Analytical Expressions for Positioning Uncertainty Propagation in Networks of Robots

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Abstract— **In this paper we present an analysis of the positioning uncertainty increase rate for a group of mobile robots. The simplified version for a group of N robots moving along one dimension is considered. The one dimension restriction permits us to extract an exact expression for the accumulation of positioning uncertainty in a group of robots equipped with proprioceptive (odometric in this case) and exteroceptive (relative distance between robots) sensors. The solution obtained provides insight in the structure of the multirobot localization problem. In addition, it serves both as an approximation and a starting point for examining the more realistic case of N robots moving on a plane. Our derivation is based on a Kalman filter estimator that combines all measurements from all robots in the group. Furthermore, we analyze the effect of initial uncertainty, number of robots (N) and sensor noise on the rate of positioning uncertainty increase. The analytical results derived in this paper and the impact of the different parameters are validated in simulation.**

Keywords— **Multirobot Localization, Cooperative Localization, Networks of Robots.**

I. Introduction

This paper studies the localization accuracy of a team of mobile robots that closely cooperate while navigating an area. We consider primarily the most challenging scenario where the absolute positions of the robots cannot be measured or inferred, and no external positioning information is obtained from a GPS receiver or a map of the environment. One of the advantages of multi-robot systems is that robots can accurately localize by measuring their relative position and/or orientation and communicating localization information throughout the group. In this case the uncertainty in the position estimates for all robots will continuously increase. However, previous work on *cooperative localization* [1], [2], [3] has demonstrated that the localization uncertainty increase across groups of robots is lower compared to the situation where each robot is estimating its position without cooperation with the rest of the team.

The theoretical analysis of the positioning uncertainty propagation during cooperative localization has been an open problem to this date. In this paper we present the necessary first step for determining upper bounds on the position uncertainty accumulation for a group of N robots. The 1-D case is examined where the N robots move only along one dimension and they continuously measure the relative distance to each other. The upper bound is calculated by solving the continuous time Riccati equation for the covariance of the errors in the position estimates. The key element in our derivation is the separation of the covariance matrix into two sets of submatrices, those that converge to steady state values and those that capture the time dependence of the uncertainty increase during cooperative localization. Throughout the paper we assume that all robots move at the same time.

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The 1-D case is interesting by itself as it provides insights in the solution of the Riccati equation by examining the evolution of the covariance matrix until it reaches a steady state. The behaviour of the Riccati equation in 1-D also serves as a starting point for solving the more general 2-D case. Discussion of the 2-D case is out of the scope of this paper and can be found in [4].

In the following section we outline the main approaches to cooperative localization. In Section III we present the formulation of the multi-robot localization problem in 1D and study the effect of consecutive relative position updates on the structure of the Riccati equation which describes the time evolution of the uncertainty in the position estimates. In Section IV simulation results are presented that validate the derived analytical expressions for the rate of localization increase. Finally, Section V draws the conclusions from this analysis and suggests directions of future work.

II. Related Work

An example of a system designed for cooperative localization was first reported in [1]. A group of robots is divided into two teams in order to perform cooperative positioning. At each instant, one team is in motion while the other team remains stationary and acts as a landmark. Improvements over this system and optimum motion strategies are discussed in [5], [6] and [7]. Similarly, in [8], only one robot moves, while the rest of the team of small-sized robots forms an equilateral triangle of localization beacons in order to update their pose estimates. Another implementation of cooperative localization is described in [2] and [9]. In this approach a team of robots moves through the free space systematically mapping the environment. These approaches have the following limitations: (a) Only one robot (or team) is allowed to move at any given time, and (b) The two robots (or teams) must maintain visual (or sonar) contact at all times.

A Kalman filter-based implementation of a cooperative navigation schema is described in [10]. In this work the effect of the orientation uncertainty in both the state propagation and the relative position measurements is ignored resulting in a simplified distributed algorithm. In [3], [11] a Kalman filter pose estimator is presented for a group of simultaneously moving robots. The Kalman filter is decomposed into a number of smaller communicating filters, one for every robot, processing sensor data collected by its host robot. It has been shown that when every robot senses and communicates with its colleagues at all times, every member of the group has less uncertainty about its position than the robot with the best (single) localization results.

To the best of our knowledge there exist only two cases in the literature where uncertainty propagation has been considered in the context of cooperative localization. In [10] the improvement in localization accuracy is computed after only a *single* update step with respect to the previous values of uncertainty. In [12] the authors have explored the effect of different robot tracker sensing modalities on the effectiveness of cooperative localization. Statistical properties were derived from simulated results for groups of robots of increasing size N when only one robot moved at a time.

In the following sections we present the details of our approach for estimating the uncertainty propagation during cooperative localization in 1-D. Our initial formulation is based on the algorithm described in [11].

III. Cooperative Localization in 1-D

Consider the case of N robots moving randomly along one dimension (without loss of generality let the dimension be the $x - axis$). Each robot is equipped with odometric sensors that measure the velocity of the robot and a robot tracker sensor that measures the relative position between any two robots, and possesses no other sensing ability. A Kalman Filter estimator is employed to combine the two sensor measurements and provide an optimal position estimate for each robot together with a position uncertainty estimate. In the next subsections we provide an analytical expression for the position uncertainty as it grows over time.

A. Motion Model

For the case of a single robot moving in a 1-D environment, its motion is described (in discrete time) by the following equation

$$x_i(k+1) = x_i(k) + V_i(k)\delta t \tag{1}$$

If a sensor on board the robot measures its velocity then the estimate for this motion is given by

$$\hat{x}_i(k+1) = \hat{x}_i(k) + V_{mi}(k)\delta t \tag{2}$$

with

$$V_{mi}(k) = V_i(k) - w_{V_i}(k) \tag{3}$$

where w_{V_i} is a zero-mean white Gaussian process, the noise in the velocity measurement for robot i, with covariance $q = E\{w_{V_i}^2\}$. Since each robot carries its own proprioceptive sensor that measures the velocity of the vehicle, these measurements are independent, i.e. $E\{w_{V_i} w_{V_j}\} = 0$ for $i \neq j$. The error in the position estimate is given by:

$$\begin{aligned}
\widetilde{x}_i(k+1) &= x_i(k+1) - \hat{x}_i(k+1) \\
&= x_i(k) - \hat{x}_i(k) + (V_i(k) - V_{mi}(k))\,\delta t \\
&= \widetilde{x}_i(k) + w_{V_i}(k)\delta t \\
&= [1]\widetilde{x}_i(k) + [\delta t] w_{V_i}(k) \\
&= F_i \widetilde{x}_i(k) + G_i w_{V_i}(k)
\end{aligned}$$

Expanding the previous expression to the case of N robots, we have

$$\begin{bmatrix} \widetilde{x}_1(k+1) \\ \vdots \\ \widetilde{x}_N(k+1) \end{bmatrix} = \begin{bmatrix} \widetilde{x}_1(k) \\ \vdots \\ \widetilde{x}_N(k) \end{bmatrix} + \delta t \begin{bmatrix} w_{V_1}(k) \\ \vdots \\ w_{V_N}(k) \end{bmatrix} \Leftrightarrow$$

$$\begin{aligned}
\widetilde{X}(k+1) &= \mathbf{\Phi}\widetilde{X}(k) + \mathbf{G}W(k) \Leftrightarrow \\
\widetilde{X}(k+1) &= \mathbf{I}\widetilde{X}(k) + \delta t \mathbf{I} W(k)
\end{aligned} \tag{4}$$

The same equation in continuous time is given by

$$\begin{aligned}
\dot{\widetilde{X}}(t) &= \mathbf{F}\widetilde{X}(t) + \mathbf{G_c} W(t) \Leftrightarrow \\
\dot{\widetilde{X}}(t) &= \mathbf{0}\widetilde{X}(t) + \mathbf{I} W(t)
\end{aligned} \tag{5}$$

B. Measurement Model

When one of the robots in the group detects another robot and measures their relative position, the measurement equation is:

$$z_{ij}(k+1) = x_j(k+1) - x_i(k+1) + n_{ij}(k+1) \tag{6}$$

where $n_{ij}(k+1)$ is the noise associated with the relative position measurement. This noise is assumed to be a zero-mean white Gaussian process with known variance $E\{n_{ij}^2(k+1)\} = r$. Note also that the measurements n_{ij} are independent, i.e. $E\{n_{ij} n_{kl}\} = 0$ for $ij \neq kl$. The estimated measurement would be

$$\hat{z}_{ij}(k+1) = \hat{x}_j(k+1) - \hat{x}_i(k+1) \tag{7}$$

and the residual (error) for this measurement is

$$\begin{aligned}
\widetilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1) \\
&= x_j(k+1) - x_i(k+1) - \\
&\quad (\hat{x}_j(k+1) - \hat{x}_i(k+1)) + n_{ij}(k+1) \\
&= \widetilde{x}_j(k+1) - \widetilde{x}_i(k+1) + n_{ij}(k+1) \\
&= \begin{bmatrix} 0 \ldots -1 \ldots 1 \ldots 0 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(k+1) \\ \vdots \\ \widetilde{x}_i(k+1) \\ \vdots \\ \widetilde{x}_j(k+1) \\ \vdots \\ \widetilde{x}_N(k+1) \end{bmatrix} + n_{ij}(k+1) \\
&= H_{ij}\widetilde{X}(k+1) + n_{ij}(k+1)
\end{aligned} \tag{8}$$

If each of the robots measures its relative position with respect to all robots in the group then the resulting $N(N-1)$ measurement error equations can be written in a compact form as:

$$\begin{bmatrix} \widetilde{z}_{12}(k') \\ \vdots \\ \widetilde{z}_{1N}(k') \\ \vdots \\ \widetilde{z}_{N1}(k') \\ \vdots \\ \widetilde{z}_{N(N-1)}(k') \end{bmatrix} = \mathbf{H} \begin{bmatrix} \widetilde{x}_1(k') \\ \vdots \\ \widetilde{x}_N(k') \end{bmatrix} + \begin{bmatrix} n_{12}(k') \\ \vdots \\ n_{1N}(k') \\ \vdots \\ n_{N1}(k') \\ \vdots \\ n_{N(N-1)}(k') \end{bmatrix} \Leftrightarrow$$

$$\widetilde{Z}(k') = \mathbf{H}\widetilde{X}(k') + N(k') \tag{9}$$

where $k' = k+1$ and

$$\mathbf{H} = \begin{bmatrix} -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{bmatrix}$$

By employing the previous expression, the covariance of the residual (measurement error) is given by:

$$S = E\{\widetilde{Z}\widetilde{Z}^T\} = \mathbf{H}\mathbf{P}(k+1/k)\mathbf{H}^\mathbf{T} + r\mathbf{I} \tag{10}$$

where $\mathbf{P}(k+1/k)$ is the covariance for the position estimate at time $k+1$ using measurements up to time k and $\mathbf{I}$ is the $(N(N-1))^2$ identity matrix. Here we have assumed that all relative position measurements between the robots have the same level of accuracy.

C. Riccati Equation

Since none of the robots in the group receives absolute positioning information, the system of the N robots is not observable and the covariance matrix will increase without bound. In order to determine the rate of increase of the covariance matrix we first write the discrete time propagation and update equations for it:

$$\mathbf{P}(k+1/k) = \mathbf{\Phi}\mathbf{P}(k/k)\mathbf{\Phi}^\mathbf{T} + \mathbf{G}\mathbf{Q}\mathbf{G}^\mathbf{T} = \mathbf{P}(k/k) + \mathbf{Q_d} \tag{11}$$

where we substituted $\mathbf{\Phi} = \mathbf{I}$ from Eq. (4). The covariance update equation in its inverse form is given by:

$$\begin{aligned} \mathbf{P^{-1}}(k+1/k+1) &= \mathbf{P^{-1}}(k+1/k) + \mathbf{H^T}\mathbf{R^{-1}}\mathbf{H} \\ &= (\mathbf{P}(k/k) + \mathbf{Q_d})^{-1} + \mathbf{H^T}\mathbf{R^{-1}}\mathbf{H} \end{aligned} \tag{12}$$

or, in order to simplify the notation

$$\mathbf{P}_{k+1}^{-1} = (\mathbf{P}_k + q_d\mathbf{I})^{-1} + \frac{1}{r}\mathbf{H^T}\mathbf{H} \tag{13}$$

where we substituted $\mathbf{R} = r\mathbf{I}$ and $\mathbf{Q_d} = q_d\mathbf{I} = q\delta t^2\mathbf{I}$. This equation computes the inverse covariance of the system of N robots after two consecutive steps of propagation and update. Matrix $\mathbf{H^T}\mathbf{H}$ is constant and it only depends on the number of robots N. It can be computed as:

$$\begin{aligned} \mathbf{H^T}\mathbf{H} &= \begin{bmatrix} 2N-2 & -2 & \dots & -2 & -2 \\ -2 & 2N-2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2N-2 & -2 \\ -2 & -2 & \dots & -2 & 2N-2 \end{bmatrix} \\ &= 2N\mathbf{I} - 2\mathbf{1} \end{aligned} \tag{14}$$

where

$$\mathbf{1} = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{1^2} = N\mathbf{1} \tag{15}$$

Assume that at step $k = 0$ all robots know their position with the same level of accuracy, that is $\mathbf{P}_0 = p_0\mathbf{I}$. The covariance matrix $\mathbf{P}$ is symmetric with equal non-diagonal terms (here all zero) and also equal diagonal terms (here all p_0). We will prove that after any number of steps the covariance matrix sustains this structure. By substituting $\mathbf{P}_0 = p_0\mathbf{I}$ in Eq. (13) we have

$$\begin{aligned} \mathbf{P}_1^{-1} &= (\mathbf{P}_0 + q_d\mathbf{I})^{-1} + \frac{1}{r}\mathbf{H^T}\mathbf{H} \\ &= \frac{1}{q_d+p_0}\mathbf{I} + \frac{2N}{r}\mathbf{I} - \frac{2}{r}\mathbf{1} \\ &= \left(\frac{1}{q_d+p_0} + \frac{2N}{r}\right)\mathbf{I} + \left(-\frac{2}{r}\right)\mathbf{1} \\ &= v_1'\mathbf{I} + u_1'\mathbf{1} \end{aligned} \tag{16}$$

By employing the relations in the Appendix the covariance matrix can be computed as:

$$\mathbf{P}_1 = \frac{1}{v_1'}\mathbf{I} - \frac{u_1'}{v_1'(v_1' + Nu_1')}\mathbf{1} = v_1\mathbf{I} + u_1\mathbf{1} \tag{17}$$

Note again that both the diagonal and non-diagonal elements of this matrix are equal between themselves (i.e. $P_{ii} = P_{jj},\ \forall i,j$, $P_{ij} = P_{kl},\ \forall\ i \neq j,\ k \neq l$). Assume that after a certain number of propagation and update steps, at step $k = m$ the covariance matrix has still equal diagonal and equal non-diagonal elements. That is

$$\mathbf{P}_m = v_m\mathbf{I} + u_m\mathbf{1} \tag{18}$$

We will prove that the covariance matrix $\mathbf{P}_{m+1}$ also has equal diagonal and non-diagonal elements. By substituting in Eq. (13) we have:

$$\begin{aligned} \mathbf{P}_{m+1}^{-1} &= (\mathbf{P}_m + q_d\mathbf{I})^{-1} + \frac{1}{r}\mathbf{H^T}\mathbf{H} \\ &= ((v_m+q_d)\mathbf{I} + u_m\mathbf{1})^{-1} + \frac{1}{r}\mathbf{H^T}\mathbf{H} \\ &= \left(\frac{1}{v_m+q_d} + \frac{2N}{r}\right)\mathbf{I} + \\ &\quad \left(\frac{u_m}{(v_m+q_d)(v_m+q_d+Nu_m)} - \frac{2}{r}\right)\mathbf{1} \\ &= v_{m+1}'\mathbf{I} + u_{m+1}'\mathbf{1} \end{aligned} \tag{19}$$

By employing the relations in the Appendix the covariance matrix can be computed as:

$$\begin{aligned} \mathbf{P}_{m+1} &= \frac{1}{v_{m+1}'}\mathbf{I} - \frac{u_{m+1}'}{v_{m+1}'(v_{m+1}' + Nu_{m+1}')}\mathbf{1} \\ &= v_{m+1}\mathbf{I} + u_{m+1}\mathbf{1} \end{aligned}$$

We have proven the following:

Lemma 1: The covariance matrix for a group of N robots with the same level of uncertainty for their proprioceptive and exteroceptive measurements when performing cooperative localization is a matrix with equal diagonal and equal non-diagonal terms.

A direct result of the previous lemma is the following:

Corollary 2: A group of N robots with the same level of uncertainty for their proprioceptive and exteroceptive measurements experience when performing cooperative localization the same level of positioning uncertainty and they share the same amount of information.

The amount of information shared by two robots is captured in the cross-correlation terms (non-diagonal) of the covariance matrix. At this point we employ Lemma 1 to derive an analytical expression for the rate of increase in the localization uncertainty for the group of robots.

Lemma 3: For a group of $N \geq 2$ robots with the same level of uncertainty for their proprioceptive q and exteroceptive r measurements, when they perform cooperative localization their positioning uncertainty and cross-correlation terms grow linearly with respect to time. i.e.

$$\mathbf{P}_{ij}(t) = \begin{cases} \frac{q}{N}t + \frac{1}{N}\left(p(0) + (N-1)\sqrt{\frac{qr}{2N}}\right) & i = j \\ \frac{q}{N}t + \frac{1}{N}\left(p(0) - \sqrt{\frac{qr}{2N}}\right) & i \neq j \end{cases} \tag{20}$$

Here we use the continuous time Riccati equation for the propagation and update of the covariance matrix, that is:

$$\dot{\mathbf{P}} = \mathbf{FP} + \mathbf{PF^T} + \mathbf{G_c Q G_c^T} - \mathbf{PH^T R^{-1} HP} \tag{21}$$

where $\mathbf{F} = \mathbf{0}$, $\mathbf{G_c} = \mathbf{I}$, $\mathbf{Q} = q\mathbf{I}$, $\mathbf{R} = r\mathbf{I}$, $\mathbf{H^T H} = (2N)\mathbf{I} - 2\mathbf{1}$ and $\mathbf{P} = v\mathbf{I} + u\mathbf{1}$. Substituting in the previous equation we have

$$\begin{aligned} \dot{\mathbf{P}} &= q\mathbf{I} - \frac{1}{r}\mathbf{PH^T HP} \\ &= q\mathbf{I} - \frac{1}{r}\left(v\mathbf{I} + u\mathbf{1}\right)\left(2N\mathbf{I} - 2\mathbf{1}\right)\left(v\mathbf{I} + u\mathbf{1}\right) \Leftrightarrow \\ \dot{v}\mathbf{I} + \dot{u}\mathbf{1} &= \left(q - \frac{2Nv^2}{r}\right)\mathbf{I} + \frac{2v^2}{r}\mathbf{1} \end{aligned}$$

The last relation can be divided into two differential equations. The first one is:

$$\dot{v} = -\frac{2N}{r}v^2 + 0v + q \tag{22}$$

It can be shown [4] that the solution of this Riccati equation in steady state is given by

$$v(t) = \sqrt{\frac{qr}{2N}} \tag{23}$$

where v_c is a constant. The second differential equation is

$$\dot{u} = \frac{2v^2}{r} \tag{24}$$

By rewritting Eq. (22) as

$$v^2(t) = \frac{r}{2N}(q - \dot{v}(t)) \tag{25}$$

and substituting in Eq. (24), for $u(0) = 0$ we have

$$\begin{aligned} u(t) &= \int_0^t \frac{2v^2}{r}dt' \\ &= \frac{1}{N}\int_0^t (q - \dot{v}(t'))dt' \\ &= \frac{1}{N}qt - \frac{1}{N}(v(t) - p(0)) \end{aligned} \tag{26}$$

where we have employed the initial condition $v(0) = p(0)$. The covariance matrix can now be written as

$$\mathbf{P}(t) = v(t)\mathbf{I} + \left[\frac{q}{N}t - \frac{1}{N}\left(v(t) - p(0)\right)\right]\mathbf{1} \tag{27}$$

and after sufficient time this would be

$$\mathbf{P}(t) = \sqrt{\frac{qr}{2N}}\mathbf{I} + \left[\frac{q}{N}t - \frac{1}{N}\left(\sqrt{\frac{qr}{2N}} - p(0)\right)\right]\mathbf{1} \tag{28}$$

where we employed the results from Eq. (23). From this last equation it is evident that after sufficient time, the positioning uncertainty for this group of robots increases linearly with time, i.e.

$$\dot{\mathbf{P}}(t) = \frac{q}{N}\mathbf{1} \tag{29}$$

From this last equation the following are true:

- The rate of increase is proportional to the odometric uncertainty q of each robot and inversely proportional to the number N of robots.
- The rate of uncertainty increase in steady state does *not* depend on the accuracy (determined by their covariance r) of the relative position measurements.
- The time for the system to reach steady state is determined by the time constant of the system

$$\tau = \frac{1}{\sqrt{D}} = \frac{1}{2}\sqrt{\frac{r}{2Nq}} \tag{30}$$

which depends on the odometric accuracy q of the robots, their number N and the accuracy r of the relative position measurements. Inaccurate relative position measurements will delay the system reaching steady state. On the other hand large teams of robots will quickly reach steady state. Finally, robots with very precise odometric information (small q) will depend less on the relative position measurements and more on their own odometry for a longer period of time.

Up to this point, we have assumed that all robots have the same odometric uncertainty $q_i = q,\ \forall i$ and relative position measurement uncertainty $r_i = r,\ \forall i$. This was done in order to facilitate the previous derivations and thus gain insight in the structure of the cooperative localization problem. Nevertheless, Eq. (28) can be used to determine the bounds for the expected uncertainty growth for $q = \max(q_i)$, $r = \max(r_i)$. That is

$$\mathbf{P}(t) \leq \sqrt{\frac{qr}{2N}}\mathbf{I} + \left[\frac{q}{N}t - \frac{1}{N}\left(\sqrt{\frac{qr}{2N}} - p(0)\right)\right]\mathbf{1} \tag{31}$$

IV. Simulation Results

We performed a series of experiments in simulation to verify the performance of cooperative localization and the effect of the different parameters (e.g. number of robots, initial uncertainty and sensor uncertainty) on the increase in the uncertainty of the system. The results obtained validate the theoretical analysis presented in the previous section.

Figure 1 presents the actual (solid line) and theoretical (dash-dotted line) covariance values over time for different numbers of robots. As expected, after the robots reach steady state the values are very similar. It is worth noting that as the number of robots increases the rate of uncertainty increase reduces following a law of diminishing returns: the larger the group the less value each additional robot adds to the localization. The effect of the initial position uncertainty $P(0)$ is presented in Figure 2

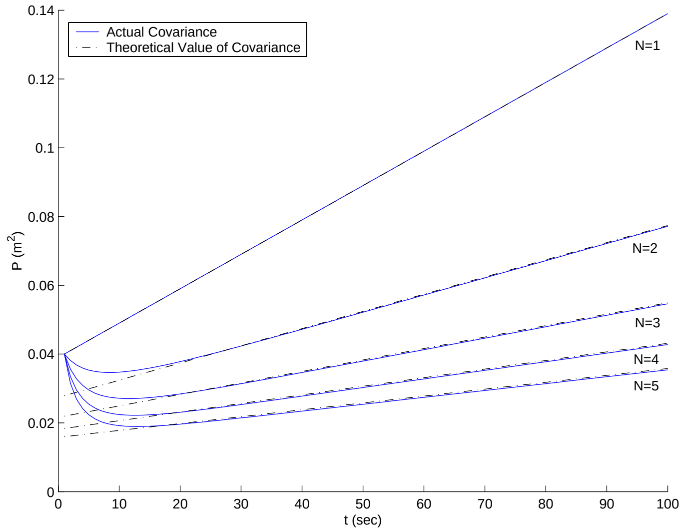

Fig. 1. Actual and theoretical covariance values for robot groups of different size N.

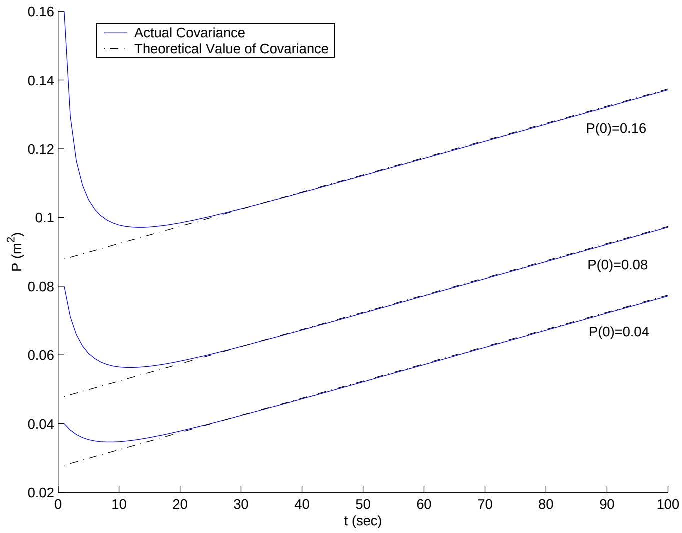

Fig. 2. Actual and theoretical covariance values for a group of $N = 2$ robots for different values of initial uncertainty $P(0)$.

for a group of two robots; again, after an initial period the theoretical and the simulated uncertainty agree. Clearly, as predicted by the theoretical analysis, the initial position uncertainty $P(0)$ does not affect the rate of uncertainty increase.

The final significant result is illustrated in Figure 3: the quality of the robot tracker measurements does not influence the rate of position uncertainty increase. In other words, in the case where all the robots move simultaneously, cooperative localization enables the uncertainty of each robots position to decrease inversely proportional to number of the robots in the group but the accuracy of the robot tracker adds only an initial (constant) improvement. Figure 3 shows for a group of two robots that the simulated uncertainty matches the theoretically calculated values.

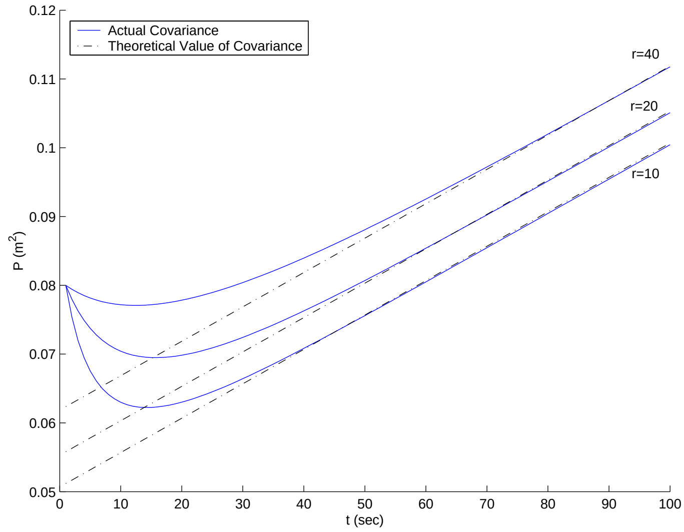

Fig. 3. Actual and theoretical covariance values for a group of $N = 2$ robots for different values of relative position measurement uncertainty r.

V. Conclusions

This paper presented a theoretical analysis for the propagation of position uncertainty for a team of mobile robots moving in one dimension. The most challenging case of localization was considered based only on inter-robot observations (cooperative localization) and dead reckoning estimates. Furthermore, all robots moved simultaneously, in contrast to previous work where often the robots take turns to move and to act as landmarks. An analytical formula was derived that expresses the upper bound of the uncertainty accumulation as a function of time and the noise characteristics of the robot sensors.

The analysis presented here is a required first step for the treatment of the more general case of N robots moving simultaneously on flat terrain. The analytical solution presented here generalizes to the 2-D case under certain assumptions [4] as an upper bound of the accumulation of position uncertainty.

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APPENDIX

Special Case of Matrix Inversion: The inverse of the $N \times N$ symmetric matrix

$$T = \begin{bmatrix} \tau+v & v & \dots & v & v \\ v & \tau+v & \dots & v & v \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v & v & \dots & \tau+v & v \\ v & v & \dots & v & \tau+v \end{bmatrix} = \tau\mathbf{I} + v\mathbf{1} \tag{32}$$

is

$$\begin{aligned} T^{-1} = & \frac{1}{\tau(\tau+Nv)} \\ & \begin{bmatrix} \tau+(N-1)v & -v & \dots & -v & -v \\ -v & \tau+(N-1)v & \dots & -v & -v \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -v & -v & \dots & \tau+(N-1)v & -v \\ -v & -v & \dots & -v & \tau+(N-1)v \end{bmatrix} \\ = & \quad \frac{1}{\tau}\mathbf{I} - \frac{v}{\tau(\tau+Nv)}\mathbf{1} \end{aligned} \tag{33}$$

This can be shown by computing the product TT^{-1} and substituting from the previous expressions.