A Worst-Case Framework for Perfect Reconstruction of Discrete Data Transmissions

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***Abstract*— In this paper we present a deterministic worst-case framework for reconstruction of discrete (source) data transmissions. This framework can be explored based on robust control ideas and formulations and can be viewed as a complement to the traditional probabilistic approaches in the communications area. The particular problems touched upon are: (i) necessary and sufficient conditions for causal and noncausal reconstruction under deterministic magnitude bounded noise for single-input single-output (SISO) and multi-input multi-output (MIMO) channels, (ii) reconstruction based on decision feedback equalizer (DFE) structures, and (iii) performance optimization under channel fading. The ℓ^1 control theory and linear programming emerge as the natural key player in this framework.**

Keywords: Equalization, ℓ^1 optimality, worst case, discrete data reconstruction.

I. Introduction

The topic of data transmission and reconstruction is based almost entirely on stochastic formulations of the various problems involved (e.g., [1], [2]). In these formulations, the main measure of performance of a communication system is characterized primarily in terms of the probability of error under various stochastic assumptions on the noise and channel behavior. Designing a system that minimizes this probability is a rather hard problem and proposed algorithms are characterized by high complexity (e.g., Viterbi's algorithm [1]).

In this paper we present an alternative, deterministic worst-case framework for accurate reconstruction of discrete (source) data. This framework, which can be explored based on robust control ideas and formulations, mainly addresses the question of when perfect reconstruction of a discrete sequence of source symbols (e.g., $+1$ or -1) is possible if the magnitude of the additive noise is allowed to be anything as long as it is bounded by an *a priori* known bound. In other words, this framework is a complementary worst-case, deterministic approach that provides necessary and sufficient conditions for an error to occur. The main motivation for this worst-case approach comes from applications were security to attacks by malicious agents (e.g., jammers [3]) is of paramount importance and therefore "hard" guarantees are required.

The framework leads to a set of necessary and sufficient conditions on the maximum noise level so that perfect reconstruction is possible by some receiver without specifying any structure to it. We then consider decision feedback equalizer (DFE) structures and prove that they form an optimal structure for some (but not all) of the formulated problems. In doing so, we also furnish a procedure for designing a perfectly reconstructing DFE based on linear programming (LP) and ℓ^1-optimization methods [4]. Although DFE analysis and design has received considerable attention over the last forty years, there are many issues of current interest (e.g., [5] and references therein). A common assumption in the DFE design literature is that of correct past decisions, something that is arguably a strong assumption to coexist with the notion of optimality of a design procedure [6]. In our approach, however, since we provide the exact conditions for the existence of perfectly reconstructing DFEs, as well as explicit constructions of such DFEs (if at all possible), this assumption is unnecessary. Of course, if the noise level is higher than the maximum allowed for perfect reconstruction, errors will occur and one has to analyze how these errors propagate in the system. This is not done in this paper, although we touch some relevant issues in the case of first-order FIR channels. For a detailed discussion on these matters we refer to the works in [7], [8], [9]. The paper also considers linear equalizers as a special case of DFEs and certain performance characterizations are given in terms of ℓ^1-optimization formulations. Finally, closed-form performance results are obtained for the case of first-order FIR channels.

The notation is as follows: $\|x\| := \sup_k |x(k)|$ is the ℓ^∞ norm of a sequence $x = \{x(k)\}_{k=0}^{\infty}$; $\|T\|_1 := \sum_{k=0}^{\infty} |t(k)|$ is the ℓ^1 norm of the linear time-invariant (LTI) system T having unit pulse response $\{t(k)\}_{k=0}^{\infty}$; $\hat{T}(\lambda) := \sum_{k=0}^{\infty} t(k)\lambda^k$ is the λ-transform of T. For a vector valued signal $x = (x_1, x_2, \ldots, x_n)^T$, $\|x\| := \max_i \|x_i\|$ and for MIMO systems $T = \{T_{ij}\}$, $\|T\|_1 := \max_i \sum_j \|T_{ij}\|_1$; $\|S\|_{\infty-\infty} := \sup_{x \neq 0} \frac{\|Sx\|}{\|x\|}$ is the ℓ^∞-induced norm of a possibly time-varying and/or nonlinear system S (note that $\|S\|_{\infty-\infty} = \|S\|_1$ if S is LTI).

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II. Problem Definition and Solution

The basic problem we are concerned with is depicted in Figure 1 where s is a binary signal to be transmitted with $s(k) \in \{-1,1\}$ for all $k = 0,1,\ldots$; n is the noise with $|n(k)| \leq b$, where b is a known constant; $H = \{h_0, h_1, \ldots\}$ is an LTI system that represents the channel dynamics which are assumed known *a priori* for now. We want to accurately and causally reconstruct s via the receiver structure R, i.e., we would like to find what are the necessary and sufficient conditions for $\hat{s}(k) = s(k)$ for all times $k = 0,1,\ldots$, and what should the receiver structure R be.

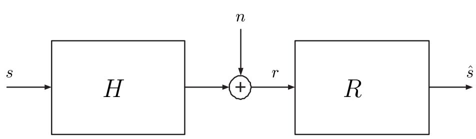

Fig. 1. Basic set-up

A. How to construct a perfectly reconstructing R

Our analysis in [10] showed that perfect reconstruction requires $|h_0| > b$. In this case, the construction of a receiver R (refer to Figure 1) can be obtained as the decision feedback equalizer (DFE) shown in Figure 2, where $F := H - h_0$ and Λ is the unit step delay operator.

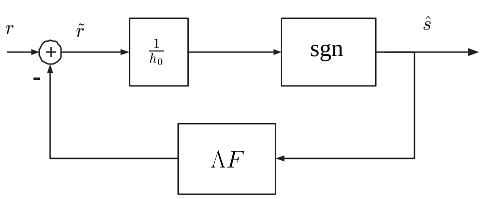

Fig. 2. DFE structure

The above setup and analysis can be generalized to cases where $s(k)$ belongs to a set of equally spaced numbers in $[-1,1]$. For instance, if $s(k) \in \{j/N,\ j = -N, -N+1, \ldots, 0, \ldots, N\}$, i.e., if there are $2N+1$ numbers equally spaced by intervals of size $1/N$, the condition for perfect reconstruction becomes $|h_0| > 2Nb$; the decision structure is an obvious extension of the structure in Figure 2.

B. Non-causal reconstruction

The case of non-causal reconstruction (smoothing) can also be considered in the same framework. In this case we are allowed to estimate $s(k)$ by incorporating K future receptions $r(k+1), \ldots, r(k+K)$ as well as $r(0),\ r(1), \ldots, r(k)$. In other words, we allow a delay of K steps in reconstructing s. The necessary and sufficient condition for perfect reconstruction is that there are no sequences s_1 and s_2 such that, if they are indistinguishable at any time t, they remain so for the next K time steps. Following the same line of argument as in [10], we obtain that the necessary and sufficient condition is

$$\min_{v(0)\neq 0, v(i)\in\{-1,0,1\}} \max\{|a(0)|, |a(1)|, \ldots, |a(K)|\} > b \tag{1}$$

where

$$\begin{pmatrix} a(0) \\ a(1) \\ \vdots \\ a(K) \end{pmatrix} := \begin{pmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & & \ddots & \\ h_K & \ldots & \ldots & h_0 \end{pmatrix} \begin{pmatrix} v(0) \\ v(1) \\ \vdots \\ v(K) \end{pmatrix}.$$

The above test for perfect reconstructability requires solving a mixed integer linear program [11]. For certain instances, such as one-step delay, analytical results can be obtained (see Section V). Also, one can prove the special case below.

Proposition 2.1: Assume that $|h_0| \geq |h_i|$ for all $i = 1,2,\ldots,K$. Then, s is perfectly reconstructible with delay K if and only if $|h_0| > b$.

The essence of the above proposition is that the use of non-causal reconstruction ($K \neq 0$) in this case does not offer any improvement (increase) on the maximum allowable noise bound for perfect reconstruction.

C. MIMO channels

Generalizations are also possible in the case of MIMO channels. In the case of m transmitters and p receivers the (equivalent) channel dynamics can be represented by a $p \times m$ transfer function H with pulse response $H = \{H_0, H_1, \ldots\}$, where each H_i is a $p \times m$ matrix. The motivation for problems of this sort comes from multiple antenna systems designed to combat fading channels and the detection of multiuser code division multiple access (CDMA) signals. The set-up is as before, i.e., $r = Hs + n$, where s and n are vector valued sequences such that $s_i(k) \in \{-1,1\}$ for all components s_i of s and $|n_i(k)| \leq b$ for all components n_i of n.

Under this setup, one can prove the following: if $v_i \in \{-1,0,1\}$, $i = 1,\ldots,m$, then the necessary and sufficient for perfect reconstruction causally in time is

$$\min_{v_i\in\{-1,0,1\}\text{ , not all equal } 0} \max\{|a_0|, |a_1|, \ldots, |a_p|\} > b, \tag{2}$$

where

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{pmatrix} = H_0 \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix}.$$

Checking the above condition is a mixed integer linear program (MILP) and a closed-form solution is, in general, hard to obtain. Finally, we would like to note that, as in the the SISO case, one can look at noncausal reconstruction for MIMO channels; the problem to solve is again an MILP.

D. Some remarks

In the case of noncausal reconstruction and/or MIMO channels the test for perfect reconstructability requires solving an MILP. In general, the construction of a perfectly reconstructing R can be quite complex. This is the motivation for the specific (DFE and linear) structures of reconstructors that we consider in the next section.

Finally, we mention that the case when the noise n enters through a "filter" W (sometimes called a coloring filter), i.e., when $r = Hs + Wn$, is analogous and has results similar in flavor. For example, in the case of causal reconstruction in a SISO channel, if $W = \{w_0, w_1, \ldots\}$, then the necessary and sufficient condition for perfect reconstruction is

$$|h_0| > b \sum_{i=0}^{\infty} |w_i| = b \, \|W\|_1 \, . \tag{3}$$

III. Reconstruction Based on DFEs

In the previous section, causal perfect reconstruction in SISO channels led naturally to a DFE structure. Herein we elaborate more on the optimality of such a structure for noncausal and MIMO problems.

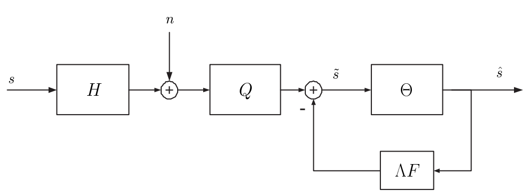

Fig. 3. General DFE structure for causal reconstruction

The general DFE structure is depicted in Figure 3 where Q is a (stable) linear forward filter $Q = \{q_0, q_1, \ldots\}$; ΛF is a feedback filter with Λ being the one-step delay operator (i.e., $\hat{\Lambda}(\lambda) = \lambda$) and $F = \{f_0, f_1, \ldots\}$; Θ is a thresholding operator that produces -1 or 1 depending on which one has the closest distance to $\tilde{s}$. In this particular case, $(\Theta\tilde{s})(k) = \text{ sgn } [\tilde{s}(k)]$. It should be clear that for this structure to perfectly reconstruct $s(k)$ causally in time for each time k it is necessary and sufficient that $\tilde{s}(k) > 0$ iff $s(k) > 0$ for all possible signal sequences s and all possible noise sequences n.

Under the perfect reconstruction requirement, the situation in Figure 3 can also be interpreted as in Figure 4: there are (stable) Q and F and an (arbitrarily small) $\epsilon > 0$ such that for all time-steps k with $s(k) = 1$

$$\tilde{s}(k) = ((QH - \Lambda F)s + Qn)(k) > \epsilon > 0$$

for all sequences s and $\|n\| \leq b$. Denoting by $X := QH - \Lambda F = \{x_0, x_1, \ldots\}$, we have that the above condition equivalently means

$$\tilde{s}(k) = \sum_{i=0}^{k} x_{k-i} s(i) + \sum_{i=0}^{k} q_{k-i} n(i) > \epsilon > 0$$

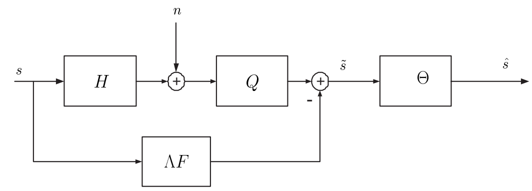

Fig. 4. Equivalent DFE structure under perfect reconstruction

for all times k and for all sequences s with $s(k) = 1$ and $\|n\| \leq b$ (for $s(k) = -1$ same condition can be easily obtained). Equivalently, we have that

$$x_0 \quad > \quad \sum_{i=1}^{k} |x_i| + b \sum_{i=0}^{k} |q_i| + \epsilon \quad \forall k = 1, 2, \ldots \tag{4}$$

Note that the above is also equivalent to

$$x_0 \quad > \quad \sum_{i=1}^{\infty} |x_i| + b \sum_{i=0}^{\infty} |q_i|. \tag{5}$$

Since the x_i's are (linear) functions of the q_i's and the f_i's, the problem of checking the above condition for a given $b > 0$ is a linear programming (LP) feasibility problem with infinite variables $\{q_i, f_i\}_{i=0}^{\infty}$. We now reduce Condition (5) to equivalent conditions.

Proposition 3.1: The following are equivalent:

(a) Condition (4) is satisfied for some Q and F.
(b) $\|(I - QH + \Lambda F \quad Qb)\|_1 < 1$ for some Q and F.
(c) There are Q and F so that, for all k, $|s(k) - \tilde{s}(k)| < 1 - \epsilon'$ for some $\epsilon' > 0$ and all s and n with $\|n\| \leq b$.

Based on the above proposition we can define the relevant ℓ_1-optimization

$$\mu \quad := \quad \inf_{Q,F} J_b,$$

where

$$J_b \quad = \|(I \quad 0) - (QH - \Lambda F \quad Qb)\|_1 \, .$$

Then, the problem of perfect reconstruction has a DFE solution if and only if $\mu < 1$. If we denote $G := QH = \{g_0, g_1, \ldots\}$, it follows from the structure of the ℓ^1 norm, that to minimize J_b, the optimal F for any Q should be selected as $F = \{g_1, g_2, \ldots\}$ so that ΛF cancels all terms in G except the feed-through term $g_0 = q_0 h_0$ (any other choice for F will increase J_b). Given that, it also follows that the minimizing Q has to be a constant $Q = q_0$, where q_0 minimizes

$$\mu = \min_{q_0} |1 - q_0 h_0| + b|q_0| \, .$$

This gives us $\mu = 1$ if $|h_0| \leq b$ and $\mu = b/|h_0|$ if $|h_0| > b$. The minimizing Q is $Q = 0$ and $Q = \frac{1}{h_0}$ respectively. As expected, the DFE structure leads to the same conclusion for perfect reconstructability as in the previous section.

A point to be made here is that having $J_b > 1$ does not necessarily imply that the DFE structure with the particular Q and F does not perfectly reconstruct s, unless $0 < g_0 = q_0 h_0 \leq 1$. If $g_0 > 1$, Condition (5) needs to be checked to determine whether perfect reconstruction is possible at the given noise level b.

A. Noncausal reconstruction

To capture K-step noncausal reconstruction we allow the forward filter to be of the form $\Lambda^{-K}Q$ (i.e., noncausal). It then follows that the problem of interest is to find

$$\mu \;=\; \inf_{Q,F} J_b \;,$$

where now

$$J_b \;=\; \sup_{\|s\|\le 1, \|n\|\le b} \left\| [(I \quad 0) - (\Lambda^{-K}QH - \Lambda F \quad \Lambda^{-K}Q)] \begin{pmatrix} s \\ n \end{pmatrix} \right\| .$$

Again, it is true that the best F, for any choice of Q in the above optimization, is to cancel the coefficients of $G = QH$ of order $K+1$ and above, i.e., $F = \{g_{K+1}, g_{K+2}, \ldots\}$. With this done, it also follows that the minimizing choice for Q is FIR of order K, i.e., $Q = \{q_0, q_1, \ldots, q_K, 0, \ldots\}$ with the parameters $q_0, \ldots, q_K$ solving the linear program

$$\mu = \min_{q_0,\ldots,q_K} |1 - g_K| + b|q_K| + \sum_{i=0}^{K-1} (|g_i| + b|q_i|). \qquad (6)$$

Hence the DFE structure will perfectly reconstruct s with a delay of K steps if and only if the linear programming problem in (6) leads to a cost $\mu < 1$.

A natural question to answer is how closely the maximum allowable noise bound b for perfect reconstructability with DFE structure relates to the absolute bound of Condition (1). For the one-step delay case ($K = 1$) it can be shown that the bounds are the same.

Proposition 3.2: With one-step delay, perfect reconstruction is possible with a DFE if and only if

$$b \;<\; \begin{cases} |h_0| & \text{if} \quad |h_1| \le 2|h_0| \;, \\ |h_1| - |h_0| & \text{if} \quad |h_1| > 2|h_0| \;. \end{cases}$$

Moreover, the DFE structure is an optimal one for the problem.

For higher order channels however, these bounds can differ as the following example illustrates.

Example 3.1: Consider a (possibly IIR) channel H with the first three coefficients being $h_0 = 1$, $h_1 = 1.5$, $h_2 = -1.75$. By solving for Condition (1) for a two-step delay ($K = 2$), we obtain that $b < 1.5$ for perfect reconstruction (this can be done by hand in this simple case by checking all the possibilities —essentially nine choices for $v(1)$ and $v(2)$— in the underlying MILP).

On the other hand, if we restrict ourselves to the DFE structure and check for $\mu < 1$ in Condition (6) we obtain the condition $b < 1.2$ (this can also be done by hand).

B. MIMO channels

In the case of MIMO channels with a DFE receiver structure one needs to check whether $\|s_i - \tilde{s_i}\| < 1$ for *all* of the i source data s_i transmitted. In particular, the relevant problem to solve is

$$\mu \;=\; \inf_{Q,F} \|(I \; 0) - (QH - \Lambda F \quad QB)\|_1,$$

where all systems are MIMO and B is a scaling noise matrix with $B := \mathrm{diag}(b_i)$, where b_i is the noise bound on channel i. The DFE perfectly reconstructs s causally in time iff $\mu < 1$.

Using the same arguments as in the previous two subsections, a minimizing F is $F = \{G_1, G_2, \ldots\}$ where $G = QH$, and a minimizing Q is $Q = Q_0$, a constant matrix that solves the finite-dimensional LP

$$\mu \;=\; \min_{Q_0} \|(I - Q_0 H_0 \qquad - Q_0 B)\|_1 \,. \qquad (7)$$

In general, assuming (for simplicity) that $B = \mathrm{diag}(b)$, the maximum bound on the noise b for perfect reconstructability obtained by requiring that $\mu < 1$ can be different (smaller) than the absolute bound of Condition (2) as the following example illustrates.

Example 3.2: Consider a MIMO 2×2 channel H with the feed-through term $H_0 = \begin{pmatrix} 8 & 4 \\ 12 & -3 \end{pmatrix}$. Condition (2) leads to $b < 4$ for perfect reconstruction (the checking can be done by hand by checking all possibilities in the the MILP of Condition (2)).

On the other hand, employing the DFE structure and checking for $\mu < 1$ in (7) leads to the existence of $Q_0 = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$ satisfying

$$\begin{aligned} 8q_1 + 12q_2 &> |4q_1 - 3q_2| + b(|q_1| + |q_2|) \;, \\ 4q_3 - 3q_4 &> |8q_3 + 12q_4| + b(|q_3| + |q_4|) \;. \end{aligned}$$

It can be easily checked (again by hand) that, while the first inequality can be satisfied for any $b < 4$, the second inequality requires that $b < 3.6$. In other words, s_1 (the first component of s) is perfectly reconstructible causally with a DFE for $3.6 \le b < 4$, while s_2 (the second component of s) is not.

Note that noncausal reconstruction problems lead to analogous results as in the scalar case. The relevant problem to consider for K-step delayed reconstruction is the (finite-dimensional) LP

$$\begin{aligned} \mu \;=\; \min_{Q_0,\ldots,Q_K} \|(I - G_K \quad - G_{K-1} \quad \ldots & \qquad (8) \\ \ldots \quad -G_0 \quad - Q_K B \quad \ldots \quad - Q_0 B)\|_1 < 1 \,. & \end{aligned}$$

C. Reconstruction Based on Linear Equalizers

In this case we restrict the structure of R to be $R = Q$ where $Q = \{q_0, q_1, \ldots\}$ is a linear time-invariant filter. This is a special case of the DFE structure when F is set to zero, thus all of the previous discussion carries over.

IV. Robustness to Channel Uncertainty

In the previous section we investigated DFE structures under the assumption that the channel H is known. We

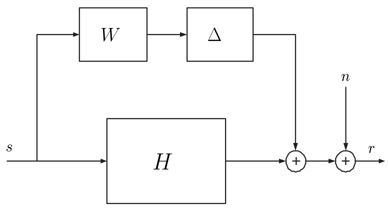

Fig. 5. A fading channel model

now consider the case of uncertain channel dynamics which we model as in Figure 5. The uncertainty here is given in terms of an additive weighted block ΔW, where Δ is assumed to be an unknown perturbation, possibly time-varying and even nonlinear, that has a bounded ℓ^∞ to ℓ^∞ norm $\|\Delta\|_{\infty-\infty} \leq 1$. The weight W is a known stable LTI dynamical system that may reflect magnitude normalizations and partial information on the magnitude of the uncertainty over different frequencies (i.e., it "shapes" the uncertainty block).

As an example, consider the actual channel H_a as $H_a = H + E$ where $H = \{h_0, h_1, \ldots\}$ is the nominal LTI channel and E represents time-varying perturbations on the parameters of H leading to a response

$$(H_a s)(k) = \sum_{i=0}^{k} (h_{k-i} + \epsilon_{ki}) s(i)$$

with the perturbation ϵ_{ki} bounded as $|\epsilon_{ki}| \leq \epsilon_i$ for all $k = 0, 1, 2, \ldots$, but otherwise arbitrary. If $\sum_{i=0}^{\infty} \epsilon_i = \epsilon$, then this amounts to modeling E as $E = \Delta W$ with $W = \epsilon$ and $\|\Delta\|_{\infty-\infty} \leq 1$. Similarly, if the first N channel coefficients are not changing ($\epsilon_0 = \ldots = \epsilon_{N-1} = 0$) but there is uncertainty in the higher order terms, then $\hat{W}(\lambda) = \epsilon \lambda^N$.

We note that this uncertainty formulation is different in nature than what is typically assumed in the stochastic framework (e.g., Chapter 14 in [1]). However, we believe that it captures a number of relevant fading phenomena due to time variations and can be used to design reliable reconstruction algorithms.

Let J_a represent the ℓ^∞ induced norm of the map $\binom{s}{n} \to s - \tilde{s}$, i.e., $J_a = \left\| \binom{s}{n} \to s - \tilde{s} \right\|_{\infty-\infty}$. This is a function of the uncertainty Δ. We assume that when no uncertainty is present, i.e., when $\Delta = 0$, $J_a < 1$ and hence perfect reconstruction is achieved for the specific DFE parameters Q and F. Note that in this case $J_a = J$ where J represents the ℓ^1 norm of the (nominal) map $\binom{s}{n} \to s - \tilde{s}$, i.e., $J = \left\| \binom{s}{n} \to s - \tilde{s} \right\|_1$. What we would like to ensure is robust performance (RP) in the presence of all possible $\|\Delta\|_{\infty-\infty} \leq 1$, i.e., we want to find conditions such that

$$J_a < 1 \quad \text{for all} \quad \|\Delta\|_{\infty-\infty} \leq 1.$$

If we denote $\Phi_1 := I - QH + \Lambda F$ and $\Phi_2 := -Qb$, then from the definition of the 1-norm we have

$$J \;=\; \|\Phi_1\|_1 + \|\Phi_2\|_1 = \|(\Phi_1 \;\; \Phi_2)\|_1 .$$

Let $\bar{M}$ be

$$\begin{aligned} \bar{M} \;&:=\; \begin{pmatrix} 0 & \|(W\;0)\|_1 \\ \|-Q\|_1 & \|(\Phi_1\;\Phi_2)\|_1 \end{pmatrix} \\ &=\; \begin{pmatrix} 0 & \|W\|_1 \\ \|Q\|_1 & J \end{pmatrix}. \end{aligned}$$

By redrawing the set up as in Figure 6, it can be shown [4] that RP is obtained if and only if

$$\rho(\bar{M}) < 1 \quad \Leftrightarrow \quad \frac{J + \sqrt{J^2 + 4\|W\|_1\|Q\|_1}}{2} < 1 ,$$

where $\rho(\bar{M})$ is the spectral radius of $\bar{M}$ (largest eigen-

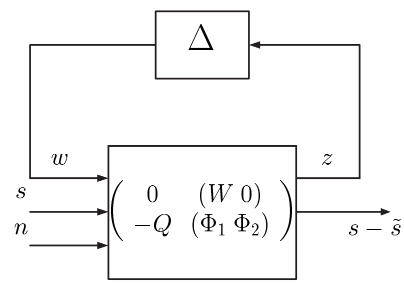

Fig. 6. Robustness analysis loop

value.) Equivalently, $\|W\|_1\|Q\|_1 + J < 1$. Note that the left side is equal to

$$\begin{aligned} \bar{J} \;&:=\; \|(I\;0\;0) - (QH - \Lambda F \quad Qb \quad \|W\|_1 Q)\|_1 \\ &=\; \|(I\;\;0) - (QH - \Lambda F \quad (b + \|W\|_1)Q)\|_1. \end{aligned}$$

Hence, RP optimization is equivalent to minimizing $\left\| \binom{s}{\bar{n}} \to s - \tilde{s} \right\|_1$, where $\bar{n}$ is bounded as $\|\bar{n}\| \leq b + \|W\|_1$. That is, one has to consider the nominal channel with the "equivalent" noise bound given as $b + \|W\|_1$. Based on the results of the previous section, perfect causal reconstruction is possible with a DFE iff $|h_0| > b + \|W\|_1$.

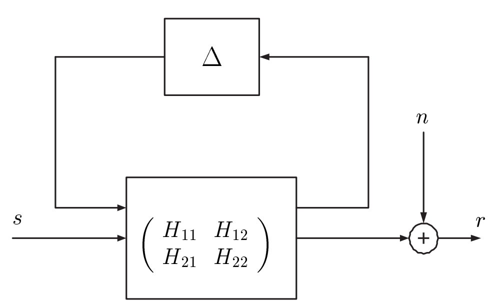

Fig. 7. A more general fading model

A more general situation is depicted in Figure 7 where $H_{22} = H$ and H_{ij} can be general (stable) dynamical LTI systems that connect the nominal channel with the sources of dynamical uncertainty lumped in Δ. For example, consider a channel $H = N_H D_H^{-1}$, where $N_H = N + W_N \Delta_N$ and $D_H = D + W_D \Delta_D$ with D, N being the nominal "numerator" and "denominator" respectively (i.e., $H = ND^{-1}$). Assume that D and N are coprime [4], and that Δ_N and Δ_D are normalized perturbations with known shaping weights W_N, W_D. Then, putting it in the mold of Figure 6 the following identification is obtained

$$H_{11} = (0 \;\; -D^{-1}W_D) \quad , \quad H_{12} = D^{-1}$$
$$H_{21} = (W_N \;\; -HW_D) \quad , \quad H_{22} = H, \; \Delta = \begin{pmatrix} \Delta_N \\ \Delta_D \end{pmatrix}.$$

To analyze robustness, we can redraw the system and use the same spectral radius condition as before [4]. By letting

$$\begin{aligned} \bar{M} &= \begin{pmatrix} \|H_{11}\|_1 & \|(H_{12} \; 0)\|_1 \\ \|QH_{21}\|_1 & \|(\Phi_1 \; \Phi_2)\|_1 \end{pmatrix} \\ &= \begin{pmatrix} \|H_{11}\|_1 & \|H_{12}\|_1 \\ \|QH_{21}\|_1 & J \end{pmatrix}, \end{aligned}$$

we conclude that for perfect reconstruction for all possible perturbations $\|\Delta\|_{\infty-\infty} \le 1$ (i.e., $\|\Delta_N\|_{\infty-\infty} \le 1$, $\|\Delta_D\|_{\infty-\infty} \le 1$), it is necessary and sufficient that there exist Q , F such that $\rho(\bar{M}) < 1$, which equivalently leads to $\|H_{11}\|_1 < 1$ and

$$\begin{aligned} \bar{J} \;\; = \;\; & \|(I \; 0 \; 0) - \\ & \left.(QH - \Lambda F \quad Qb \quad \frac{\|H_{12}\|_1}{1 - \|H_{11}\|_1} QH_{21})\right\|_1 < 1. \end{aligned}$$

For the case of noncausal reconstruction with K-step delay, the same arguments hold and lead to the following necessary and sufficient condition for RP: $\|H_{11}\|_1 < 1$.

To check whether there exist Q, F so that the DFE perfectly reconstructs s in the presence of unmodeled dynamics, the above conditions lead to the following ℓ^1-optimization problem:

$$\mu = \inf_{Q,F} \bar{J} < 1 \;,$$

assuming $\|H_{11}\|_1 < 1$ holds (which can be easily checked since H_{11} does not depend on the DFE).

For this problem it is clear that, for any choice of Q, filter F should cancel the coefficients of $G = QH$ of order $K+1$ and above, i.e., $F = \{g_{K+1}, g_{K+2}, \ldots\}$. In general, the optimal Q may not be FIR as the LP is infinite-dimensional. Nonetheless, arbitrarily close to optimal solutions can be obtained using standard ℓ^1 methods.

V. First-Order FIR Channels

In this section we consider the case of a first-order FIR channel given by $H = \{h_0, h_1, 0, 0, \ldots\}$. This could represent simple channel dynamics in a wireless-communications scenario. We provide some analytical results on perfect reconstructability as well as some analytical and simulation results on the probability of error.

A. Linear equalization for first-order FIR channels

Herein we provide a closed-form result for perfect linear reconstruction. Recall that the underlying problem to solve for K-step delayed reconstruction is

$$\mu \;\; = \;\; \inf_Q \left\| \Lambda^K (I \quad 0) - Q(H \quad bI) \right\|_1 . \tag{9}$$

As indicated earlier; this is a standard ℓ^1-optimization problem. We will call Q° an optimal ℓ^1 linear reconstructor if it solves the above optimization.

1) Causal reconstruction: For the case of causal reconstruction ($K = 0$) we have the following result.

Proposition 5.1: Let H be a first-order channel $\hat{H}(\lambda) = h_0 + h_1\lambda$ and let $\omega := \frac{|h_1|}{|h_0|}$, $\alpha := \frac{b}{|h_0|}$. Then, the optimal estimator Q° and its associated optimal cost μ is

$$\begin{array}{lll} Q^\circ = H^{-1}, & \mu = \frac{\alpha}{1-\omega} < 1, & \text{whenever } \; \omega + \alpha < 1, \\ Q^\circ = 0, & \mu = 1, & \text{whenever } \; \omega + \alpha \ge 1. \end{array}$$

From the above proposition, we see that the noise level b should be $b < |h_0| - |h_1|$ for perfect reconstruction as opposed to $b < |h_0|$ obtained from the unrestricted nonlinear R or the optimal DFE. The conservatism is expected due to the restricted structure considered.

When $s(k)$ belongs to a set of equally spaced numbers in $[-1, 1]$ (for instance, if $s(k) \in \{j/N, \; j = -N, -N+1, \ldots, 0, \ldots, N-1, N\}$), the same approach leads to the condition $b < \frac{|h_0| - |h_1|}{2N}$ (the thresholding device now changes to produce the closest j/N to $\tilde{s}(k)$). Again, this is more conservative than the condition $b < \frac{|h_0|}{2N}$ in the unrestricted case.

2) Noncausal reconstruction: Results for one-step delayed reconstruction ($K = 1$) can also be obtained.

Proposition 5.2: Let H be a first-order channel $\hat{H}(\lambda) = h_0 + h_1\lambda$ with $\omega := \frac{|h_1|}{|h_0|}$, $\alpha := \frac{b}{|h_0|}$. Then, the optimal Q° and its associated optimal cost μ is

$$\begin{array}{lll} Q^\circ = \Lambda H^{-1}, & \mu = \frac{\alpha}{1-\omega} < 1, & \text{whenever } \; \omega + \alpha < 1, \\ Q^\circ = \frac{1}{h_1}, & \mu = \frac{1+\alpha}{\omega} < 1, & \text{whenever } \; \omega + \alpha \ge 1 \\ & & \text{and } \omega - \alpha > 1, \\ Q^\circ = 0, & \mu = 1, & \text{otherwise.} \end{array}$$

Recall that in the case of one-step delayed reconstruction using a DFE structure, perfect reconstruction is achieved if and only if $b < \nu$ where

$$\nu \;\; := \;\; \begin{cases} |h_0| & \text{if} \quad |h_1| \le 2|h_0|, \\ |h_1| - |h_0| & \text{if} \quad |h_1| > 2|h_0|. \end{cases}$$

Note that both linear and nonlinear reconstruction give the same bound for $|h_1| \ge 2|h_0|$. In the region $0 < |h_1| < 2|h_0|$, the linear is clearly outperformed. In fact, for $|h_1| = |h_0|$ the linear is very sensitive to any nonzero noise.

Finally, we would also like to mention that, in the case of a first-order channel, any additional delay K does not improve the bound ν.

Proposition 5.3: Let H be a first-order channel $\hat{H}(\lambda) = h_0 + h_1\lambda$. Then, s is perfectly reconstructible with K-step delay if and only if it is perfectly reconstructible with one-step delay. Moreover, if $|h_1| > 2|h_0|$ reconstruction can be performed using a linear structure.

We compare the performance of the above filters using simulations based on 30000 trials; the results are depicted in Figure 8. We can see that the DFE structure reconstructs perfectly in the given bounds and performs better than the other two structures.

B. Explicit probability of error calculations

As we see from Figure 2, in binary pulse amplitude modulated (PAM) input signal, our decoder (which obtains $\hat{s}(k)$ from $\tilde{r}(k)$) is just a sign function. Without loss of generality

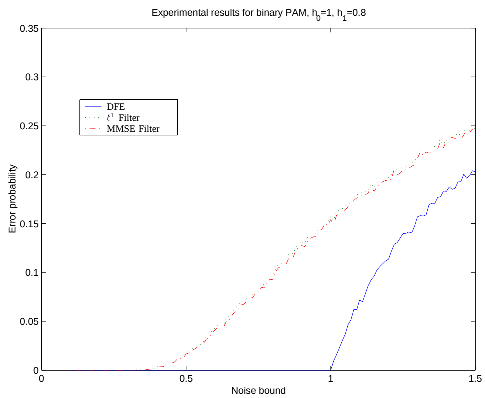

Fig. 8. Error Probability as a function of the uniform noise bound b for binary PAM.

we can assume that $h_0 > 0$ and $\hat{s}(k) = \operatorname{sgn}[\tilde{r}(k)]$. Note that "probability of error" is meaningless unless b is larger than the bound required for perfect reconstruction (so that errors occur).

Proposition 5.4: In the long run, the probability of getting an error for binary PAM with uniform noise bound b and first-order FIR channel dynamics using the DFE structure of Figure 2 is given by

$$P_e = \begin{cases} 0, & b \le h_0, \\ \frac{2(b-h_0)}{5b-h_0-2h_1}, & h_0 + 2h_1 \ge b > h_0, \\ \frac{b-h_0}{2b}, & b > h_0 + 2h_1. \end{cases} \tag{10}$$

Our experimental results verify the validity of Proposition 5.4.

VI. Concluding Remarks

We have presented a deterministic formulation of various communication problems. Our approach leads to exact magnitude bounds on the noise level for which perfect causal or delayed reconstruction of the transmitted symbols is possible. It also allows for the synthesis of perfect reconstructing structures, despite possibly time-varying uncertainty that is present in the channel. The main structure that was studied was a DFE structure; our framework provides connections between the limiting performance of DFEs and ℓ^1-optimization, a subject that has been thoroughly studied in the context of robust control. We parenthetically mention here that the (optimal) linear equalizer given for first-order FIR channels is, by itself, a contribution to ℓ^1 model-matching theory as it is a closed-form solution to a so-called two-block problem for which, to the best of our knowledge, no closed-form solutions are available.

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