On actuators/sensors placement for collocated flexible plates

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Abstract— **The problem of optimal placement of actuators and sensors for a collocated flexible structure is studied. For a given number of available piezoelectric patches of fixed dimensions to be bonded on the plate, the best locations are looked for in order to maximize the modal controllability and observability of the structure. A related problem, leading to an optimal actuator and sensor placement for a reduced order model, is also taken into account.**

Keywords— **Piezoelectric sensor-actuator, optimal placement, flexible plates.**

I. Introduction

The problem of optimal placement of actuators and sensors for flexible structures has been the subject of much research in recent years (see *e.g.* [2], [3], [4] and references therein). As usual in optimization problems, the actual meaning of "optimal placement" is strongly related to the criterion of optimality adopted, which in turn ultimately relies on the real world problem under study: for example, the problem studied in [2] aims at choosing the actuators and sensors locations optimizing an LQR or LQG performance index, *i.e.*, the eventual aim of actuators/sensors placement is in connection with the design of an LQR or LQG controller; in [3], a structural testing problem is studied under the assumption that the actuators and sensors used for testing cannot be placed in the actual locations on which disturbances act and performance output is measured, so that the considered optimization aims at identifying actuators/sensors locations such that the modal controllability and observability in testing conditions is as close as possible to the modal controllability and observability experienced from the disturbance input to the performance output in operation conditions; in [4], the optimization of spatial $\mathcal{H}_2$ controllability was considered in order to obtain a highly controllable structure in an average sense, with the constraint of guaranteeing some minimum modal controllability and observability on each low frequency mode (*i.e.*, on each mode in the bandwidth of interest for vibration reduction) and the additional constraint of guaranteeing some maximum modal controllability and observability on few additional high frequency modes (in order to attenuate spillover effects from the modes outside the bandwidth of interest for vibration reduction).

In this paper, a numerical study is performed with the aim of designing an experimental setup for testing both active and passive control algorithms for steel plates actuated and sensed via collocated piezoelectric actuators/sensors. For a given number of available piezoelectric patches of fixed dimensions to be bonded on the plate, the best locations are looked for in order to maximize the modal controllability and observability of the structure. In particular, given a model considering several low frequency modes of the plates, two cases often encountered in real world applications are considered: in the first case the optimal locations are chosen as the ones guaranteeing the highest modal controllability/observability for all the modes in the model, whereas in the second case it is assumed that a lower order model is sought for control purposes, and then in order to limit spillover effects the optimal locations are identified as those providing the largest gap between the controllability and observability of the low frequency, reduced order model and the controllability and observability of the neglected modes.

This work was supported by ASI and MIUR. Corresponding author L. Menini. Tel. +39-06-7259 7432. Fax +39-06-7259 7460.

II. Simplified modelling

A simplified model based on Kirchhoff-Love theory is here considered in order to describe the dynamic of the piezoactuated plate, whose thickness is h, Young modulus E, Poisson ratio ν and whose density is ρ. It is assumed that the piezoelectric patches bonded on the plate are thin enough, and then have negligible influence on the shape of the eigenmodes and on the values of the eigenfrequencies.

A modal analysis of the dynamical problem is performed, by expanding all the unknown quantities as a linear combination of the structural eigenfunctions. For simplicity, simply supported boundary conditions are chosen for the plate, so that the eigenfunctions can be analytically determined. For different boundary conditions, the modal shapes can be numerically evaluated using a finite element code, or the analysis can be performed using an appropriate set of Rayleigh functions [6].

The effect of the piezoelectric actuators consists of shear forces, exerted on the plate along the actuator boundaries and proportional to a control voltage V_a applied between the piezoelectric electrodes; these forces result in line moments applied on the plate. In particular, by neglecting the mechanical stiffness of the thin piezoelectric patches, the actuating shear force per unit length f_a^i exerted by the $i-$th actuator is given by

$$f_a^i = -\bar{e}_{31} V_a^i, \qquad (1)$$

where $R^i(\cdot,\cdot)$ is the characteristic function relevant to the $i-$th piezoelectric actuator, which is equal to 1 if the point x, y belongs to the $i-$th piezoelectric patch and zero elsewhere, V_a^i is the control voltage on the $i-$th piezoelectric actuator and $\overline{e}_{31} = e_{31} - c_{31}e_{33}/c_{33}$; e_{31}, e_{33}, c_{31} and c_{33} are piezoelectric constants according to Voigt notation and the usual assumption of negligible normal stresses in the piezoelectric patches has been enforced [1].

The sensing voltage V_s which can be measured between the electrodes of a piezoelectric sensor is given by the following expression

$$V_s^i = -\frac{1}{A^i}\frac{h}{2}\int [\frac{h^i}{\overline{\epsilon}_{33}}\overline{e}_{31}\Delta(w)]dA, \tag{2}$$

where A^i is the area of the i piezoelectric sensor, h^i is its thickness, w is the plate mid plane deflection and $\overline{\epsilon}_{33} = \epsilon_{33} + e_{33}^2/c_{33}$ is the clumped piezoelectric capacity. Moreover Δ is the Laplace operator and e_{33} is another piezoelectric constant. The actuated plate equation can be written as

$$D\Delta^2 w + \rho h \frac{\partial^2 w}{dt^2} = \sum_{k=1}^{n_a}(-\frac{h}{2}\overline{e}_{31}V_a^k\Delta(R^k(x,y))), \tag{3}$$

where $D = Eh^3/(12(1-\nu^2))$ is the plate stiffness.

The solution of (3) can be given the following expansion:

$$w(x,y,t) = \sum_{p=1}^{+\infty}\sum_{q=1}^{+\infty}\alpha_{pq}(x,y)\beta_{pq}(t), \tag{4}$$

where, for simply supported boundary conditions and a plate of dimensions $a \times b$, the $(p,q)-$th eigenmode can be analytically obtained as

$$\alpha_{pq}(x,y) = C_0 \sin\left(\frac{\pi p x}{a}\right)\sin\left(\frac{\pi q y}{b}\right), \tag{5}$$

and C_0 is a normalization constant. Taking $C_0 = \frac{2}{\sqrt{\rho h a b}}$, the modal coordinate $\beta_{pq}(t)$ is the solution of

$$\ddot{\beta}_{pq}(t) + K_{pq}\beta_{pq}(t) = \sum_{i=1}^{N_{PZT}}\gamma_{pq}^i V_a^i, \tag{6}$$

and the output voltages can be expressed as

$$V_s^i = \sum_{p=1}^{+\infty}\sum_{q=1}^{+\infty}\eta_{pq}^i\beta_{pq}(t), \tag{7}$$

where, defining $\Gamma(x_1,x_2,w) := \cos(w\pi x_1) - \cos(w\pi x_2)$ and denoting by a_p, b_p the length and the width of the piezoelectric patch and by h_p its thickness,

$$K_{pq} = \frac{D}{\rho h}\left(\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2\right)^2,$$

$$\gamma_{pq}^i = -C_0 K_P\left(\frac{pb}{qa} + \frac{qa}{pb}\right)\Gamma(x_{i,1}, x_{i,2}, p/a)\Gamma(y_{i,1}, y_{i,2}, q/b),$$

$$\eta_{pq}^i = -C_0 K_S\left(\frac{pb}{qa} + \frac{qa}{pb}\right)\Gamma(x_{i,1}, x_{i,2}, p/a)\Gamma(y_{i,1}, y_{i,2}, q/b),$$

$$K_P = -\frac{h}{2}\overline{e}_{31},$$

$$K_S = -\frac{h_p}{a_p b_p}\frac{\overline{e}_{31}}{\overline{\epsilon}_{33}}\frac{h}{2}.$$

TABLE I
PHYSICAL PARAMETER VALUES.

h	[m]	0.0015
E	[GPa]	210
μ	[]	0.3
ρ	[kg/m^3]	7850
h_a	[m]	0.000127
c_{11}	[GPa]	133
c_{12}	[GPa]	77.5
$c13$	[GPa]	87
$c33$	[GPa]	127
$c44$	[GPa]	26.7
$e15$	[C/m^2]	13.37
$e31$	[C/m^2]	-7.22
$e33$	[C/m^2]	15.10
ϵ_{11}	[nF/m]	9.97
ϵ_{33}	[nF/m]	8.70

TABLE II
FIRST EIGENFREQUENCIES AND GAPS.

p	q	Freq. (Hz)	Δ freq.	Δ freq. (rel.)
1	1	322	328	1.01
2	1	650	309	0.47
1	2	960	237	0.24
3	1	1198	90	0.07
2	2	1289	547	0.42
3	2	1836	128	0.06
4	1	1964	59	0.03
1	3	2024	328	0.16
2	3	2353	250	0.10
4	2	2603	297	0.11
3	3	2900	49	0.01
5	1	2950	-	-

Considering only a finite number of modes in (6) and (7) by imposing $p = 1, \ldots, N_x$ and $q = 1, \ldots, N_y$, a finite dimensional linear time invariant system describing the plate is obtained. In such a modal model, in order to take into account the natural damping of the structure, it is customary to assume a small proportional damping (though this damping does not appear in (6); in this paper, $\zeta = 0.01$). Actual values of the physical parameters for the plate and piezoelectric patches under study are given in table I.

Using the model described above, the eigenfrequencies (equal to $\sqrt{K_{pq}}$) of the first modes have been obtained; in particular, Table II shows the first twelve lowest eigenfrequencies, the corresponding values of (p,q), the gap between each eigenfrequency and the following one, and the relative gap between each eigenfrequency and the following one (obtained as the elementwise ratio between the forth and the third column).

From the reported values, it seems reasonable to focus the control efforts on frequencies below 1.5 kHz, *i.e.*, to consider $M = 5$ modes in a model of the system to be used for control purposes. The resulting state space model will have a state of dimension 10 and will be called the *"reduced* *model"* in the sequel. The eigenfunctions relative to the five lowest frequency modes are represented in figures 1-5.

In fact, for the analysis to follow another model (which will be called the *"higher order model"*) will also be of interest. As a matter of fact, when a controller designed on the basis of the reduced model is applied to the real structure, spillover effects can appear due to the dynamics neglected in the reduced model; such spillover effects will be particularly evident if the actuators/sensors locations leading to high controllability/observability levels for the low frequency modes actually make some of the higher frequency neglected modes even more controllable/observable than the low frequency modes considered in the reduced model. In order to study the effects of actuators/sensors placement on higher frequency modes too, a representative number of additional eigenmodes has been taken into account by choosing the values of the parameters $N_x = 4$ and $N_y = 3$; hence, the total number of modes contained in the state space description of the higher order model is $N = N_x \cdot N_y = 12$, and its state has dimension $2N = 24$.

In order to select the best locations for the collocated piezoelectric actuators/sensors, the surface of the plate has been divided in $N_{COL} = \frac{a}{a_p} = 10$ columns and $N_{ROW} = \frac{b}{b_p} = 10$ rows, so that the rectangles resulting at the intersection of each couple (column,row) have exactly the size of a piezoelectric patch and can be considered as a possible location. It should be noticed that even if such a choice makes available only $N_{COL} \cdot N_{ROW} = 100$ possible locations for each patch, the number of total possible configurations that are to be checked when more actuator/sensor pairs are to be placed rapidly increases with the number N_{PZT} of patches: in particular, taking into account that a patch cannot be placed on top of another patch and that the patches are indistinguishable from each other, $\binom{100}{2} = 4950$ configurations are to be checked when $N_{PZT} = 2$, and $\binom{100}{3} = 161700$ configurations are to be checked when $N_{PZT} = 3$. In general, the number of configurations increases as $\binom{N_{ROW} \cdot N_{COL}}{N_{PZT}}$, and clearly such order of magnitude cannot be significantly lowered even if symmetries among configurations are taken into account.

Depending on the chosen positions for the piezoelectric patches, the resulting simplified models are characterized by different controllability/observability levels. The next two sections are devoted to choose the "best" positions for the patches according to two indexes of interest in control applications.

III. Optimal placement $\max(\sigma_{2M})$: maximization of the minimum singular value

The first approach considered in order to select the "best" locations for the piezoelectric patches is simply aimed to get a highly controllable/observable reduced model (taking into account only the $M = 5$ lowest frequency modes of the system). In order to achieve this goal, the patch locations are selected which guarantee the highest controllability/observability of the least controllable/observable mode, *i.e.*, a maximization of the smallest singular value of the controllability/observability gramians

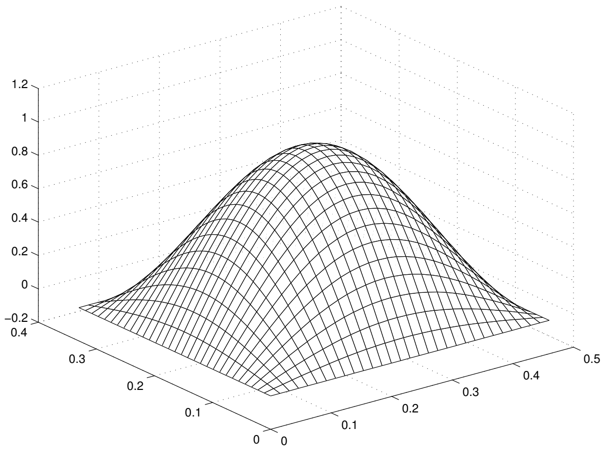

Fig. 1. The first eigenmode of the plate.

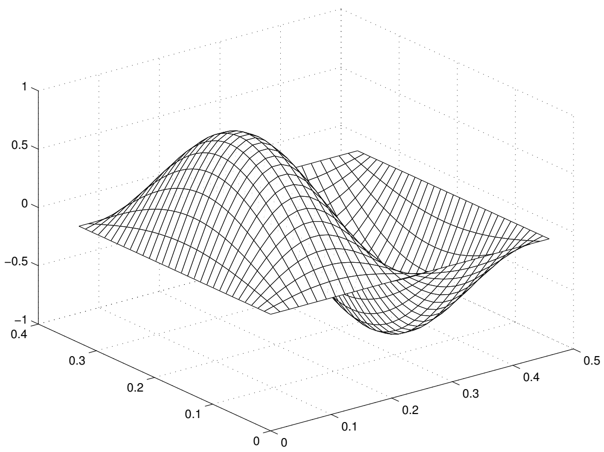

Fig. 2. The second eigenmode of the plate.

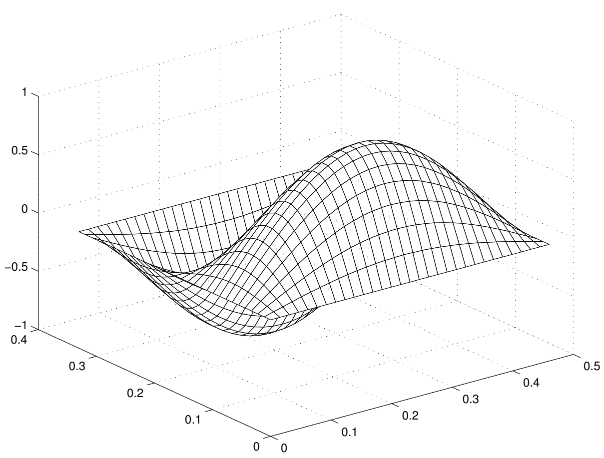

Fig. 3. The third eigenmode of the plate.

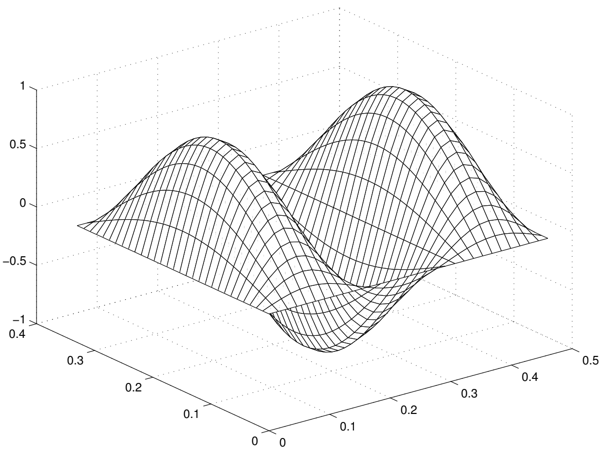

Fig. 4. The fourth eigenmode of the plate.

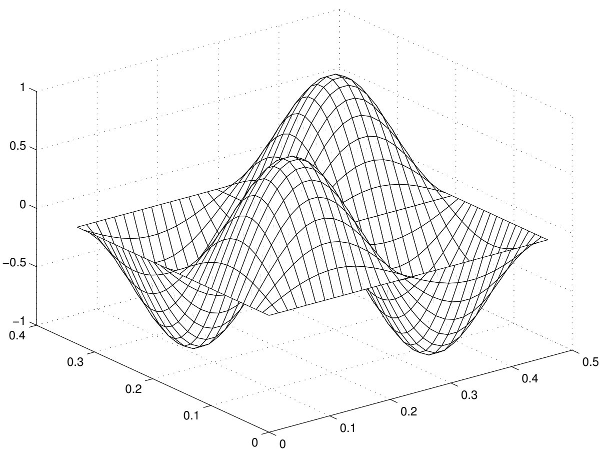

Fig. 5. The fifth eigenmode of the plate.

corresponding to each actuator/sensor configuration is performed.

For a given realization (A, B, C, D) of an asymptotically stable system, the controllability gramian W_C and the observability gramian W_O can be obtained as the unique symmetric, positive semidefinite solutions of the following Lyapunov equations:

$$AW_C + W_C A^T + BB^T = 0,$$
$$A^T W_O + W_O A + C^T C = 0.$$

It is a well known fact [7], [5] that in order to have a consistent indication of the levels of controllability and observability of the modes of a system, it is not enough to look at the controllability/observability gramians of a generic realization of the system. In fact, in general a given mode can be highly controllable and poorly observable according to the gramians obtained for a given realization, while being poorly controllable and highly observable according to the gramians obtained for another realization (such conclusion is immediate in the single input, single output case: in such a case, it is easy to see that (A, B, C, D) and (A^T, C^T, B^T, D^T) are both realizations of the same system, but the controllability/observability gramians of the second realization are, respectively, the observability/controllability gramians of the first realization).

	4'	2'					2"	4"	
	3'	1'					1"	3"	
	3	1					1"'	3"'	
	4	2					2"'	4"'	

Fig. 6. $\max(\sigma_{\mathbf{2M}})$, $\mathbf{N_{PZT} = 1}$. Optimal location (1), corresponding symmetric locations (1'), (1"), (1"'); next three suboptimal locations (2)(3)(4) and their symmetric locations.

However, it can be proven that among the class of equivalent realizations of a system, there always exists a *balanced realization*, *i.e.*, a realization whose controllability and observability gramians are equal; hence, consistent information can be obtained by considering a balanced realization, for which the singular values of the controllability (equivalently, observability) gramian simultaneously measure the controllability and observability of the modes. As customary, the singular values of the controllability gramian of a balanced realization with state of dimension $2M$ will be denoted as $\sigma_1, \ldots, \sigma_{2M}$, with σ_{2M} being the smallest one.

Since the flexible plate has two axes of symmetry (the two lines connecting the centers of the opposite sides of the plate), for each configuration of the piezoelectric patches (optimal or not) it will be possible to find either one or three configurations (which in the sequel will be called *"symmetric locations"* for brevity) leading to the same performance: in particular, such symmetric locations can be identified by a rotation of π radians around one axis of symmetry, possibly followed by another rotation of π radians around the other axis of symmetry; moreover, there will be only one symmetric location if the given location is invariant under a single rotation of π radians around one symmetry axis, or if it is invariant under two consecutive rotations of π radians around two different symmetry axes, whereas there will be three symmetric location otherwise. The symmetry between optimal locations can be appreciated in figure 6, where optimal locations are indicated with (1), $(1')$, $(1'')$, $(1''')$; the first suboptimal locations are indicated as (2), $(2')$, $(2'')$, $(2''')$ and so on. Notice that in the subsequent figures, symmetric locations are not shown in order to make the figure more readily understandable (but they are of course present and can be obtained by means of the previously described rotations).

The optimal configurations for the criterion $\max(\sigma_{2M})$ and the cases $N_{PZT} = 1, 2$ are studied in the next two subsections.

A. $N_{PZT} = 1$

In the simplest possible case where just one piezoelectric collocated actuator/sensor pair has to be placed, the results summarized in table III are obtained. From the reported values, the following conclusions can be drawn:

- as could be expected, for any level of guaranteed control-

TABLE III

Maximum smallest singular value σ_{2M}, corresponding largest singular value σ_1 for $N_{PZT} = 1$. [First 4 (sub)optimal locations]

σ_{2M}	σ_1
0.3599×10^{-5}	0.9014×10^{-5}
0.2496×10^{-5}	1.0949×10^{-5}
0.1482×10^{-5}	0.8296×10^{-5}
0.1174×10^{-5}	0.7160×10^{-5}

lability/observability (as measured by σ_{2M}) four different equivalent configurations can be considered, corresponding to the two basic symmetries of the structure under study;

- the level of controllability/observability of the most controllable/observable mode (as measured by σ_1) for each "optimal" configuration is not necessarily increasing with the level of controllability/observability of the least controllable/observable mode (as measured by σ_{2M}).

It should also be stressed that, as will be discussed in greater detail in section IV, the singular values σ_i, $i = 1, \ldots, 2M$ are not necessarily sorted according the eigenfrequencies of the plate, *e.g.* the most controllable/observable mode in each considered configuration is not necessarily the one associated with the lowest frequency.

B. $N_{PZT} = 2$

When two piezoelectric collocated actuator/sensor pairs have to be placed, the results summarized in table IV are obtained for the configurations. In particular, each row of the table is obtained for four different configurations of the piezoelectric patches, that can be obtained as a basic configuration and its symmetric locations. It turns out that, among the 32 configurations corresponding to the values in table IV, 8 configurations with different optimality levels are characterized by a patch in position $(3, 4)$ (marked "A" in figure 7). Among these 8 configurations, (A-1) is the best (corresponding to the first row of table IV), (A-2) is the second best (corresponding to the second row of table IV) and so on. The other 24 configurations corresponding to the values in table IV are the symmetric locations of those pictured in figure 7).

It may be interesting to notice that the position marked "A" in figure 7 corresponds to the optimal location for the optimization of σ_{2M} with $N_{PZT} = 1$.

IV. Optimal placement $\max(\Delta\sigma_{2M})$: maximum gap after σ_{2M}

The second approach considered in order to select the "best" locations for the piezoelectric patches takes into account the possible occurrence of spillover effects. In order to limit the spillover effects, the information provided by the high order model is used: in particular, the patch locations are selected in such a way to maximize the gap $\Delta\sigma_{2M} = \sigma_{2M} - \sigma_{2M+1}$ between the $2M-$th and the $(2M + 1)-$th singular value of the resulting balanced realization.

When using $\Delta\sigma_{2M}$ to measure the insensitivity of the system to spillover effects, an additional constraint needs to be enforced: in fact, since the singular values of the gramian are sorted in decreasing order of magnitude, in general there is no guarantee that the first $2M$ singular values correspond to the $2M$ eigenmodes having the lowest eigenfrequencies. Hence, the "optimal" configuration (leading to a reduced order system having minimum spillover effects) can be identified by looking for the configuration with the highest $\Delta\sigma_{2M}$ and satisfying the additional constrain that the corresponding reduced order system (obtained by balanced truncation [7], *i.e.*, by considering only the first $2M$ states of the balanced realization) contains exactly the M lowest frequency modes of the high order system.

From a physical point of view, the "pathological" situation depicted in the previous paragraph (where a high frequency mode substitutes a low frequency mode in the reduced model obtained by balanced truncation) can be associated with an actuators/sensors configuration such that the controllability and observability of the high frequency mode is strongly enhanced, at the expenses of the controllability/observability of the lower frequency mode; for the plate under study, this situation occurs for example when $N_{PZT} = 1$ and the patch is placed in the locations (2), (3) and (4) in figure 6.

By means of an exhaustive search over the admissible configurations identified in section II, the following results have been obtained for the placement of $N_{PZT} = 1, 2$ piezoelectric patches.

TABLE IV

Maximum smallest singular value σ_{2M} and corresponding largest singular value σ_1 for $N_{PZT} = 2$. [First 8 (sub)optimal locations]

σ_{2M}	σ_1
0.7997×10^{-5}	1.4762×10^{-5}
0.7983×10^{-5}	1.4783×10^{-5}
0.7732×10^{-5}	1.3644×10^{-5}
0.7716×10^{-5}	1.3646×10^{-5}
0.7596×10^{-5}	1.4758×10^{-5}
0.7596×10^{-5}	1.4799×10^{-5}
0.7513×10^{-5}	1.3646×10^{-5}
0.7509×10^{-5}	1.3643×10^{-5}

				5	1				
				8	3				
		A		4	7				
				2	6				

Fig. 7. $\max(\sigma_{\mathbf{2M}})$, $\mathbf{N_{PZT} = 2}$. Optimal location (A-1) and next seven suboptimal locations (A-2)(A-8) [the symmetric locations having the same level of optimality are not shown].

TABLE V

Maximum gap after smallest singular value.

$\Delta\sigma_{2M}$
0.4489×10^{-5}
0.4468×10^{-5}
0.4429×10^{-5}
0.3926×10^{-5}
0.3908×10^{-5}
0.3886×10^{-5}
0.3877×10^{-5}

				4	7				
		1					2		
		A					3		
				5	6				

Fig. 8. $\max(\boldsymbol{\Delta\sigma_{2M}})$, $\mathbf{N_{PZT} = 2}$. Optimal location (A-1) and next seven suboptimal locations (A-2)(A-8) [the symmetric locations having the same level of optimality are not shown. Notice that (A-1), (A-2) and (A-3) have just one symmetric location].

A. $N_{PZT} = 1$

In the simplest possible case where just one piezoelectric collocated actuator/sensor pair has to be placed, only one location (and its three associated symmetric locations) satisfies the constraint of having only low frequency modes in the reduced order model, so that the optimization is trivial. Quite remarkably, the resulting location agrees with the best location identified for the case $N_{PZT} = 1$, and then can be considered optimal with respect to both criteria $\max(\sigma_{2M})$ and $\max(\Delta\sigma_{2M})$. On the other hand, the next three suboptimal locations (and their symmetric locations) listed in table III do not satisfy the mentioned constraint, and then can be supposed prone to spillover effects unless a suitable design of the controller is performed.

B. $N_{PZT} = 2$

When two piezoelectric collocated actuator/sensor pairs have to be placed, the results summarized in table V and figure 8 are obtained for the configurations (for the interpretation of the table and the figure, see section III.B). In particular, the first three rows of table V correspond to the configurations (A-1), (A-2) and (A-3) of figure 8, which admit only one symmetric location; on the contrary, each of the following four configuration (A-4), (A-5), (A-6) and (A-7) (listed in decreasing order of optimality) admits three symmetric location providing the same level of optimality. Contrary to what happens in the case $N_{PZT} = 1$, when $N_{PZT} = 2$ there is no configuration (between the ones considered in this paper) being optimal according to both $\max(\sigma_{2M}$ and $\max(\Delta\sigma_{2M})$.

V. Conclusions

The numerical study conducted in this paper has lead to the individuation of candidate "optimal" locations for the placement of collocated piezoelectric sensor/actuator pairs on a flexible plate. Two different optimality indexes have been considered, one simply aimed at achieving a highly controllable/observable reduced order model, and the other also taking into account possible spillover effects. The cases of one and two pairs have been investigated in detail.

When a single pair has to be placed, it turns out that the optimal location obtained is such for both criteria, whereas no solution which is suboptimal for the first index is feasible in the optimization of the second index.

For the case with two pairs, the optimal solutions obtained according to the first index are not optimal with respect to the second index (leading to a structure probably prone to spillover effects). A possible approach in the second case is then to choose the optimal solution with respect to the second index, which is actually only slightly suboptimal with respect to the first index.

Further work will be along the following paths:

- finer optimization: using a more refined finite element formulation (taking into account additional information about the structure, *e.g.* the increased mechanical stiffness due to the piezoelectric patches), the optimal locations found in this work will be validated, and then the sensitivity of the given locations to small displacement will be explored, possibly leading to improved locations;
- study of configurations involving more piezoelectric patches: as pointed out in the introduction, the quantity of computation time and storage space needed when optimal locations are sought for multiple pairs dramatically increases with the number of pairs. As a consequence, developments along this line will also include the generation of enhanced code (possibly to be run on parallel machines) to deal with the above mentioned difficulties;
- realization of the described experimental setup in our laboratory.

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