On the Control Synthesis for Target Problems in Continuous and Hybrid Systems using Level Set Methods

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***Abstract*— The focus of this paper is on the numerical solution of target control problems in continuous systems using level set methods. Such control problems appear naturally in hybrid control synthesis when specifications with respect to reachable states of the system are considered. To account for the existence of disturbance inputs the problem is studied as a pursuit-evasion differential game. The boundary of the reachable set, i.e. the set of states from which the problem is feasible, is characterized as the zero level set of the viscosity solution of a Hamilton-Jacobi PDE. Our contribution is the detailed presentation of the procedure for the computation of the control and worst-case disturbance policies together with the reachable set using level set methods.**

I. Introduction

The computation of the set of states from which there exists a control policy such that the controlled trajectories of a continuous system reach a given target set in the presence of set-valued disturbance is paramount for the extraction of controllers in safety-critical hybrid systems. Typically, such controllers are obtained through dynamic programming iterations over the reachable states of the system. Moreover, the computation of reachable sets is of current research interest in the differential game, optimal and robust control communities.

Recently, the developments in real-time automation have promoted new interest in the reachability problem and a number of methodologies have been proposed within the hybrid system community. The bulk of the published results concern the computation of reachable sets for continuous systems with fixed or absent control and disturbance inputs for the needs of computer-aided verification. However, significant results on the computation of reachable sets in control synthesis problems have also appeared [1], [3], [7], [8], [9], [13], [14], [15]. In [13], [14] a methodology based on quantifier elimination is proposed for the symbolic computation of reachable sets in a class of linear and triangular pursuit-evasion differential games. In [7], it shown that the reachable set of a controlled linear system can be internally approximated by a series of ellipsoids, whose parameters are given in analytical form. A static Hamilton-Jacobi partial differential equation (PDE) was considered in [1], where the boundary of the reachable set is encoded by the minimum time to reach function. The same Hamilton-Jacobi PDE was also considered in [3] and a numerical scheme based on viability theory was employed to yield an over-approximation of the reachable set. In [8], [9], [15] the boundary of the reachable set of a pursuit-evasion differential game is characterized by the zero level set of the viscosity solution to a different, time dependent Hamilton-Jacobi PDE. Level set methods [12] are employed for the numerical computation of the solution yielding results with higher accuracy than those based on the static formulation.

Although level set methods have been tested extensively in computing reachable sets in [8], [9], due to the nature of the considered safety specifications therein, the extraction of the least restrictive control policy was a trivial problem. More specifically, the maximal invariant set was computed first and then the control policy was obtained by logic elaboration in a post-processing step. However, this is not the case with target control or eventuality synthesis problems in hybrid systems [16], [17], where the continuous control policy must be computed directly from the Hamilton-Jacobi PDE simultaneously with the maximal initial set and not in a post-processing step. Hence, the focus of this paper is the extraction of continuous feedback policies in target control problems and pursuit-evasion games using level set methods. Our main contribution is the detailed presentation of the procedure and the application of the results in two benchmark pursuit-evasion differential games, namely the double integrator and the homicidal chauffeur.

The format of the paper is as follows: In section 2 we present the Hamilton-Jacobi PDE, the solution of which specifies the reachable set together with the control and worst-case disturbance policies. In section 3 we briefly review the numerical scheme of the level set method in interface propagation problems. In section 4 we present the control synthesis procedure. In section 5 we illustrate the application of the methodology in the systems of the double integrator and the homicidal chauffeur. Conclusions and directions for further research are given in section 6.

II. Reachable Sets of Pursuit-Evasion Differential Games

A. Modelling and Design Objective

The key assumption in order to derive controllers for safety-critical systems is that the goal of the disturbance is directly orthogonal to that of the controller. This entails that the controller must protect against worst case uncertainty in the actions of the disturbance. Therefore, the natural framework to study such synthesis problems is that of differential game theory [2]. Unlike stochastic models, designs based on game models can guarantee specification satisfaction under worst case disturbance.

Hence, the systems considered in this paper are modeled by the time-invariant ordinary differential equation:

$$\dot{x} = f(x, u, d), \quad x(0) = x_0 \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is the state of the differential game, $u \in U \subset \mathbb{R}^{m_u}$ is the input of the first player and $d \in D \subset \mathbb{R}^{m_d}$ is the input of the second player. U and D are assumed closed. In the literature, the first player is called the *pursuer* while the second is called the *evader*. However, in order to be intuitive, we refer to the two players as the control and the disturbance respectively. The spaces of the control and disturbance signals are denoted by the spaces of piecewise continuous functions:

$$\mathcal{U} = \{u(\cdot) \in PC \mid u(t) \in U, \ \forall t \in \mathbb{R}\} \tag{2}$$

and

$$\mathcal{D} = \{d(\cdot) \in PC \mid d(t) \in D, \ \forall t \in \mathbb{R}\} \tag{3}$$

With specified input signals $u(\cdot) \in \mathcal{U}$ and $d(\cdot) \in \mathcal{D}$, the vector field f is assumed to be Lipschitz continuous over X and continuous in u and d, which entails that there exists a unique trajectory solving (1).

The final time T of the game is defined by:

$$T = \min\{t \in \mathbb{R}^+ \mid x(t) \in F\} \tag{4}$$

where $F \subseteq X$ is the target set of the pursuit-evasion game. F is closed and can be represented as the zero sub-level set of a differentiable function $l : \mathbb{R}^n \to \mathbb{R}$:

$$F = \{x \in X \mid l(x) \leq 0\} \tag{5}$$

The design objective is to compute the maximal set of initial states W^* and the corresponding control policy $u^*(\cdot)$ such that all the trajectories of the differential game (1) reach the target set F in finite time for any $d(\cdot) \in \mathcal{D}$.

B. The Hamilton-Jacobi PDE

Let us consider the system (1) over the time interval $[t, 0]$, where $t < 0$, with initial condition $x(t) = x$ and target set F defined by (5). Although the considered problem is obviously not an optimization problem, an optimization methodology is employed to obtain a feasible solution. The Hamilton-Jacobi PDE whose solution specifies the reachable set W^* and the control and disturbance signals $u^*(\cdot)$ and $d^*(\cdot)$ is defined with respect to the cost function $J : X \times \mathcal{U} \times \mathcal{D} \times \mathbb{R}^- \to \mathbb{R}$ with:

$$J(x, u(\cdot), d(\cdot), t) = \inf_{s \in [t,0]} l(x(s)) \tag{6}$$

J represents the cost of a trajectory $x(\cdot)$ which initiates from state x at initial time $t < 0$, evolves according to (1) with input signals $u(\cdot)$ and $d(\cdot)$ and terminates at the final state $x(0)$ with cost $\inf_{s \in [t,0]} l(x(s))$. Note that there is no running cost and the cost function depends only on whether the trajectory $x(\cdot)$ enters the target set or not. If there exists an $s \in [t, 0]$ such that $x(s)$ is in F, i.e. $l(x(s)) \leq 0$, then $J(x, u(\cdot), d(\cdot), t) \leq 0$, the controller wins the game and the initial state x is said to be a winning state, i.e. $x \in W^*$. Otherwise, the disturbance wins the game.

In a two-player differential setting it is important to address what information the players know about each other's decisions. To further increase the robustness of our results, the informational advantage should be given to the disturbance, which is assumed to employ a *non-anticipating* strategy. This means that while the control chooses $u(s)$ based on the current state $x(s)$ of the system for all $s \in [t, 0]$, the disturbance chooses $d(s)$ using not only the feedback information but also the current control input $u(s)$. Therefore, according to the standard game-theoretic convention, the value function V of the game is defined as:

$$V(x, t) = \inf_{u(\cdot) \in \mathcal{U}} \sup_{d(\cdot) \in \mathcal{D}} J(x, u(\cdot), d(\cdot), t) \tag{7}$$

whereas for the corresponding optimal control and disturbance signals it holds:

$$u^*(\cdot) = \arg \inf_{u(\cdot) \in \mathcal{U}} \sup_{d(\cdot) \in \mathcal{D}} J(x, u(\cdot), d(\cdot), t) \tag{8}$$

$$d^*(\cdot) = \arg \sup_{d(\cdot) \in \mathcal{D}} J(x, u^*(\cdot), d(\cdot), t) \tag{9}$$

By making the standard assumptions that the value function $V(x, t)$ exists and is continuously differentiable[1], application of the Bellman's principle of optimality, leads to the following Hamilton-Jacobi equation:

$$\frac{\partial V(x,t)}{\partial t} + \min[0, H\left(\frac{\partial V(x,t)}{\partial x}, x\right)] = 0 \qquad \text{for all}\, x \in X \tag{10}$$

with Hamiltonian defined by:

$$H\left(\frac{\partial V(x,t)}{\partial x}, x\right) = \min_{u \in U} \max_{d \in D} \frac{\partial V(x,t)}{\partial x} f(x, u, d) \tag{11}$$

and with boundary condition:

$$V(x, 0) = l(x) \tag{12}$$

The definition of the value function V allows the efficient characterization of the reachable set W^*. At this point let us denote by $W(t)$ the set of states $x \in X$ which can reach the target set in at most $|t|$ time units. We refer to this set as the *reachable set at time t*. In [9] the authors formally proved the following proposition, which characterizes $W(t)$ by means of the value function $V(x, t)$:

Proposition 1: Let $V : X \times [T, 0] \to \mathbb{R}$, with $T < 0$, be the viscosity solution of the terminal value Hamilton-Jacobi equation (10). Then, the zero sublevel set of V describes $W(t)$ as:

$$W(t) = \{x \in X \mid V(x, t) \leq 0\} \tag{13}$$

for all $t \in [T, 0]$.

Thus, one can obtain the reachable set W^* at the limit of $W(t)$ as $T \to -\infty$. In practice, however, we do not need to compute $W(t)$ at the limit to get W^*. Besides, we assume $W^* = \{x \in X \mid V(x, t) \leq 0\}$, with $t \in [T, 0]$, for a "large enough" T. In this case, since time has been essentially abstracted away, the value function V looses dependence on time, e.g. $V = V(x)$. Finally, for the optimal control and disturbance feedback inputs it holds:

$$u^*(x) = \arg \min_{u \in U} \max_{d \in D} \frac{\partial V(x)}{\partial x} f(x, u, d) \tag{14}$$

$$d^*(x) = \arg \max_{d \in D} \frac{\partial V(x)}{\partial x} f(x, u^*, d) \tag{15}$$

Note that in this paper the target control problem is considered without state constraints. However, from the analysis in [16] it becomes clear that the solution to the constrained problem depends on our ability to solve the Hamilton-Jacobi PDE (10).

III. Level Set Methods

Level set methods [12] is a family of numerical tools for the solution of Hamilton-Jacobi PDEs of the form:

$$\begin{aligned} \phi_t + H(\nabla\phi, x) &= 0 \\ \phi(x, 0) &= \phi_0(x) \end{aligned} \tag{16}$$

where:

$$H(\nabla\phi, x) = f(x)\nabla\phi \tag{17}$$

Such PDEs appear in interface propagation problems, where the level set function $\phi_0(x)$ encodes the initial position of the front at $t = 0$ and $\phi(x, t)$, with $t \geq 0$, captures its evolution under the velocity field $f(x)$. Points $x \in X$ with $\phi(x, t) \leq 0$ belong to the propagated interface at time t, while points with $\phi(x, t) > 0$ do not. It is clear that the boundary of the propagated interface at time t is given indirectly by the zero level set of $\phi(x, t)$. Also, for numerical stability $\phi(x, t)$ is assumed to be the signed distance function to the interface, i.e. $|\nabla\phi(x, t)| = 1$.

It is well known that the solutions of (16) develop discontinuous derivatives even if the initial conditions are smooth. However, level set methods account for these *shocks* in a natural way and they produce the unique viscosity solution with high accuracy.

[1]The differentiability assumption is used for the derivation of (10) and is relaxed afterwards.

A. The Numerical Scheme

Here we briefly present the basic numerical scheme for the solution of (16), which forms the basis for the numerical solution of the original Hamilton-Jacobi PDE (10). In [4] it was shown that numerical consistent monotone schemes of the form

$$\phi^{n+1} = \phi^n - \Delta t\, \hat{H}(\nabla^+ \phi^n, \nabla^- \phi^n, x) \tag{18}$$

converge to the viscosity solution, where $\hat{H}$ is a consistent numerical approximation to the Hamiltonian H: $\hat{H}(\nabla\phi, \nabla\phi, x) = H(\nabla\phi, x)$. A collection of consistent approximations $\hat{H}$ is given in [10]. For our purposes, the Lax Friedrichs (LF) numerical approximation is employed

$$\begin{aligned}\hat{H}^{LF}(\nabla^+\phi, \nabla^-\phi, x) = H\left(\frac{\nabla^+\phi + \nabla^-\phi}{2}, x\right) \\ - \frac{1}{2}\,\alpha^T(\nabla^\pm\phi)(\nabla^+\phi - \nabla^-\phi)\end{aligned} \tag{19}$$

where the vector $\alpha(\nabla^\pm\phi)$ is a numerical dissipation term used to dump out spurious oscillations in the solution. $\nabla^+\phi$ and $\nabla^-\phi$ denote the right and left approximation to the gradient $\nabla\phi$ depending on which neighboring grid points are used in the finite difference scheme. For the computation of $\nabla^+\phi$ and $\nabla^-\phi$ a basic first order and a highly accurate fifth order weighted essentially non–oscillatory (WENO) approximation [6] are used. The WENO scheme performs a fifth order accurate approximation of the gradient in the smooth parts of the solution and it turns to first order near the discontinuities so that spurious numerical oscillations are avoided.

The temporal derivative is approximated using the method of lines which leads to the ODE:

$$\frac{d\phi}{dt} = -\widehat{H}^{LF}(\phi) \tag{20}$$

The solution of (20) is obtained by applying either a basic first order scheme (Euler approximation), which is matched with the first order approximation of the spatial derivative to obtain a (1,1) scheme for the solution of (16), or a second order Total Variation Diminishing (TVD) Runge-Kutta scheme [5], which is matched with the WENO approximation to obtain a (5,2) scheme. Finally, in order to avoid spurious oscillations, some extra care should be taken so that the grid spacing Δx and the time step Δt satisfy the CFL condition $f_{max}\Delta t \leq \Delta x$, with $f_{max} = \max_x\{\|\bar{f}(x)\|_\infty\}$.

Note that in most cases it is impossible to maintain the level set function as a signed distance function to the evolving interface. For numerical reasons we need to resurrect the level set function to be close to the distance function from time to time. This is the so called distance *reinitialization* of the level set function. In our computations reinitialization is performed by solving the following Hamilton-Jacobi PDE [11] to steady state

$$\begin{aligned} d_\tau + s(d)\,(|\nabla d| - 1) = 0 \\ d(x,0) = d_0(x) = \phi(x,t) \end{aligned} \tag{21}$$

where:

$$s(d) = \frac{d}{\sqrt{d^2 + |\nabla d|^2\,\Delta x}} \tag{22}$$

is a smooth approximation to the sign function. The distorted level set function $\phi(x,t)$ is used as the initial condition of (21), which iterates over the auxiliary variable τ. As long as (21) iterates the real time t is frozen. When $|\nabla d| \simeq 1$ then we substitute $\phi(x,t) = d$, which means that $\phi(x,t)$ has become the signed distance function again.

Although only first order (1,1) schemes of the form (18) have been formally proven convergent to the viscosity solution, the (5,2) scheme has been experimentally proven to be a very accurate scheme for a variety of problems [6], [8]. Due to the generality and the complexity of the numerical scheme, a priori error analysis does not exist. Besides, the error is quantified numerically by pointwise calculations in a post-processing step.

IV. Control Synthesis

At this point, before we present the adaptation of the level set method to obtain the numerical solution of (10), let us focus on the vector field f. Our main result is based on the assumption that f is affine in $u \in U$ and $d \in D$, where $U = [U_1, U_2]$ and $D = [D_1, D_2]$ are compact and convex sets, i.e.:

$$f(x,u,d) = f_1(x) + f_2(x)u + f_3(x)d \tag{23}$$

and, consequently, the optimal control u^* and disturbance d^* are of *bang-bang* type. Indeed, if $f(x,u,d) = f_1(x) + f_2(x)u + f_3(x)d$ then:

$$\begin{aligned} &H\left(\frac{\partial V(x,t)}{\partial x}, x\right) = \\ &\min_{u\in U}\max_{d\in D}\left[\frac{\partial V(x,t)}{\partial x}f_1(x) + \frac{\partial V(x,t)}{\partial x}f_2(x)u + \frac{\partial V(x,t)}{\partial x}f_3(x)d\right] \end{aligned} \tag{24}$$

which yields:

$$u^*(x,t) = \begin{cases} U_1 & \text{if } \frac{\partial V(x,t)}{\partial x}f_2(x) > 0 \\ [U_1, U_2] & \text{if } \frac{\partial V(x,t)}{\partial x}f_2(x) = 0 \\ U_2 & \text{if } \frac{\partial V(x,t)}{\partial x}f_2(x) < 0 \end{cases} \tag{25}$$

and:

$$d^*(x,t) = \begin{cases} D_1 & \text{if } \frac{\partial V(x,t)}{\partial x}f_3(x) < 0 \\ [D_1, D_2] & \text{if } \frac{\partial V(x,t)}{\partial x}f_3(x) = 0 \\ D_2 & \text{if } \frac{\partial V(x,t)}{\partial x}f_3(x) > 0 \end{cases} \tag{26}$$

That is, u^* and d^* switch between their extreme values whenever $\frac{\partial V(x,t)}{\partial x}f_2(x)$ and $\frac{\partial V(x,t)}{\partial x}f_3(x)$, respectively, change sign. Note that at the moment the sign changes the optimal control and disturbance inputs are unspecified, which results in loss of differentiability of $V(x,t)$. Also, note that:

$$\begin{aligned} &\min_{u\in U}\max_{d\in D}\left[\frac{\partial V(x,t)}{\partial x}f_1(x) + \frac{\partial V(x,t)}{\partial x}f_2(x)u + \frac{\partial V(x,t)}{\partial x}f_3(x)d\right] = \\ &\max_{d\in D}\min_{u\in U}\left[\frac{\partial V(x,t)}{\partial x}f_1(x) + \frac{\partial V(x,t)}{\partial x}f_2(x)u + \frac{\partial V(x,t)}{\partial x}f_3(x)d\right] \end{aligned} \tag{27}$$

which means that in smooth areas the order the optimal control and disturbance inputs are computed does not affect the solution.

For simplicity of exposition, we further assume that U_1, U_2 and D_1, D_2 are singletons, which implies that $u(\cdot)$ and $d(\cdot)$ are one-dimensional signals, and that the state space is two-dimensional, i.e. $X = \mathbb{R}^2$. It becomes clear that the extension of the procedure for multi-dimensional signal and state spaces is straightforward at the expense of some extra notation.

A. Basic Method

Considering an orthogonal grid with mesh sizes Δx, Δy and Δt, the basic method to obtain the numerical solution to (10) is as follows: For each grid-point $(x_i, y_j) \in X$, (10) can be written as:

$$\begin{aligned} &\frac{\partial V(x_i,y_j,t)}{\partial t} + \min[0, \widehat{H}(V_x^\pm(x_i,y_j,t), V_y^\pm(x_i,y_j,t))] = 0 \\ &V(x_i,y_j,0) = l(x_i,y_j) \end{aligned} \tag{28}$$

As in the interface propagation problem, (28) is solved by the method of lines, which involves the iteration over t of the following three-step procedure:

1) Computation of the left $V_x^-(x_i, y_j, t)$, $V_y^-(x_i, y_j, t)$ and right $V_x^+(x_i, y_j, t)$, $V_y^+(x_i, y_j, t)$ approximations to the spatial derivatives of $V(x_i, y_j, t)$.
2) Computation of the optimal control and disturbance inputs $u^*(x_i, y_j, t)$ and $d^*(x_i, y_j, t)$, respectively, and construction of the numerical approximation $\widehat{H}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$ to the Hamiltonian. Note that the Hamiltonian is computed as: $\widehat{H}(\cdot,\cdot) = -\widehat{H}(\cdot,\cdot)|_{f(\cdot)=-f(\cdot)}$ because the solution is obtained backwards in time.
3) Approximation of the temporal derivative $\frac{\partial V(x_i,y_j,t)}{\partial t}$ and computation of the value function $V(x_i, y_j, t-\Delta t)$, $\Delta t > 0$, which specifies the reachable set $W(t-\Delta t)$ at time $t-\Delta t$.

The major differentiation from the standard schemes for the interface evolution problem lies in the computation of the numerical approximation to the Hamiltonian $\widehat{H}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$. The reason is that, in order to compute $\widehat{H}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$, at each grid-point (x_i, y_j), the value of the vector field $f(x_i, y_j, u^*(x_i, y_j, t), d^*(x_i, y_j, t))$ needs to be known. However, the optimal values $u^*(x_i, y_j, t)$ and $d^*(x_i, y_j, t)$ depend on the value of $\widehat{H}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$. To solve this problem, we make use of the fact that the optimal control and disturbance signals are of bang-bang type, which means that optimization is achieved at the boundaries of the intervals $[U_1, U_2]$ and $[D_1, D_2]$. This means that for every $t < 0$ and grid-point (x_i, y_j), there are only 4 "candidate" optimal Hamiltonians:

$$H_1\left(\frac{\partial V(x_i,y_j,t)}{\partial x}, x_i, y_j\right) = \frac{\partial V(x_i,y_j,t)}{\partial x} f(x_i, y_j, U_1, D_1) \tag{29}$$

$$H_2\left(\frac{\partial V(x_i,y_j,t)}{\partial x}, x_i, y_j\right) = \frac{\partial V(x_i,y_j,t)}{\partial x} f(x_i, y_j, U_2, D_1) \tag{30}$$

$$H_3\left(\frac{\partial V(x_i,y_j,t)}{\partial x}, x_i, y_j\right) = \frac{\partial V(x_i,y_j,t)}{\partial x} f(x_i, y_j, U_1, D_2) \tag{31}$$

and

$$H_4\left(\frac{\partial V(x_i,y_j,t)}{\partial x}, x_i, y_j\right) = \frac{\partial V(x_i,y_j,t)}{\partial x} f(x_i, y_j, U_2, D_2) \tag{32}$$

Therefore, assuming that the Lax-Friedrichs numerical approximation (19) is employed, algorithm 1 yields the optimal control and disturbance signals $u^*(x_i, y_j, t)$ and $d^*(x_i, y_j, t)$ at each grid-point as well as the approximation to the Hamiltonian $\widehat{H}^{LF}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$. Considering that $t = n\Delta t$ for $n \in \mathbb{Z}^- \cup \{0\}$, we introduce the shorthand notation, $V_x^{\pm n}{}_{i,j} = V_x^\pm(x_i, y_j, t)$, $V_y^{\pm n}{}_{i,j} = V_y^\pm(x_i, y_j, t)$, $u^{*n}_{i,j} = u^*(x_i, y_j, t)$ and $d^{*n}_{i,j} = d^*(x_i, y_j, t)$.

At each grid point (x_i, y_j) algorithm 1 computes the optimal control and disturbance inputs $u^{*n}_{i,j}$ and $d^{*n}_{i,j}$, respectively, by checking the values of the four possible optimal Hamiltonians. For points which satisfy the condition $V(x_i, y_j, t) \le 0$, that is $(x_i, y_j) \in W(t)$, the optimal inputs need not be computed again, since the reachability problem for those points has been solved at a previous time step.

When the solution to (10) has been computed for all $t \in [T, 0]$, the iteration terminates and the reachable set W^* is obtained together with the optimal control inputs $u^*(x_i, y_j)$, which will eventually drive every grid-point $(x_i, y_j) \in W^*$ to the target set F, for every disturbance $d \in D$. The optimal control policy is obtained as a finite map, which corresponds every grid-point $(x_i, y_j) \in W^*$ to a value $u^*(x_i, y_j)$. In order to derive a continuous map $u^* : X \to \mathbb{R}$ such that every state $(x, y) \in W^*$ is assigned with a value $u^*(x, y)$, one can interpolate the values $u^*(x_i, y_j)$ with a continuous function. However, this is an implementation issue and we will not consider it any further in this paper.

Algorithm 1 (Computation of u^*, d^* and $\widehat{H}^{LF}$ at $t = n\Delta t$):

for all $(x_i, y_j) \in X$
  if $V(x_i, y_j, t) \le 0$
    compute $\widehat{H}^{LF}(V_x^{\pm n}{}_{i,j}, V_y^{\pm n}{}_{i,j})$ using the known values of $u^{*n+1}_{i,j}$ and $d^{*n+1}_{i,j}$ from previous time steps
  else
    if $\widehat{H}_3^{LF}(\cdot,\cdot) \le \widehat{H}_1^{LF}(\cdot,\cdot) \le \widehat{H}_2^{LF}(\cdot,\cdot)$
      $u^{*n}_{i,j} = U_1$; $d^{*n}_{i,j} = D_1$; $\widehat{H}^{LF}(\cdot,\cdot) = \widehat{H}_1^{LF}(\cdot,\cdot)$
    else if $\widehat{H}_4^{LF}(\cdot,\cdot) \le \widehat{H}_2^{LF}(\cdot,\cdot) < \widehat{H}_1^{LF}(\cdot,\cdot)$
      $u^{*n}_{i,j} = U_2$; $d^{*n}_{i,j} = D_1$; $\widehat{H}^{LF}(\cdot,\cdot) = \widehat{H}_2^{LF}(\cdot,\cdot)$
    else if $\widehat{H}_1^{LF}(\cdot,\cdot) < \widehat{H}_3^{LF}(\cdot,\cdot) \le \widehat{H}_4^{LF}(\cdot,\cdot)$
      $u^{*n}_{i,j} = U_1$; $d^{*n}_{i,j} = D_2$; $\widehat{H}^{LF}(\cdot,\cdot) = \widehat{H}_3^{LF}(\cdot,\cdot)$
    else if $\widehat{H}_2^{LF}(\cdot,\cdot) < \widehat{H}_4^{LF}(\cdot,\cdot) < \widehat{H}_3^{LF}(\cdot,\cdot)$
      $u^{*n}_{i,j} = U_2$; $d^{*n}_{i,j} = D_2$; $\widehat{H}^{LF}(\cdot,\cdot) = \widehat{H}_4^{LF}(\cdot,\cdot)$
    end
  end
end

Observation 1: Although the computed reachable set W^* at the limit $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta t \to 0$ converges to the actual reachable set, this does not hold for the computed control and disturbance policies $u^*(\cdot)$ and $d^*(\cdot)$.

Indeed, the value function $V(x, t)$ in (10) is defined with respect to the level sets of the target set F. However, in our level set approach, $V(x, t)$ is treated as the level set function of the reachable set $W(t)$. Therefore, at each iteration of the procedure, instead of F, we consider $W(t)$ as the target set in order to compute $W(t-\Delta t)$. This may result to a different control policy than the one specified by (10). Nevertheless, by induction, it is straightforward to prove the following proposition, which states the main property of the solution obtained from the numerical scheme.

Proposition 2: At the limit of discretization and for each $t = n\Delta t$ the control inputs $u^{*n}_{i,j}$ drive each trajectory of (23) to the target set F in no more than $|t|$ time units for every initial condition $x_0 = (x_i, y_j) \in W(t)$ and for any disturbance action.

B. An Alternative Method

The basic method is based exclusively on the well tested numerical schemes for the interface propagation problem, which means that, in theory, the computation of u^* and d^* does not introduce any spurious oscillations in the solution. However, the simultaneous computation of the optimal inputs with the Hamiltonian $\widehat{H}^{LF}$ increases the complexity of the method, especially when equation (23) does not hold. An alternative, more efficient, approach is to decouple this computation.

More specifically, having obtained the value function $V(x, t)$ at time t, instead of determining the optimal inputs and the Hamiltonian simultaneously, we can compute the optimal inputs first by considering directly the equations:

$$u^*(x, t) = \arg\min_{u\in U}\max_{d\in D} \frac{\partial V(x,t)}{\partial x} f(x, u, d) \tag{33}$$

$$d^*(x, t) = \arg\max_{d\in D} \frac{\partial V(x,t)}{\partial x} f(x, u^*, d) \tag{34}$$

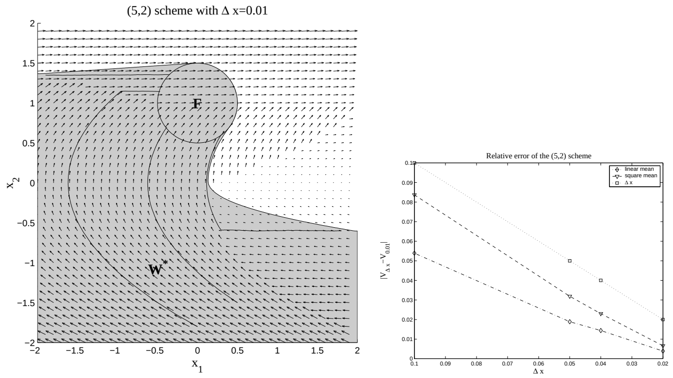

Fig. 1. The reachable set $W^* = W(-10)$ for the double integrator, together with simulated trajectories, and relative error analysis.

where for the approximation of the gradient $\frac{\partial V(x,t)}{\partial x}$ at time $t \leq 0$ and each grid-point (x_i, y_j) we employ a central difference scheme, i.e.:

$$V_x^0(x_i, y_j, t) = \frac{V(x_{i+1}, y_j, t) - V(x_{i-1}, y_j, t)}{2\Delta x} \tag{35}$$

$$V_y^0(x_i, y_j, t) = \frac{V(x_i, y_{j+1}, t) - V(x_i, y_{j-1}, t)}{2\Delta y} \tag{36}$$

Hence, the solution to (28) is obtained by iterating over t the following 4-step procedure:

1) Computation of the central approximations $V_x^0(x_i, y_j, t)$ and $V_x^0(x_i, y_j, t)$ to the spatial derivatives of $V(x_i, y_j, t)$, and extraction of the optimal inputs $u^*(x_i, y_j, t)$ and $d^*(x_i, y_j, t)$, for $(x_i, y_j) \notin W(t)$, from equations (33) and (34) respectively. If $(x_i, y_j) \in W(t)$, it means that the optimal inputs have been computed in a previous iteration.
2) Computation of the left $V_x^-(x_i, y_j, t)$, $V_y^-(x_i, y_j, t)$ and right $V_x^+(x_i, y_j, t)$, $V_y^+(x_i, y_j, t)$ approximations to the spatial derivatives of $V(x_i, y_j, t)$.
3) Construction of the numerical approximation $\widehat{H}(V_x^\pm(x_i, y_j, t), V_y^\pm(x_i, y_j, t))$ to the Hamiltonian using the optimal values $u^*(x_i, y_j, t)$ and $d^*(x_i, y_j, t)$ obtained from the first step and previous iterations.
4) Approximation of the temporal derivative $\frac{\partial V(x_i, y_j, t)}{\partial t}$ and computation of the value function $V(x_i, y_j, t - \Delta t)$, $\Delta t > 0$, which specifies the reachable set $W(t - \Delta t)$ at time $t - \Delta t$.

The drawback of this approach is that the "brute force" computation of the gradient $\frac{\partial V(x,t)}{\partial x}$, using the second order central difference approximations, may deteriorate the quality of the approximation to the optimal inputs. Unlike the basic method, where the optimal inputs are computed through sophisticated essentially non-oscillatory schemes, here the approximation of the gradient by the central difference schemes may introduce spurious oscillations which, in the sequel, will affect the computation of the optimal inputs. However, in problems where the gradient $\frac{\partial V(x,t)}{\partial x}$ does not become very steep, this approach can produce reliable results.

V. Numerical Examples

A. The Double Integrator

Consider the benchmark linear system of the double integrator, whose state space representation is as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \tag{37}$$

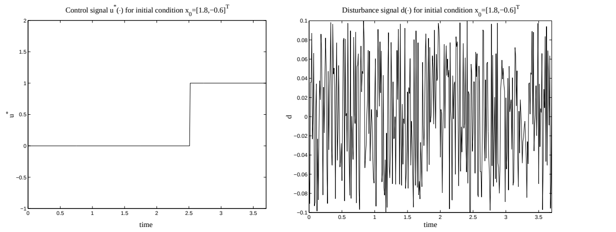

Fig. 2. Control and disturbance signals for initial condition $x_0 = [1.8, -0.6]^T$.

For the control and disturbance inputs it holds $u \in [0, 1]$ and $d \in [-0.1, 0.1]$ respectively, whereas the target set F is defined by:

$$F = \{x \mid x_1^2 + (x_2 - 1)^2 \leq 0.5^2\} \tag{38}$$

Since the vector field (37) is affine in the control and disturbance inputs the basic method for the computation of the reachable set W^* and the inputs u^* and d^* does apply. We have implemented a fifth order in space and second order in time (5,2) accurate scheme. The computational domain is defined by $-2 \leq x_1 \leq 2$, $-2 \leq x_2 \leq 2$ and the scheme was tested on grids with grid spacing: $\Delta x_1 = \Delta x_2 = \Delta x = 0.1, 0.05, 0.04, 0.02$ and 0.01. The time window over which the problem is solved is $t \in [-10, 0]$. Therefore, for the reachable set W^* we assume it holds, $W^* = W(-10)$.

In figure 1 we illustrate the reachable set W^* computed using grid spacing $\Delta x = 0.01$, and the relative error of the scheme with respect to the finest solution obtained for $\Delta x = 0.01$. Linear interpolation on the finest grid has been employed to evaluate the error. It is clear that the scheme converges to the finest solution as Δx decreases. Also, both the linear mean error and the square mean error remain less than the grid spacing.

To demonstrate the practical value of the approach, we have implemented the control policy obtained by the (5,2) scheme and we have simulated trajectories, shown in figure 1, of the controlled system in the presence of disturbance. The disturbance signal $d(\cdot)$ is supplied by a uniformly distributed random signal generator. Naturally, the signal fluctuates in the $[-0.1, 0.1]$ interval. As for the implementation of the controller, at each time step of the simulation the controller determines the closest grid point to the current state of the system. Finally, figure 2 depicts the temporal profile of the optimal control and disturbance signals for a trajectory initiating from $x_0 = [1.8, -0.6]^T$.

B. The Homicidal Chauffeur

The homicidal chauffeur problem [2] is a benchmark example of a pursuit-evasion differential game. A pursuer and an evader are both moving with constant speeds in a two-dimensional plane, with the pursuer trying to catch the evader. The pursuer moves faster but is less manuverable than the evader, which moves slower but is able to change direction instantaneously. If we normalize both pursuer's speed and minimum turn radius to one and consider the problem in relative coordinates, we obtain the following differential description of the game:

$$\begin{aligned} \dot{x}_1 &= -x_2 u + v_2 \cos d \\ \dot{x}_2 &= -1 + x_1 u + v_2 \sin d \end{aligned} \tag{39}$$

where u and d are the control actions of the the pursuer and the evader respectively and v_2 is the speed of the evader. In the sequel u will be referred to as the control and d as the disturbance. The control

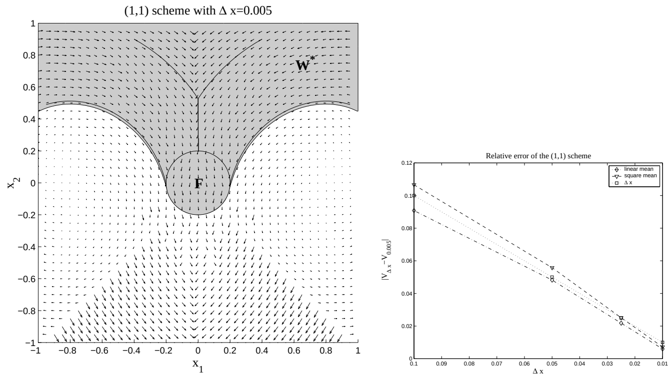

Fig. 3. The reachable set $W^* = W(-10)$ for the homicidal chauffeur game, together with simulated trajectories, and relative error analysis.

input must satisfy the constraint $u \in [-1, 1]$ while for the disturbance it holds, $d \in [0, 2\pi)$. Also, $v_2 = 0.2$ speed units. The game is won by the pursuer (controller) if the state of the game reaches the target set F defined as:

$$F = \{x \mid x_1^2 + x_2^2 \leq 0.2^2\} \tag{40}$$

From (39) we observe that the control signal is of bang-bang type. However, this is not the case with the disturbance signal. This means that the solution to the problem must be obtained using the alternative method presented in the previous section. For that we have implemented a (1,1) first order accurate in space and time scheme. The computational domain is defined by $-1 \leq x_1 \leq 1$, $-1 \leq x_2 \leq 1$ and the scheme was tested on grids with grid spacing: $\Delta x_1 = \Delta x_2 = \Delta x = 0.1, 0.05, 0.025, 0.01$ and 0.005. As in the previous example, the time window over which the problem is solved is $t \in [-10, 0]$. Therefore, for the reachable set W^* we assume it holds, $W^* = W(-10)$.

Figure 3 illustrates the reachable set W^* and the relative error of the (1,1) scheme with respect to the finest solution obtained for $\Delta x = 0.005$. It is clear that the scheme converges to the finest solution as Δx decreases. The simulated trajectories, shown also in figure 3, verify the correctness of our design. Unlike the previous example, here we employ the worst-case disturbance inputs obtained from the numerical scheme. The temporal profile of the control and disturbance signals for the trajectory initiating from $x_0 = [0.95, 0.49]^T$ is depicted in figure 4. Observe that since the worst-case disturbance signal is not of bang-bang type it is more sensitive to numerical errors that occur during the computation.

VI. Discussion

The major disadvantage of the approach is that it produces convergent approximations to the real reachable set. This means that away from the discretization limit there is no guarantee that the computed set is an under or over-approximation. Given that in target control problems we seek to under-approximate the reachable set, the next research step in this direction should be the investigation of numerical schemes based on level set methods that compute conservative under-approximations of reachable sets. Also, as any other numerical scheme based on the principle of optimality, the level set approach suffers from the *curse of dimensionality*. That is the computational cost grows exponentially with dimension.

Nevertheless, the advantages of the approach should not be overlooked. Level set methods are thoroughly studied and tested, and

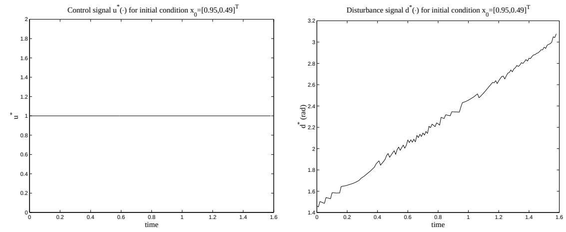

Fig. 4. Control and disturbance signals for initial condition $x_0 = [0.95, 0.49]^T$.

provide highly accurate results even when abnormalities, like shocks or voids, appear in the solution. Moreover, level set methods deal with nonlinear dynamics and they pose no constraints on the topology of the sets involved.

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