+\gamma I)x, \quad \forall \gamma \in \mathbb{R}, \quad \forall t \in [t_0\ t_1].$$

Therefore it is possible to derive the following expression for the state dynamics

$$\dot{x}(t) = [(I + bK)A + \gamma bK]x(t) := Fx(t)\,. \qquad (2)$$

By pre-multiplying (2) by K, if $Kb = -1$ we obtain

$$\begin{aligned} K\dot{x}(t) &= [K(I + bK)A + \gamma(Kb)K]x(t) \\ &= [KA + (Kb)KA - \gamma K]x(t) \\ &= -\gamma(Kx(t)) \end{aligned}$$

and therefore

$$\frac{d(Kx(t))}{dt} = -\gamma(Kx(t))$$

so that

$$Kx(t) = Kx(t_0)e^{-\gamma(t-t_0)}\,.$$

As a consequence, if $Kx(t_0) = 0$, then $Kx(t) = 0\ \forall t \geq 0$, as it has to be in R. Indeed, if we took $u(t) = K(A+\gamma I)x(t)$ even outside such manifold, the feedback law would still guarantee that $Kx(t) \to 0$ if $\gamma > 0$.

To analyze the structure of the matrix F in (2), we assume without loss of generality that the pair (A, b) is in canonical controllability form and $Kb = -1$, i.e., $K = [k_1\ k_2 \ldots k_{n-1}\ \ -1]$. A simple computation shows that F is in companion form and its n-th row is given by the row vector

$$[\gamma k_1\,, \gamma k_2 + k_1\,, \gamma k_3 + k_2\,, \ldots\,, \gamma k_{n-1} + k_{n-2}\,, k_{n-1} - \gamma]\,.$$

Let us now introduce an useful change of basis in the state space. If we choose

$$T = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ k_1 & k_2 & k_3 & \ldots & k_{n-1} & 1 \end{bmatrix}$$

so that the state vector in the new basis is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = T^{-1}x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ -Kx \end{bmatrix},$$

then we obtain

$$T^{-1}FT = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ k_1 & k_2 & k_3 & \ldots & k_{n-1} & & 1 \\ \hline 0 & 0 & 0 & \ldots & 0 & & -\gamma \end{array}\right] \qquad (3)$$

Therefore, if we define

$$\tilde{x}(t) := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

we have that the evolution of $\tilde{x}$ is given by

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ k_1 & k_2 & k_3 & \ldots & k_{n-1} \end{bmatrix} \tilde{x}(t) - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} Kx(t) \qquad (4)$$

with

$$\frac{d(Kx(t))}{dt} = -\gamma(Kx(t))\,. \qquad (5)$$

Recalling that equation (2) can be used only when $x(t) \in R$ (and, therefore, $Kx(t) = 0$), (4)-(5) reduce to

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ k_1 & k_2 & k_3 & \ldots & k_{n-1} \end{bmatrix} \tilde{x}(t)$$

and $Kx(t) = 0$. Therefore the matrix K determines the $(n-1)$-dimensional state dynamics in the region R once $x(t)$ has reached (in a finite time) the manifold given by $Kx = 0$ and remains in such manifold. In particular, if $\gamma > 0$ and K is appropriately chosen, F can be made asymptotically stable given that its eigenvalues can be arbitrarily fixed via K.

An interesting characterization of the region R can be obtained as follows. Evaluation of KA gives

$$KA = [\, a_0 \quad a_1 + k_1 \quad a_2 + k_2 \quad \ldots \quad a_{n-1} + k_{n-1}\,]$$

So, noting that $Kx = 0$ gives $x_n = [\, k_1 k_2 \ldots k_{n-1}\,]\,\tilde{x}$, substitution of this expression for x_n into $|KAx| < 1$ yields

$$R = \{x \in \mathbb{R}^n : x_n = [\, k_1 k_2 \ldots k_{n-1}\,]\,\tilde{x} \text{ and } | < w, \tilde{x} > | < 1\}$$

where

$$\begin{aligned} w^T \ := \ & [\, a_0 \quad a_1 + k_1 \quad a_2 + k_2 \quad \ldots \quad a_{n-2} + k_{n-2}\,] \\ & +(a_{n-1} + k_{n-1})\,[\, k_1 \quad k_2 \quad \ldots \quad k_{n-1}\,]\,. \end{aligned} \qquad (6)$$

III. High-gain feedback

In this Section we analyze the effect of the high-gain feedback law $u = \gamma Kx$ (with $\gamma \gg 1$) on the system $\dot{x} = Ax + bu$, assuming such a control law holds in the whole state space $\mathbb{R}^n$.

We introduce two simplifying assumptions:

- the pair (A, b) is in canonical control form (if not, it suffices to resort to a suitable basis change, since (A, b) is assumed to be a controllable pair);
- the pair (K, b) satisfies $Kb = -1$.

As far as the second assumption is concerned, we observe that the actual necessary assumption needed in the following is $Kb \neq 0$, but in this case there is no loss of generality in assuming $Kb = -1$. If not, it suffices to simply redefine the value of γ. Note that, if $Kb > 0$, the sign of γ has to be changed, so in the following we will not impose the constraint $\gamma > 0$, and γ will be taken as an arbitrary real number.

As a consequence of the previous assumptions, recalling that $K = [\,k_1 \quad k_2 \quad \ldots \quad k_{n-1} \quad -1\,]$, and denoting by $\{a_0,\, a_1,\, \ldots,\, a_{n-1},\, 1\}$ the coefficients of the characteristic polynomial of A, the closed loop matrix $F := A + \gamma b K$ is in companion form and its n-th row is given by the row vector

$$[\gamma k_1 - a_0\,, \gamma k_2 - a_1\,, \ldots\,, \gamma k_{n-1} - a_{n-2}\,, -(\gamma + a_{n-1})]\,.$$

In order to better understand the modal properties of F, we look for a suitable basis change $z = T^{-1}x$ such that

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} := T^{-1}x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ -Hx \end{bmatrix}$$

where $H := [\,h_1 \quad h_2 \quad \ldots \quad h_{n-1} \quad -1\,]$ has to be determined and

$$T = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ h_1 & h_2 & h_3 & \ldots & h_{n-1} & 1 \end{bmatrix}.$$

Simple computations leads to

$$T^{-1}FT = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ h_1 & h_2 & h_3 & \ldots & h_{n-1} & & 1 \\ \hline v_1 & v_2 & v_3 & \ldots & v_{n-1} & & v_n \end{array}\right]$$

where

$$\left\{ \begin{array}{lcl} v_1 & = & (\gamma k_1 - a_0) - h_1(\gamma + a_{n-1} + h_{n-1}) \\ v_i & = & (\gamma k_i - a_{i-1} - h_{i-1}) - h_i(\gamma + a_{n-1} + h_{n-1}), \\ & & i = 2, 3, \ldots, n-1 \\ v_n & = & -(\gamma + a_{n-1} + h_{n-1}) = -\gamma(1 + \delta_n) \end{array} \right.$$

and $\delta_n := \frac{a_{n-1} + h_{n-1}}{\gamma}$. We want to determine the h_i's in such a way that $v_1 = v_2 = \ldots = v_{n-1} = 0$. In this case $T^{-1}FT$ assumes a block triangolar form which closely reminds the matrix associated with the linear mode of the ideal relay feedback scheme (see (3)).

Introducing the vector $\Delta := [\,\delta_1 \quad \delta_2 \quad \ldots \quad \delta_{n-1}\,]$ and searching for a solution of the form

$$\begin{array}{rcl} [\,h_1 \quad h_2 \quad \ldots \quad h_{n-1}\,] & = & [\,k_1 \quad k_2 \quad \ldots \quad k_{n-1}\,] \\ & & + [\,\delta_1 \quad \delta_2 \quad \ldots \quad \delta_{n-1}\,] \end{array}$$

we obtain that imposing $v_1 = v_2 = \ldots = v_{n-1} = 0$ is equivalent to choosing the δ_i's satisfying

$$\left\{ \begin{array}{lcl} \delta_1 & = & -\frac{1}{\gamma}[(a_{n-1} + k_{n-1} + \delta_{n-1})(k_1 + \delta_1) + a_0] \\ \delta_i & = & -\frac{1}{\gamma}[(a_{n-1} + k_{n-1} + \delta_{n-1})(k_i + \delta_i) \\ & & +(a_{i-1} + k_{i-1} + \delta_{i-1})]\,, i = 2, 3, \ldots, n-1 \end{array} \right. \tag{7}$$

In general, (7) can have more than one solution. For instance, if $n = 4, a_0 = a_2 = k_1 = -1, a_1 = k_2 = k_3 = -2, a_3 = 0, \gamma = \frac{1}{2}$, we have (at least) two solutions

$$\Delta = [\,\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\,] \Rightarrow ||\Delta|| = \frac{\sqrt{3}}{2} < 1$$

and

$$\Delta = [\,2 \quad 2 \quad 2\,] \Rightarrow ||\Delta|| = 2\sqrt{3} > 1\,.$$

However, we are interested in finding sufficiently small solutions (for instance satisfying $||\Delta|| \le 1$) corresponding to sufficiently high values of $|\gamma|$. In this case, we will prove both that a solution always exists and that it is unique. Such a solution will allow us to evaluate the vector H and therefore the basis change matrix T which makes $T^{-1}FT$ the block triangolar matrix

$$T^{-1}FT = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ h_1 & h_2 & h_3 & \ldots & h_{n-1} & 1 \\ \hline 0 & 0 & 0 & \ldots & 0 & v_n \end{array}\right]$$

Let us note that (7) can be rewritten as

$$\Delta = \frac{1}{\gamma} L(\Delta) \tag{8}$$

where $L(\cdot)$ is a suitable continuous operator depending on coefficients a_i's and k_i's, which are assumed to be fixed. If $\Xi := [\,\epsilon_1 \quad \epsilon_2 \quad \ldots \quad \epsilon_{n-1}\,]$, we have

$$[L(\Xi) - L(\Delta)]^T = \begin{bmatrix} (a_{n-1} + k_{n-1} + \delta_{n-1})(\delta_1 - \epsilon_1) \\ +(k_1 + \epsilon_1)(\delta_{n-1} - \epsilon_{n-1}) \\ \\ (a_{n-1} + k_{n-1} + \delta_{n-1})(\delta_2 - \epsilon_2) \\ +(k_2 + \epsilon_2)(\delta_{n-1} - \epsilon_{n-1}) + (\delta_1 - \epsilon_1) \\ \vdots \\ (a_{n-1} + k_{n-1} + \delta_{n-1})(\delta_i - \epsilon_i) \\ +(k_i + \epsilon_i)(\delta_{n-1} - \epsilon_{n-1}) \\ +(\delta_{i-1} - \epsilon_{i-1}) \\ \vdots \\ (a_{n-1} + k_{n-1} + \delta_{n-1})(\delta_{n-2} - \epsilon_{n-2}) \\ +(k_{n-2} + \epsilon_{n-2})(\delta_{n-1} - \epsilon_{n-1}) \\ +(\delta_{n-3} - \epsilon_{n-3}) \\ \\ (a_{n-1} + 2k_{n-1} + \delta_{n-1} + \epsilon_{n-1})(\delta_{n-1} - \epsilon_{n-1}) \\ +(\delta_{n-2} - \epsilon_{n-2}) \end{bmatrix}$$

from which it is easy to see that

$$||L(\Xi) - L(\Delta)||^2 = (\Xi - \Delta)P(\Xi, \Delta)(\Xi - \Delta)^T \tag{9}$$

where $P(\Xi, \Delta) = P(\Xi, \Delta)^T \ge 0$ is a suitable matrix continuously depending on Ξ and Δ. The greatest eigenvalue of $P(\Xi,\Delta)$, denoted by $\lambda_{MAX}(P(\Xi,\Delta))$, continuously depends on the same vectors, too. Therefore, once the a_i's and the k_i's are fixed, the expression

$$\lambda_0 := \max_{||\Xi||\leq 1, ||\Delta||\leq 1} \lambda_{MAX}(P(\Xi,\Delta))$$

defines a positive (finite) real number. From (9) it follows

$$\frac{1}{|\gamma|}||L(\Xi)-L(\Delta)|| \leq \frac{\sqrt{\lambda_0}}{|\gamma|}||\Xi-\Delta|| := \mu||\Xi-\Delta||$$

which, evaluated at $\Xi = 0$, gives

$$\frac{1}{|\gamma|}||L(\Delta)-L(0)|| \leq \mu||\Delta||$$

and therefore

$$\frac{1}{|\gamma|}||L(\Delta)|| \leq \mu||\Delta|| + \frac{1}{|\gamma|}||L(0)|| \,.$$

If $|\gamma|$ is chosen sufficiently large, i.e.

$$|\gamma| \geq \max\{2\sqrt{\lambda_0}, 2||L(0)||\} \,,$$

it follows that $0<\mu\leq\frac{1}{2}$. Moreover

$$||\frac{L(\Delta)}{\gamma}|| \leq \mu||\Delta|| + ||\frac{L(0)}{\gamma}|| \leq \frac{||\Delta||+1}{2}$$

and

$$||\frac{L(\Xi)}{\gamma} - \frac{L(\Delta)}{\gamma}|| \leq \frac{1}{2}||\Xi-\Delta|| \,.$$

Therefore $\frac{1}{\gamma}L(\cdot)$ is well-defined as an operator from the unit ball $\mathcal{B}_1 := \{\Delta \in \mathbb{R}^{n-1} : ||\Delta|| \leq 1\}$ into itself, and more precisely it is a contractive operator. From the contraction theorem, it follows that, if $|\gamma|$ is sufficiently large, (8) admits in $\mathcal{B}_1$ a unique solution Δ_0. Denoting by $m>0$ the maximum value assumed by the continuous function $||L(\Delta)||$ in the compact set $\mathcal{B}_1$, (8) also implies $||\Delta_0|| = \frac{1}{|\gamma|}||L(\Delta_0)|| \leq \frac{m}{|\gamma|}$, proving that Δ_0 is infinitesimal w.r.t. $|\gamma| \to +\infty$, and consequently also that $H \to K$. Since also δ_n is infinitesimal, we have that $v_n \simeq -\gamma$, so that for $|\gamma| \gg 1$,

$$T^{-1}FT \simeq \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ k_1 & k_2 & k_3 & \dots & k_{n-1} & 1 \\ \hline 0 & 0 & 0 & \dots & 0 & -\gamma \end{array}\right]$$

as in the "ideal" relay case.

IV. Connections between relay systems and saturated linear feedback systems

Let us know consider the feedback scheme (1), where the ideal relay $u = \mathrm{sgn}(Kx)$ is replaced by a "real" one, namely the saturated linear feedback given by

$$u(t) = \begin{cases} -1 \text{ if } Kx(t) < -\frac{1}{\gamma} \\ \gamma Kx(t) \text{ if } |Kx(t)| \leq \frac{1}{\gamma} \\ +1 \text{ if } Kx(t) > \frac{1}{\gamma} \end{cases} .$$

We have that in the region

$$R(\gamma) := \{x \in \mathbb{R}^n : |Kx| < \frac{1}{\gamma}\}$$

the system behaves as a linear feedback system with $u = \gamma Kx$, $\gamma > 0$, and $Kb \neq 0$ but not necessarily equal to -1. Clearly, as stated in the Introduction, the ideal relay can be obtained as a limit case ($\gamma \to +\infty$), and the goal of this note is, in fact, that of investigating the relationships between the real case (with γ sufficiently large) and the ideal case ($\gamma = +\infty$).

From (7) and the fact that Δ_0 is infinitesimal w.r.t. $\frac{1}{\gamma}$, we can write

$$-\gamma\Delta_0 = w^T + \Xi^T \tag{10}$$

where w has been defined in (6), and $\Xi := [\,\epsilon_1\epsilon_2\dots\epsilon_{n-1}\,]^T$ is infinitesimal w.r.t. $\frac{1}{\gamma}$. Let us consider the region $R(\gamma) \cap \{x \in \mathbb{R}^n : Hx = 0\}$, which represents the portion of the manifold $Hx = 0$ where the saturating device obeys the equation

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ h_1 & h_2 & h_3 & \dots & h_{n-1} \end{bmatrix} \tilde{x}(t).$$

Imposing $Hx = 0$ is equivalent to setting $x_n = [\,k_1k_2\dots k_{n-1}\,]\,\tilde{x} + \Delta_0\tilde{x}$, which substituted into $|\gamma Kx| < 1$, recalling (10), gives

$$|<w+\Xi, \tilde{x}>| < 1 \,.$$

Therefore

$$\begin{aligned} R(\gamma) \cap \{x \in \mathbb{R}^n : Hx = 0\} = \\ \{x \in \mathbb{R}^n : x_n = [\,k_1k_2\dots k_{n-1}\,]\,\tilde{x} + \Delta_0\tilde{x} \\ \text{and } |<w+\Xi,\tilde{x}>| < 1\} \,, \end{aligned}$$

showing that $R(\gamma) \cap \{x \in \mathbb{R}^n : Hx = 0\}$ approaches R as $|\gamma|$ goes to infinity. By summarizing our results, we can say that, if $\gamma > 0$ and γ is sufficiently large:

- the role of the manifold $Kx = 0$ is taken by the manifold $(K + [\Delta_0 \mid 0])x = 0$, with Δ_0 infinitesimal w.r.t. $\frac{1}{\gamma}$;
- the characteristic polynomial of the closed loop system approaches that of the ideal relay case;
- $R(\gamma) \cap \{x \in \mathbb{R}^n : Hx = 0\}$ approaches R.

V. A numerical example

Let us consider a 3-dimensional system, with $a_0 = 1, a_1 = 3.1, a_2 = 2.1$. Choosing $K = [-2\ -2\ -1]$ leads to the stable characteristic polynomial $\lambda^2 + 2\lambda + 2$ associated with the dynamics of $\tilde{x} = [x_1\ x_2]^T$ on the manifold $Kx = 0$, i.e. $x_3 = -2x_1 - 2x_2$. Evaluation of Δ corresponding to $\gamma = 10$ gives $\Delta = [-0.1\ -0.1]$, which leads to the following regions

$$R = \{x \in \mathbb{R}^3 : x_3 = -2x_1 - 2x_2 \text{ and } |0.8x_1 + 0.9x_2| < 1\}$$

$$R(10) \quad \cap \quad \{x \in \mathbb{R}^n : Hx = 0\} = \\ \{x \in \mathbb{R}^3 : x_3 = -2.1x_1 - 2.1x_2 \text{ and } |x_1 + x_2| < 1\},$$

which are clearly sufficiently "near", despite the fact that $\gamma = 10$ is not very large. Moreover, the characteristic polynomial associated with the dynamics of $\tilde{x}$ on the manifold $Hx = 0$ is the stable polynomial $\lambda^2 + 2.1\lambda + 2.1$, which is "very close" to $\lambda^2 + 2\lambda + 2$.

VI. Conclusions

The comparison between the ideal relay case and the real relay one presented in this note shows that all the properties of the corresponding closed loop systems are preserved if the feedback gain γ is sufficiently high. The characteristic polynomial of the linear mode, the linear mode manifold, and the invariant regions of the linear mode in the real case can be seen as "perturbed" versions of the equivalent entities of the ideal case.

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