Improved Prediction of Industrial Yield Based on Tools from a Normed Linear Space of Fuzzy Interval Numbers (*FIN*s)

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Abstract -- **The set $\mathbb{F}$ of Fuzzy Interval Numbers (*FIN*s) is studied analytically in this work. A *FIN* is a set of "pulse-shaped" functions, namely *generalized intervals*. A *FIN* can be interpreted as a conventional convex fuzzy set; nevertheless a *FIN* can have either a positive or a negative membership function. The set of generalized intervals of height h is shown to be a lattice-ordered, normed linear space. It is shown that $\mathbb{F}$ is a metric lattice. In conclusion, the normed linear space $\mathbb{T}$ of *FIN*s with triangular membership functions is introduced. Mathematical tools presented here are employed for improving prediction of sugar production for Hellenic Sugar Industry (HSI), Greece from populations of measurements.**

Index terms -- **Convex fuzzy sets, Fuzzy Interval Numbers (*FIN*s), generalized intervals, normed linear spaces.**

I. Introduction

While the use of fuzzy sets keeps spreading due to good results in practical problems, the lack of analytic design tools has encouraged the proliferation of heuristics; the analytical study of fuzzy logic is still trailing its implementation [11]. This work introduces novel mathematical tools with the potential to use them for an analytic design of fuzzy systems. Furthermore this work demonstrates an application of tools presented here for improving prediction of sugar production based on populations of measurements.

More specifically, this work introduces a real linear space involving fuzzy sets. Note that various authors treat fuzzy sets as (fuzzy) numbers, furthermore they adhere to the extension principle and they introduce both an addition and a multiplication operation involving fuzzy sets. Nevertheless the definition of the multiplication operation has been cumbersome [9].

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This work introduces a real linear space involving fuzzy sets on algebraic grounds. The extension principle is abandoned. A Fuzzy Interval Number (*FIN*) is introduced, as described in this work, such that a *FIN* can have either a positive or a negative membership function. A fuzzy set is an interpretation of a *FIN*. The goal is to develop linear Fuzzy Inference System (FIS) design techniques, which effect non-linear mappings f: $\mathbb{R}^N \rightarrow \mathbb{R}^M$. The "vehicle" for rigorous mathematical analysis in this work is the theory of partially ordered vector spaces [13], [16].

This paper is organized as follows. Section II introduces normed linear spaces of *generalized intervals*. Section III introduces the metric lattice $\mathbb{F}$ of *Fuzzy Interval Numbers* (*FIN*s). Section IV introduces a normed linear space of *FIN*s with triangular membership functions. Section V shows an application of novel tools presented here to an industrial yield prediction problem. Finally, section VI summarizes the contribution of this work; it also delineates future work.

II. Normed Linear Spaces $\mathbb{M}^H$ of Generalized Intervals

This section introduces *lattice-ordered*, *normed linear spaces* $\mathbb{M}^h$ of *generalized intervals*, $h\in(0,1]$. Spaces $\mathbb{M}^h$, $h\in(0,1]$ will be employed in the following section for further mathematical analysis. Consider the following definition.

Definition 1: A *generalized interval* (*of height h*) is a real function given either by $\mu_{[x_1,x_2]_+^h}(x) = \begin{cases} h, & x_1 \le x \le x_2 \\ 0, & \text{otherwise} \end{cases}$, or by $\mu_{[x_1,x_2]_-^h}(x) = \begin{cases} -h, & x_1 \le x \le x_2 \\ 0, & \text{otherwise} \end{cases}$, where $h\in(0,1]$.

A generalized interval will simply be denoted by $[x_1,x_2]_+^h$ (*positive generalized interval*) or by $[x_1,x_2]_-^h$ (*negative generalized interval*). The set of generalized intervals of height h is denoted by $\mathbb{P}^h$. The following *partial ordering* relation has been introduced in $\mathbb{P}^h$.

(R1) $[a,b]_+^h \le [c,d]_+^h \Leftrightarrow c \le a \le b \le d$,

(R2) $[a,b]_-^h \le [c,d]_-^h \Leftrightarrow [c,d]_+^h \le [a,b]_+^h$, and

(R3) $[a,b]_-^h \le [c,d]_+^h \Leftrightarrow [a,b]\cap[c,d]\neq\emptyset$, where $[a,b]$ and $[c,d]$ denote conventional intervals (sets) of real numbers.

It has been shown that the partially ordered set P^h is a mathematical *lattice*. Furthermore, a *pseudo-metric* distance has been introduced in lattice P^h by real function $d(x,y)=v(x\vee y)-v(x\wedge y)$, $x,y\in\mathsf{P}^h$, where function $v(.)$ maps a generalized interval to the positive/negative area underneath it [6], [7], [12]. An *equivalence relation* $\sim$ in lattice P^h, given by $x \sim y \Leftrightarrow d(x,y) = 0$, $x,y\in\mathsf{P}^h$, has implied the *quotient* (set) M^h, that is $\mathsf{M}^h=\mathsf{P}^h/\sim$. Furthermore, it was shown that M^h is a lattice. To avoid redundant terminology, an element of M^h is called *generalized interval*, as well, and it is denoted by $[a,b]^h$. The set of *positive* (*negative*) *generalized intervals* is denoted by M^+ (M^-), whereas the set of *trivial generalized intervals* $[a,b]$ with $a=b$ is denoted by M_0. It is defined $\mathsf{M}_0^+ = \mathsf{M}^+ \bigcup \mathsf{M}_0$; likewise $\mathsf{M}_0^- = \mathsf{M}^- \bigcup \mathsf{M}_0$.

It follows that real function $v(.)$ defined above as the area underneath a generalized interval, is a *positive valuation function* in lattice M^h, hence the distance function $d(x,y)=v(x\vee y)-v(x\wedge y)$ is a *metric* distance in M^h. Note that even though the set M^h of generalized intervals is a metric lattice for $h > 0$, interest is focused in this work in metric lattices M^h with $h\in(0,1]$ because the latter lattices arise from *a-cuts* of conventional convex fuzzy sets. Recall that an *a-cut* has been defined as the interval $\Gamma_a= \{x|\mu(x) \ge a\}$, where $\mu(x)$ is a fuzzy set's membership function. Convenient geometric interpretations of all previous notions are shown in [6], [12]. Function *support*(.), defined in the following, will be useful below.

Definition 2: Function *support*($[a,b]^h$), $h\in(0,1]$ maps a generalized interval $[a,b]^h$ to its interval support (set); in particular *support*($[a,b]^h$)=$[a,b]$ if $a\le b$, whereas *support*($[a,b]^h$)=$[b,a]$ if $a\ge b$.

An element $[a,b]^h$ of metric lattice M^h, $h\in(0,1]$ is represented by a pair of real numbers in the Cartesian product space $\mathsf{R}\times\mathsf{R}$. Since space R^2 is a real linear space, it follows that space M^h is, likewise, a *real linear space*. More specifically,

- *addition* in M^h is defined as $[a,b]^h+[c,d]^h=[a+c,b+d]^h$,
- *multiplication* (by a real number k) is defined as $k[a,b]^h=[ka,kb]^h$.

A generalized interval in M^h is called *vector* of linear space M^h. The *zero vector* O^h in M^h equals $O^h= [0,0]^h$.

A lattice-ordered linear space, such as space M^h in this work, is called in the literature *vector lattice* or *Riesz space* ([5], section 310B). Note that a theory of vector lattices was initially introduced in [13] and it was further developed by several authors [16]. Selected properties in vector lattice M^h are shown in the following for $x,y,z\in\mathsf{M}^h$ and $\lambda\in\mathsf{R}$ [5].

(P1) $(x+z)\vee(y+z)=(x\vee y)+z$, $(x+z)\wedge(y+z)=(x\wedge y)+z$,

(P2) $\lambda x\vee\lambda y= \lambda(x\vee y)$, $\lambda x\wedge\lambda y= \lambda(x\wedge y)$, $\lambda \ge 0$,

(P2) $\lambda x\vee\lambda y= \lambda(x\wedge y)$, $\lambda x\wedge\lambda y= \lambda(x\vee y)$, $\lambda \le 0$, and

The vectors $x\vee O^h$, $(-x)\vee O^h$, and $x\vee(-x)$ are called, respectively, *positive part*, *negative part*, and *absolute value* of vector (generalized interval) x, and they are denoted, respectively, by x^+, x^-, and $|x|$. The following identities hold:

(I1) $x=x^+-x^-$ (*Jordan decomposition*)

(I2) $|x|= x^+ + x^-$,

(I3) $x\vee y + x\wedge y = x+y$, and

(I4) $|x-y| = x\vee y - x\wedge y$

Metric lattice M^h is a *normed* linear space as shown in the following.

Proposition 3: Real function $\|.\|$: $\mathsf{M}^h\to\mathsf{R}$ given by $\|x\|=v(|x|)$ defines a *norm* in real vector lattice M^h, $h\in(0,1]$, where $|x|$ is the absolute value (vector) of vector $x\in\mathsf{M}^h$ and $v([a,b]^h)=h(b-a)$, $a,b\in\mathsf{R}$.

The proof of proposition 3 will be shown elsewhere. It is well-known that a normed linear space $\mathcal{S}$ with norm $\|.\|$ implies the following metric $d(x,y)= \|x-y\|$, $x,y\in\mathcal{S}$. Hence, following proposition 3, the norm-induced metric for two vectors $x=[a,b]^h$ and $y=[c,d]^h$ in M^h is given by $d(x,y)=\|x-y\|= \|[a-b,b-d]^h\|= h(|a-c|+|b-d|)$.

The dimension of linear space M^h equals 2. A convenient *basis* is selected in M^h in the following. More specifically since $[a,b]^h= [a,0]^h+[0,b]^h= a[1,0]^h+b[0,1]^h= ae_1^h+be_2^h$, basis $(e_1^h,e_2^h)=([1,0]^h, [0,1]^h)$ has been selected in linear space M^h in the context of this work.

Note that for all equations above there exist convenient geometric interpretations on the plane [6], [12].

III. The Metric Lattice F of Fuzzy Interval Numbers (*FIN*s)

The previous section introduced normed linear spaces M^h, $h\in(0,1]$ of generalized intervals. Spaces M^h will be employed in this section to introduce analytically useful tools. Consider the following definition.

Definition 4: A *Fuzzy Interval Number*, or *FIN* for short, is a function either F: $(0,1]\rightarrow \mathbb{M}_0^+$ or F: $(0,1]\rightarrow \mathbb{M}_0^-$ such that $h_1 \le h_2 \Rightarrow support(F(h_1)) \supseteq support(F(h_2))$, where $0 < h_1 \le h_2 \le 1$.

The set of *FIN*s will be denoted by $\mathbb{F}$. In particular, $\mathbb{F}^+$ ($\mathbb{F}^-$) denotes the set of *positive* (*negative*) *FIN*s which includes *positive* (*negative*) generalized intervals, furthermore $\mathbb{F}_0$ denotes the set of *trivial FIN*s. It follows $\mathbb{F}_0^+ = \mathbb{F}^+ \cup \mathbb{F}_0$; likewise $\mathbb{F}_0^- = \mathbb{F}^- \cup \mathbb{F}_0$. Fig.1 shows examples of one negative *FIN* (F_n), one trivial *FIN* (F_t), and two positive *FIN*s (F_q, F_p). Note that a *FIN* is not a fuzzy set; rather a *FIN* is an abstract mathematical notion. However, a *FIN* can be interpreted as a fuzzy set as explained below. The advantage of negative *FIN*s is that convenient algebraic operations can be defined as explained below. An ordering relation has been introduced in the set $\mathbb{F}$ of *FIN*s as follows.

Definition 5: Let F_1, F_2 be *FIN*s, then $F_1 \le F_2 \Leftrightarrow F_1(h) \le F_2(h)$, $h\in(0,1]$.

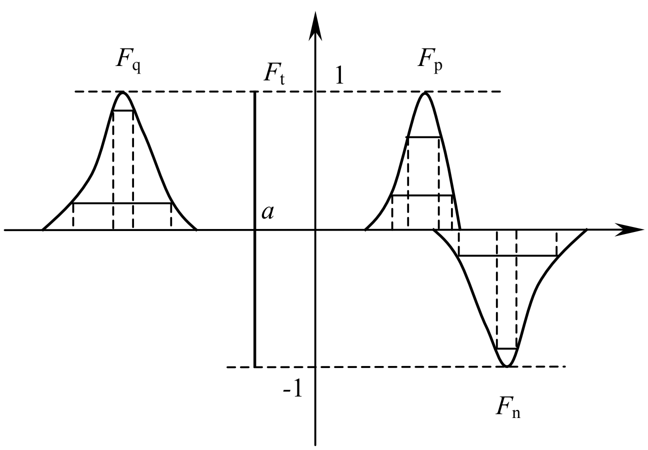

Fig. 1 Positive *FIN*s F_p, F_q include positive generalized intervals $F_p(h)$, $F_q(h)$, $h\in(0,1]$, whereas negative *FIN* F_n includes negative generalized intervals $F_n(h)$, $h\in(0,1]$. Furthermore, trivial *FIN* F_t includes trivial generalized intervals $[a,a]^h$, $h\in(0,1]$.

It can be shown that the space $\mathbb{F}$ of *FIN*s is a (mathematical) lattice. A *metric distance* function has been introduced in $\mathbb{F}$ as follows [6].

Proposition 6: Let F_1 and F_2 be *FIN*s in $\mathbb{F}$. A *metric distance* function d_K: $\mathbb{F}\times\mathbb{F}\rightarrow\mathbb{R}$ is defined in $\mathbb{F}$ by integral $d_K(F_1,F_2)=\int_0^1 c(h)d(F_1(h),F_2(h))dh$, where $c(h)>0$ and $d(.,.)$ is the norm-induced metric distance in vector lattice $\mathbb{M}^h$ of generalized intervals.

A convenient value for function $c(h)>0$ above might be $c(h)=h$ which, for two trivial generalized intervals $[a,a]^1$ and $[b,b]^1$ implies $d([a,a]^1,[b,b]^1)=|b-a|$. The calculation of a metric distance between two fuzzy sets with arbitrary-shaped membership functions has already been proposed in the literature. In particular metric d_p has been introduced in [4] based on the Hausdorf metric d_H; moreover a similar formula as in proposition 6 has been proposed in a digital image processing application [2]. Nevertheless, in both the aforementioned publications [2] and [4] a metric has been proposed based on *a-cuts* of convex, bounded, compact and upper semicontinuous fuzzy sets. There are at least two advantages for the employment of generalized intervals instead of *a-cuts*. First, an "*a-cuts* based definition" of a metric distance d_p in [4] implies $\aleph_0$ different metric distance functions, whereas a "generalized intervals based definition" of a metric distance d_K implies $\aleph_1=2^{\aleph_0}>\aleph_0$ different metric distance functions as it will be shown elsewhere, where $\aleph_1$ is the cardinality of the set $\mathbb{R}$ of real numbers. Second, on the one hand, generalized intervals imply a real linear space involving fuzzy sets, on the other hand, *a-cuts* only imply a cone. In conclusion, the well-developed linear system theory can be employed to design a Fuzzy Inference System (FIS) for approximating a non-linear mapping f: $\mathbb{R}^N\rightarrow\mathbb{R}^M$ from the inputs to the outputs of a FIS.

The employment of *FIN*s with negative membership functions should not alienate practitioners. For instance note that the expert system MYCIN, with strong influence in the design of commercial expert systems, has used *negative certainty factors* for medical diagnosis of certain blood infections [1]. Most important note that the employment of fuzzy sets with negative spreads has implied improvements in fuzzy linear regression problems [3].

IV. THE NORMED LINEAR SPACE $\mathbb{T}$ OF TRIANGULAR *FIN*S

The obvious next step is an extension of both the addition and multiplication operations from linear spaces $\mathbb{M}^h$, $h\in(0,1]$ to metric lattice $\mathbb{F}$.

Definition 7: The product kF_1, where $k\in\mathbb{R}$ and $F_1\in\mathbb{F}$, is defined as *FIN* F_p: $F_p(h)=kF_1(h)$, $h\in(0,1]$.

Note that the product kF_1 is always a *FIN*. More specifically for $F_1\in\mathbb{F}_0^+$, if $k>0$ then $kF_1\in\mathbb{F}_0^+$, whereas if $k<0$ then $kF_1\in\mathbb{F}_0^-$; and vice-versa for $F_1\in\mathbb{F}_0^-$, if $k>0$ then $kF_1\in\mathbb{F}_0^-$, whereas if $k<0$ then $kF_1\in\mathbb{F}_0^+$. For example $F_q=-F_n$ (Fig.1), where $F_q\in\mathbb{F}_0^+$, $F_n\in\mathbb{F}_0^-$.

Definition 8: The sum F_1+F_2, where $F_1,F_2\in\mathbb{F}$, is defined as *FIN* F_s: $F_s(h)=(F_1+F_2)(h)= F_1(h)+F_2(h)$, $h\in(0,1]$.

We remark that if both F_1 and F_2 are in $\mathbb{F}_0^+$ ($\mathbb{F}_0^-$) then sum F_1+F_2 is in $\mathbb{F}_0^+$ ($\mathbb{F}_0^-$). Of particular practical interest might be the linear convex combination $kF_1+(1-k)F_2$, $k\in(0,1]$. It follows that space $\mathbb{F}_0^+$ ($\mathbb{F}_0^-$) is *convex*. Moreover note that *FIN* $O=\bigcup_{h\in(0,1]}[0,0]^h$ is the *zero* element for addition in $\mathbb{F}$. However, a problem might arise in calculating the sum F_1+F_2 when one of *FIN*s F_1, F_2 is in $\mathbb{F}^+$ whereas the other is in $\mathbb{F}^-$. More specifically, the aforementioned problem occurs when generalized interval $F_1(h)+F_2(h)$ is both *positive*, for some values of $h\in(0,1]$, and *negative*, for other values of $h\in(0,1]$. In the aforementioned case, the sum F_1+F_2 is not a *FIN*. The latter problem was mended as described in the following.

Let $L(.)$ be a real non-negative real function which maps a conventional interval (set) $[a,b]$ to its length b-a, that is $L([a,b])=b-a \geq 0$. Given a *FIN*, function $L(support(F(h)))$ is, apparently, a non-increasing function of h. The sum F_1+F_2 of one positive *FIN* and one negative *FIN* implies, in essence, subtraction of the corresponding generalized interval supports. For example, Fig.2(a) displays length functions $L(support(F_1(h)))$ and $L(support(F_2(h)))$ for two *FIN*s F_1 and F_2. Since functions $L(support(F_1(h)))$ and $L(support(F_2(h)))$ in Fig.2(a) intersect each other, there follows that F_1+F_2 is not a *FIN*.

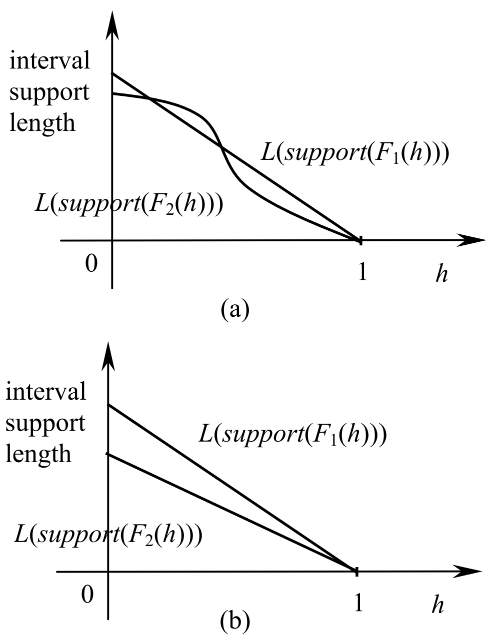

Fig. 2 (a) The difference of functions $L(support(F_1(h)))$ and $L(support(F_2(h)))$ can be both positive and negative, hence sum F_1+F_2 is not a *FIN*.
(b) For linear functions $L(support(F_i(h)))$, i=1,2 the difference $L(support(F_1(h)))$ - $L(support(F_2(h)))$ is also linear, hence addition of triangular *FIN*s is guaranteed to be a triangular *FIN*.

However, it can be guaranteed that sum F_1+F_2 is a *FIN* if both functions $L(support(F_1(h)))$ and $L(support(F_2(h)))$ decrease according to the same law, for instance they decrease linearly as shown in Fig.2(b). In the latter case, the sum F_1+F_2 of two triangular *FIN*s F_1 and F_2 is guaranteed to be a triangular *FIN*. In conclusion, if $\mathbb{T}$ denotes the space of *FIN*s with triangular membership functions it follows that $\mathbb{T}$ is a linear space. Furthermore, in line with the analysis in the previous section, linear space $\mathbb{T}$ is lattice-ordered. A norm can be introduced in vector lattice $\mathbb{T}$ as follows.

Proposition 9 Let F be a *FIN* in $\mathbb{T}$. Then integral $\|F\|=\int_0^1 \|F(h)\| dh$ defines a norm in $\mathbb{T}$.

V. Application For Predicting Industrial Sugar Production

A FIS (Fuzzy Inference System) has been developed using tools presented in the previous sections for improving prediction of the annual sugar production for Hellenic Sugar Industry (HSI), Greece based on populations of measurements. More specifically, populations of both *meteorological* and *production* variables, namely *input* variables, were available in this study during a time period spanning eleven years [8].

A *FIN* was constructed from a population of measurements using algorithm CALFIN [12]. For instance Fig.3 shows two *FIN*s which correspond to two populations of Roots Weights (RW) production variable measurements from a number of pilot fields in the Larisa agricultural district.

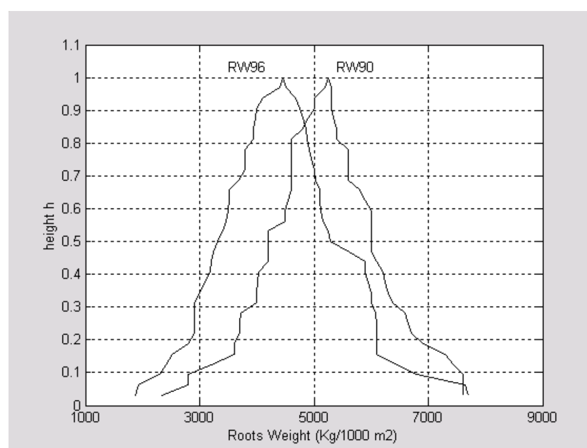

Fig. 3 *FIN*s *RW90* and *RW96* have been computed from two populations of Roots Weight (RW) production variable measurements in years 1990 and 1996, respectively, from pilot fields in the Larisa agricultural district, Greece.

For each production year a number of *FIN*s, corresponding to selected input variables, was mapped to a sugar production level using the metric d_K. In conclusion, an improved average prediction error around 2% was resulted compared to 1) an average prediction error of 5% resulted by intelligent clustering techniques, and 2) an average prediction error of 6% resulted by alternative prediction techniques including interpolation, polynomial, linear autoregression, and neural models [8].

VI. DISCUSSION AND CONCLUSION

Analytical tools have been introduced here based on *generalized intervals*, with either positive or negative characteristic functions. The set $\mathbf{M}^h$ of generalized intervals of height $h\in(0,1]$ was shown to be a lattice-ordered, normed linear space. Furthermore, Fuzzy Interval Numbers (*FIN*s) have been described as sets of generalized intervals $[a_h,b_h]^h$, $h\in(0,1]$. The integral of norm-induced metrics in linear spaces $\mathbf{M}^h$ for $h\in(0,1]$ implied a metric distance d_K in lattice-ordered space $\mathbf{F}$ of *FIN*s. Finally, a normed linear space $\mathbf{T}$ of *FIN*s with triangular membership functions was introduced. It was delineated comparatively an application of tools introduced in this work to an industrial yield prediction problem. More specifically, improved estimates of annual sugar production have been obtained.

An approach for dealing with fuzzy sets was suggested in this work based on algebra rather than based on (fuzzy) logic. More specifically, both fuzzy logic and the extension principle were abandoned here in favor of an algebraic treatment of fuzzy sets.

There is a broad range of applications where the novel tools presented in this work could potentially be advantageous for dealing with ambiguity. For instance algorithms could be developed for data clustering and classification based on convex combinations $kF_1+(1-k)F_2$, $k\in(0,1]$, $F_1,F_2\in\mathbf{F}$. Furthermore, a linear space of *FIN*s might be useful in fuzzy regression techniques [15] as well as in system modelling in the sense of Mamdani [10]. Moreover a linear space of *FIN*s might unify fuzzy system modelling in the sense of Mamdani [10] with fuzzy system modelling in the sense of Tagaki-Sugeno [14]. For instance, based on a linear space of *FIN*s, it is feasible to design systems of linguistic fuzzy if-then rules based on rigorous eigenstructure analysis.

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