Minimum Cost Optimal Control: An Application to Flight Level Tracking

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Abstract— **In earlier work we established a link between finite time viability and invariance for continuous systems and the viscosity solutions to partial differential equations which are variants of the standard Hamilton-Jacobi equation. In this paper we show how these results can be applied to address the problem of ensuring that an aircraft does not deviate from the flight level assigned to it by air traffic control. The application illustrates the advantages of the proposed viability characterisation: it makes the numerical solution of the problem easier!**

Keywords— **Optimal control, reachability, invariance, flight control, numerical methods.**

I. Introduction

Because of their importance in applications ranging from engineering to biology and economics, questions of reachability, viability and invariance have been studied extensively in the dynamics and control literature. Most recently, the study of these concepts has received renewed attention through the study of safety problems in hybrid systems. Reachability computations have been used in this context to address problems in the safety of ground transportation systems [12], [15], air traffic management systems [11], [25], [26], flight control [16], [21], etc.

Direct characterisation of reachability concepts is one of the topics addressed by viability theory [1]. The development of computational tools to support the numerous viability theory methods is an ongoing effort (see, for example, [6]). An alternative, indirect approach to reachability questions is using optimal control methods. In this case, the reachable, viable, or invariant sets are characterised as level sets of the value function of an appropriate optimal control problem. Using dynamic programming, the value function can in turn be characterised as the solution to a partial differential equation.

In earlier work we have demonstrated how reachability questions can be encoded as optimal control problems where the cost is the minimum of a function of the state over a given horizon [13], [14]. The objective of the controller is either to maximise this quantity (SupMin problem), or to minimise it (InfMin problem). We also showed how to characterise the two value functions as viscosity solutions to first order partial differential equations, which are variants of the standard Hamilton-Jacobi equation. An overview of these results is given in Section II.

The main advantage of this approach is that the resulting partial differential equations have very good properties in terms of their numerical solution. The value functions of the optimal control problems we use can be shown to be uniformly continuous. Moreover, they are characterised as solutions to partial differential equations in the standard Hamilton-Jacobi form, with continuous Hamiltonians and simple boundary conditions. Therefore, very efficient algorithms developed for this class of equations [22], [23], [19], [17], whose properties have been extensively tested in theory and in applications, can be directly applied to our problem.

In this paper we demonstrate how one can take advantage of these properties to numerically solve a problem of flight level tracking. The problem is to ensure that an airliner does not deviate excessively from the flight level assigned to it by air traffic control. Excessive deviations are dangerous since they may bring the aircraft in conflict with other aircraft moving at different flight levels (and typically different directions). At the same time, the aircraft has to ensure that it maintains certain bounds on its speed and flight path angle (to avoid stall, for reasons of passenger comfort, etc.) In Section III we show how all these constraints can be encoded in an appropriate SupMin optimal control problem. We then use the numerical algorithms of [22], [23] coded by [19], [17] to compute the solution to the problem.

II. SupMin and InfMin Optimal Control

A. Statement of the Problems

Consider a continuous time control system,

$$\dot{x} = f(x,u) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $f(\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n$, a function,

$$l(\cdot) : \mathbb{R}^n \to \mathbb{R}, \tag{2}$$

and an arbitrary time horizon, $T \geq 0$. Let $\mathcal{U}_{[t,t']}$ denote the set of Lebesgue measurable functions from the interval $[t,t')$ to U. To eliminate technical difficulties we impose the following standing assumption.

Assumption 1: $U \subseteq \mathbb{R}^m$ is compact. f and l are bounded and uniformly continuous.

Under Assumption 1 system (1) with initial condition $x(t) = x \in \mathbb{R}^n$ admits a unique solution $x(\cdot) : [t,T] \to \mathbb{R}^n$ for all $t \in [0,T]$, $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}_{[t,T]}$.

We introduce two optimal control problems with value functions $V_1 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ given by

$$V_1(x,t) = \sup_{u(\cdot)\in\mathcal{U}_{[t,T]}} \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot))) \tag{3}$$

$$V_2(x,t) = \inf_{u(\cdot)\in\mathcal{U}_{[t,T]}} \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot))). \tag{4}$$

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The minimum with respect to time is well defined by continuity. In the first problem the objective of the input u is to maximise the minimum value attained by the function l along the state trajectory over the horizon $[t, T]$. In the second problem, on the other hand, the objective of u is to minimise this minimum. For obvious reasons we will subsequently refer to the first optimal control problem as the SupMin problem and to the second problem as the InfMin problem.

B. Connection to Reachability

Given the control system of equation (1), the horizon $T \geq 0$ and a set of states $K \subseteq \mathbb{R}^n$, a number of questions can be naturally formulated regarding the relation between the set K and the state trajectories of (1) over the horizon T. Problems of interest include the following.

Viability Does there exist a $u(\cdot) \in \mathcal{U}_{[0,T]}$ for which the trajectory $x(\cdot)$ satisfies $x(t) \in K$ for all $t \in [0, T]$?

Invariance Do the trajectories $x(\cdot)$ for all $u(\cdot) \in \mathcal{U}_{[0,T]}$ satisfy $x(t) \in K$ for all $t \in [0, T]$?

Reachability Does there exist a $u(\cdot) \in \mathcal{U}_{[0,T]}$ and a $t \in [0, T]$ such that the trajectory satisfies $x(t) \in K$?

As usual, K^c stands for the complement of the set K in $\mathbb{R}^n$. One would typically like to characterise the set of initial states for which the answer to the viability/invariance/reachability questions is "yes". Or, more generally, one would like to characterise the sets

$$
\begin{aligned}
\text{Viab}(t, K) = \{x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathcal{U}_{[t,T]} \; \forall \tau \in [t, T] \\
x(\tau) \in K\} \\
\text{Inv}(t, K) = \{x \in \mathbb{R}^n \mid \forall u(\cdot) \in \mathcal{U}_{[t,T]} \; \forall \tau \in [t, T] \\
x(\tau) \in K\} \\
\text{Reach}(t, K) = \{x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathcal{U}_{[t,T]} \; \exists \tau \in [t, T] \\
x(\tau) \in K\},
\end{aligned}
$$

Notice that $\text{Reach}(t, K) = (\text{Inv}(t, K^c))^c$, therefore, the invariance and reachability problems are duals of one another and need not be treated separately.

There is in fact a direct connection between these sets and the solutions to the SupMin and InfMin optimal control problems. If we chose $l(\cdot)$ such that $K = \{x \in \mathbb{R}^n \mid l(x) > 0\}$ and $L = \{x \in \mathbb{R}^n \mid l(x) \geq 0\}$ then one can show the following.

Proposition 1: $\text{Viab}(t, K) = \{x \in \mathbb{R}^n \mid V_1(x, t) > 0\}$ and $\text{Inv}(t, L) = \{x \in \mathbb{R}^n \mid V_2(x, t) \geq 0\}$.

C. Alternative Characterisations

We first point out that the set $\text{Inv}(t, K)$ can be computed using the standard Hamilton-Jacobi-Bellman equation (see [27], [16] for more on this observation). Consider again that the closed set $L = \{x \in \mathbb{R}^n \mid l(x) \geq 0\}$ and let

$$V_3(x, t) = \inf_{u(\cdot) \in \mathcal{U}_{[t,T]}} l(x(T)).$$

A standard optimal control argument (see for example [10], [7]) shows that V_3 is a viscosity solution for the terminal value problem

$$\frac{\partial V_3}{\partial t}(x, t) + \inf_{u \in U} \frac{\partial V_3}{\partial x}(x, t) f(x, u) = 0 \tag{5}$$

with $V_3(x, T) = l(x)$ over $(x, t) \in \mathbb{R}^n \times [0, T]$.

Proposition 2: For all $(x, t) \in \mathbb{R}^n \times [0, T]$, $V_2(x, t) = \min_{\tau \in [t,T]} V_3(x, \tau)$. Moreover, $\text{Inv}(t, L) = \bigcap_{\tau \in [t,T]} \{x \in \mathbb{R}^n \mid V_3(x, \tau) \geq 0\}$.

Proposition 2 shows that one can compute $\text{Inv}(t, L)$ by solving a standard Hamilton-Jacobi-Bellman equation (5) and then taking the intersection of the level sets of the solution (or, equivalently, computing the minimum of the value function over time horizon $[t, T]$).

Unfortunately this approach will not work for the SupMin problem. There are, however, other methods in the optimal control literature that can be adapted to characterise the set $\text{Viab}(t, K)$. For example, one can treat the problem as maximising the "exit time" from the set K. It can be shown [10], [3] that this involves solving a standard Hamilton-Jacobi-Bellman equation over the set K (and possibly pieces of its boundary), with rather complicated boundary conditions (e.g. "freezing" of the value function at certain parts of the boundary of K). Moreover, the value function will not be continuous in general. These features suggest that numerical computations are likely to be difficult with this approach.

Another approach was proposed in [25] in the context of differential games. This approach involves a Hamilton-Jacobi equation with a discontinuous Hamiltonian. Continuity of the Hamiltonian is desirable, because it greatly simplifies both the theoretical analysis and the numerical solution of the partial differential equation.

The approach most closely related to the one discussed here is that of [5], [9], where a generalised version of the SupMin optimal control problem is formulated and solved. Related work on differential games includes [4] (extending the results of [5]) and [18] (based on the classical results of [8]). In [5] the value function of the problem is shown to satisfy a set of discontinuous, quasi-variational inequalities. Though this approach is conceptually appealing, the discontinuity and the implicit dependence of the Hamiltonian on the value function severely limit its usefulness from the numerical computation point of view (as the authors of [5] point out). The authors of [9] simplify this characterisation to a continuous variational inequality. In [9] specialised numerical schemes were developed to exploit the continuity of the Hamiltonian to numerically approximate the solutions of the inequalities.

While all these approaches are sound in theory, none is entirely satisfactory from the point of view of numerical computation. Some involve discontinuous Hamiltonians [5], [25], while others require discontinuous viscosity solutions to be computed over complicated domains with complicated boundary conditions [10], [3]. The best approach in this respect seems to be that of [9], which involves a continuous value function charactersied as a viscosity solution to a continuous variational inequality. The drawback is that the characterisation is not in terms of a standard

Hamilton-Jacobi equation; this implies that specialised numerical tools have to be developed. In the next section we present a solution to the SupMin problem (and hence a charactersiation of the set $\mathrm{Viab}(t, K)$) where the Hamiltonian is not only continuous, but the equation is also is in the standard Hamilton-Jacobi form and can therefore be approached numerically using well established schemes for solving these types of equation.

D. Solution to the SupMin *Problem*

The solutions to the SupMin and InfMin problems turn out to be very similar; for the most part the only difference between the two characterisations is replacing sup's by inf's in the equations. Since the numerical example in the next section relies on the SupMin characterisation, we give the results only for this case. Results in this direction were first reported in [13]; a complete discussion can be found in [14].

First, we note that the value function V_1 satisfies the following version of the optimality principle.

Lemma 1: For all $(x,t) \in \mathbb{R}^n \times [0,T]$ and all $h \in [0, T-t]$:
1. $V_1(x,t) \leq V_1(x, t+h)$ and $V_1(x,T) = l(x)$.
2. $V_1(x,t) = \sup_{u(\cdot)\in\mathcal{U}_{[t,t+h]}} [\min\{\min_{\tau\in[t,t+h]} \; l(x(\tau)), V_1(x(t+h), t+h)\}]$.

Lemma 1 makes two assertions. The first is that the "value" of a given state x can only decrease as the "time to go" increases. Starting from x the minimum value that l experiences over a certain time horizon is less than or equal to the minimum value that l would experience if we stopped the evolution at any time before the horizon expires. The second part of the lemma is a variant of the standard principle of optimality: it relates the optimal cost to go from (x,t) to the optimal cost to go from $(x(t+h), t+h)$ and the minimum value experienced by l over the interval $[t, t+h]$.

Under Assumption 1, the value function V_1 turns out to be bounded and uniformly continuous (see, for example, Proposition 3.1 of [5]).

Lemma 2: There exists a constant $C > 0$ such that $|V_1(x,t)| \leq C$ and $|V_1(x,t) - V_1(\hat{x},\hat{t})| \leq C(|x-\hat{x}| + (t-\hat{t}))$, for all $(x,t), (\hat{x},\hat{t}) \in \mathbb{R}^n \times [0,T]$.

Based on these two lemmas, the following characterisation of V_1 was derived in [13].

Theorem 1: V_1 is the unique bounded and uniformly continuous viscosity solution of the terminal value problem

$$\frac{\partial V}{\partial t}(x,t) + \min\left\{0, \sup_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u)\right\} = 0$$

over $(x,t) \in \mathbb{R}^n \times [0,T]$ with boundary condition $V(x,T) = l(x)$.

Finally, it is easy to show that if V_1 is used to characterise the set $\mathrm{Viab}(t,K)$ the result is independent of the function l used to characterise the set K.

Proposition 3: Let $l : \mathbb{R}^n \to \mathbb{R}$ and $\hat{l} : \mathbb{R}^n \to \mathbb{R}$ be two uniformly continuous, bounded functions such that $\{x \in \mathbb{R}^n \mid l(x) > 0\} = \{x \in \mathbb{R}^n \mid \hat{l}(x) > 0\}$. Let $V_1 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $\hat{V}_1 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be the viscosity solutions of the terminal value problem of Theorem 1 with boundary conditions $V_1(x,T) = l(x)$ and $\hat{V}_1(x,T) = \hat{l}(x)$ respectively. Then $\{x \in \mathbb{R}^n \mid V_1(x,t) > 0\} = \{x \in \mathbb{R}^n \mid \hat{V}_1(x,t) > 0\}$ for all $t \in [0,T]$.

III. Flight Level Control: A Numerical Study

To illustrate the application of the above results, we consider the problem of maintaining an aircraft at a desired flight level. Commercial aircraft at cruising altitudes are typically assigned a flight level by Air Traffic Control (ATC). The flight levels are separated by a few hundred feet (e.g. 500 or 1000, depending on altitude and the type of airspace). Air traffic moves in different directions at different flight levels (north to south in one level, east to west in another, etc.). This arrangement is desirable because it greatly simplifies the task of ATC: the problem of ensuring aircraft separation, which is normally three dimensional, can most of the time be decomposed to a number of two dimensional (in some places even one dimensional) problems.

Changes in the flight level happen occasionally and have to be cleared by ATC. At all other times the aircraft have to ensure that they remain within certain bounds (e.g. ± 250 feet) of their assigned level. At the same time, they also have to maintain limits on their speed, flight path angle, acceleration, etc. imposed by limitations of the engine and airframe, passenger comfort requirements, or to avoid dangerous situations such as aerodynamic stall. In this section we formulate a a SupMin optimal control problem that allows us to address such constraints.

A. Aircraft Model

We restrict our attention to the movement of the aircraft in the vertical plane and describe the motion using a point mass model. Such models are commonly used in ATC research (see, for example, [16], [20]). They are fairly simple, but still capture the essential features of aircraft flight. The analysis presented here is an extension to three dimensions of an aerodynamic envelope protection problem studied in [16].

Three coordinate frames are used to describe the motion of the aircraft: the ground frame (X_g–Y_g), the body frame (X_b–Y_b) and the wind frame (X_w–Y_w). The angles of rotation between the frames are denoted by θ (ground to body frame, known as the *pitch angle*), γ (ground to wind frame, known as the *flight path angle*) and α (wind to body frame, known as the *angle of attack*). $V \in \mathbb{R}$ denotes the speed of the aircraft (aligned with the positive X_w direction) and h its altitude. Figure 1 shows the different forces applied to the aircraft: its weight (mg, acting in the negative Y_g direction), the aerodynamic lift (L, acting in the positive Y_w direction), the aerodynamic drag (D, acting in the negative X_w direction) and the thrust exerted by the engine (T, acting in the positive X_b direction).

A force balance leads to the following equations of mo-

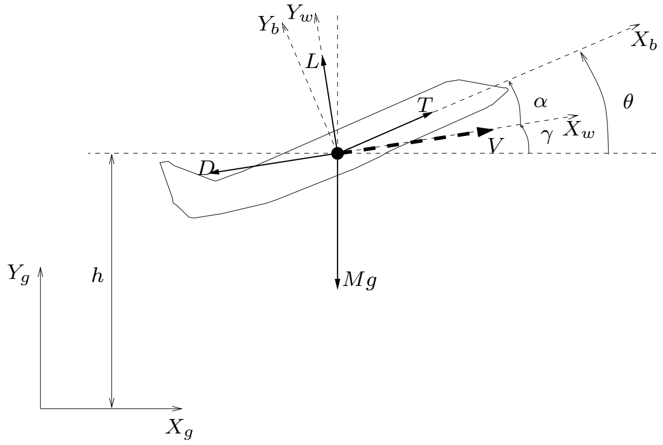

Fig. 1. Coordinate frames and forces for the aircraft model.

tion

$$m\dot{V} = T\cos(\alpha) - D - mg\sin(\gamma)$$
$$mV\dot{\gamma} = L + T\sin(\alpha) - mg\cos(\gamma).$$

From basic aerodynamics, the lift and drag can be approximated by

$$L = \frac{C_L S\rho V^2}{2}(1+c\alpha) = a_L V^2(1+c\alpha)$$
$$D = \frac{C_D S\rho V^2}{2} = a_D V^2,$$

where, C_L, C_D, and c are (dimension-less) lift and drag coefficients, S is the wing surface area, ρ is the air density and, as is commonly done in practice, the dependence of the drag on the angle of attack has been suppressed.

A three state model with $x_1 = V$, $x_2 = \gamma$ and $x_3 = h$ suffices for our purposes. The system is controlled by two inputs, the thrust, $u_1 = T$, and the pitch angle[1], $u_2 = \theta$. We assume rectangular bounds on the inputs, $u \in U = [T_{\min}, T_{\max}] \times [\theta_{\min}, \theta_{\max}]$. After a small angle approximation on α (valid for airliners, which usually operate around trimmed flight conditions) the equations of motion become

$$\dot{x} = f(x,u) = \begin{bmatrix} -\frac{a_D}{m}x_1^2 - g\sin(x_2) \\ \frac{a_L}{m}x_1(1-cx_2) - g\frac{\cos(x_2)}{x_1} \\ x_1\sin(x_2) \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \frac{a_L c}{m}x_1 \\ 0 \end{bmatrix} u_2$$

B. Cost Function and Optimal Controls

For safety reasons, certain combinations of speed and flight path angle should be avoided, because they may result in aerodynamic stall. Part of the task of the Flight Management System (FMS) is therefore to keep V and γ within a safe "aerodynamic envelope". Following [16], we consider a simplified rectangular envelope; improvements that can be introduced to make the envelope more realistic are discussed in [24], [25]. We require that $V_{\min} \leq x_1 \leq V_{\max}$ and $\gamma_{\min} \leq x_2 \leq \gamma_{\max}$, for some $V_{\min} \leq V_{\max}$ and $\gamma_{\min} \leq \gamma_{\max}$. In addition, to ensure that the aircraft does not stray away from its flight level we require that $h_{\min} \leq x_3 \leq h_{\max}$ for some $h_{\min} \leq h_{\max}$. We set[2] $K = [V_{\min}, V_{\max}] \times [\gamma_{\min}, \gamma_{\max}] \times [h_{\min}, h_{\max}]$.

To encode these constraints as a cost in a SupMin problem we define a function $l(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} l(x) = \min\{&x_1 - V_{\min}, V_{\max} - x_1 \\ &x_2 - \gamma_{\min}, \gamma_{\max} - x_2, \\ &x_3 - h_{\min}, h_{\max} - x_3\}. \end{aligned}$$

Notice that $l(x) \geq 0$ for $x \in K$ and $l(x) < 0$ for $x \notin K$. Clearly, l is Lipschitz continuous. To keep l bounded (and since we are only interested in the behaviour around the set K) we "saturate" the function l outside the set $[V_{\min} - \delta V, V_{\max} + \delta V] \times [\gamma_{\min} - \delta\gamma, \gamma_{\max} + \delta\gamma] \times [h_{\min} - \delta h, h_{\max} + \delta h]$ for some $\delta V, \delta\gamma, \delta h > 0$.

The problem is now in a form that we can apply the results of Section II-D. The Hamiltonian of Theorem 1 becomes

$$\begin{aligned} H_1(p,x) = \min\Big\{0, &p_1\left(-\frac{a_D}{m}x_1^2 - g\sin(x_2)\right) \\ &+ p_2\left(\frac{a_L}{m}x_1(1-cx_2) - g\frac{\cos(x_2)}{x_1}\right) \\ &+ p_3 x_1\sin(x_2) \\ &+ \frac{1}{m}p_1\hat{u}_1 + \frac{a_L c}{m}x_1 p_2\hat{u}_2\Big\}. \end{aligned}$$

The optimal controls are given by

$$\hat{u}_1 = \begin{cases} T_{\min} & \text{if } p_1 < 0 \\ T_{\max} & \text{if } p_1 > 0 \end{cases}$$
$$\hat{u}_2 = \begin{cases} \theta_{\min} & \text{if } p_2 < 0 \\ \theta_{\max} & \text{if } p_2 > 0 \end{cases}$$

(recall that $x_1 > 0$). The singularities at $p_1 = 0$ and $p_2 = 0$ play very little role in the numerical computation and so will not be investigated further here; a more thorough treatment (for the 2 dimensional case with state x_1 and x_2) can be found in [16].

C. Numerical Results

The resulting optimal Hamiltonian was coded in a numerical tool developed at Stanford University [19], [17] for computing viscosity solutions to Hamilton-Jacobi equations using the algorithms of [22], [23]. The results are shown in Figures 2 and 5. The parameters used were $a_L = 65.3Kg/m$, $a_D = 3.18Kg/m$, $m = 160 \cdot 10^3 Kg$, $g = 9.81 m/s^2$, $c = 6$, $\theta_{\min} = -20°$, $\theta_{\max} = 25°$,

[1] In practice, one can only control the second derivative of the pitch angle using the aerodynamic surfaces. This makes the model weakly non-minimum phase. We ignore this complication here.

[2] Strictly speaking, to follow the development on Section II-B one needs to assume that the set K is open. It is easy to see, however, that allowing K to be closed makes no difference in this case.

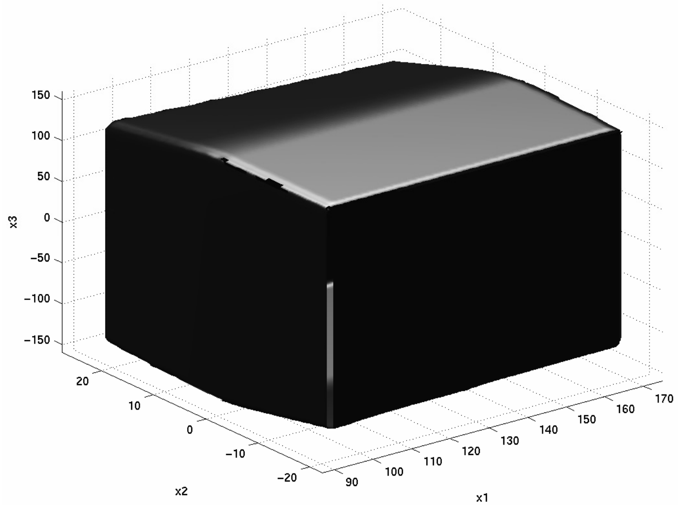

Fig. 2. Level set of $V_1(x, 0)$ for $T = 1s$.

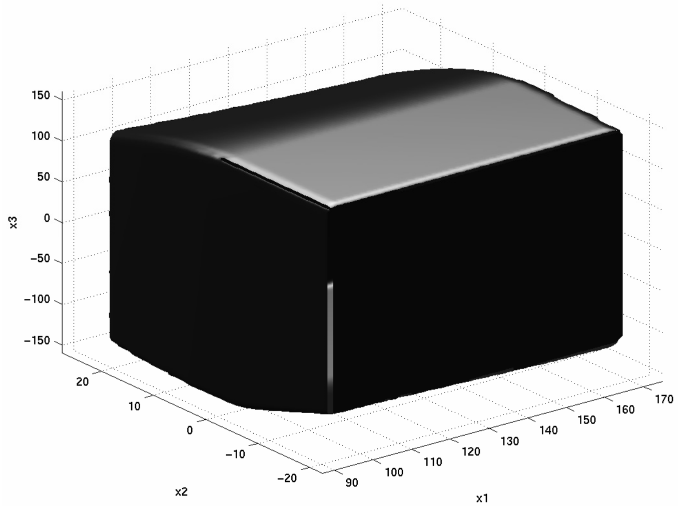

Fig. 3. Level sets of $V_1(x, 0)$ for $T = 2s$.

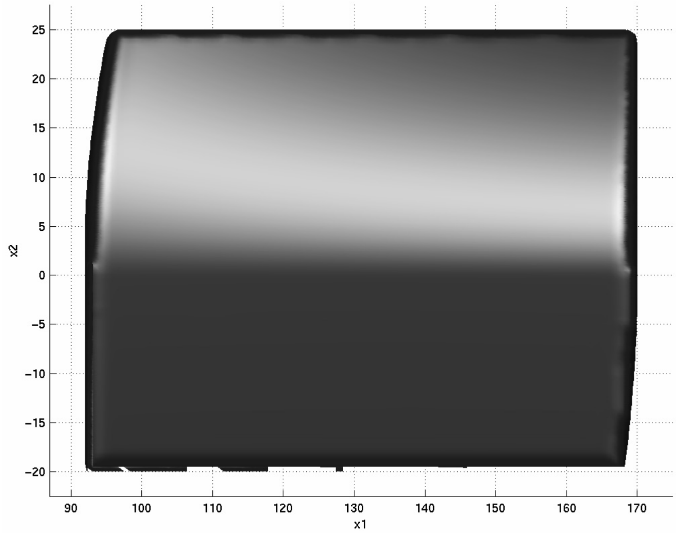

Fig. 4. Projection of $T = 2s$ level set along x_3 axis.

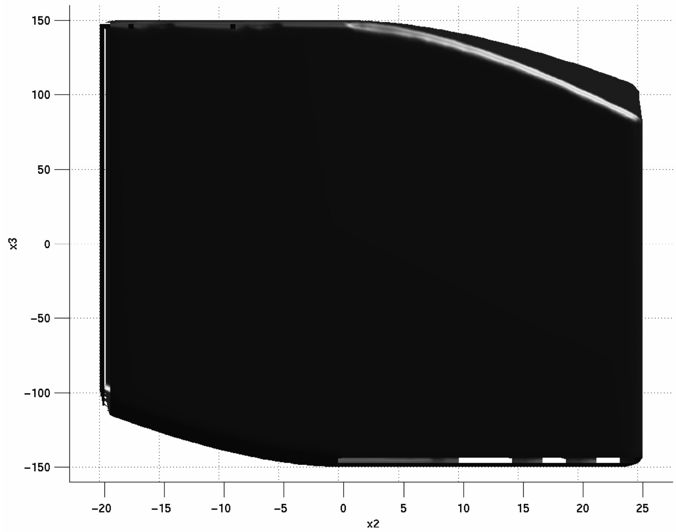

Fig. 5. Projection of $T = 2s$ level set along x_1 axis.

$T_{\min} = 60 \cdot 10^3 N$, and $T_{\max} = 120 \cdot 10^3 N$. They correspond to an Airbus A330 aircraft cruising at 35000 feet. The parameters used in the function l were $V_{\min} = 92m/s$, $V_{\max} = 170m/s$, $\gamma_{\min} = -20°$, $\gamma_{\max} = 25°$, $h_{\min} = -150m$, $h_{\max} = 150m$, $\delta V = 5m/s$, $\delta\gamma = 2.5°$, $\delta h = 10m$. The computation was performed on a $100 \times 100 \times 100$ grid and required 10298 seconds on a Pentium III, $800MHz$ processor running Red Hat Linux.

Figures 2 and 3 show the level sets $\text{Viab}(0, K) = \{x \in \mathbb{R}^3 \mid V_1(x, 0) \geq 0\}$ for two different values of the horizon, $T = 1.0s$ and $T = 2.0s$ respectively. As expected from Part 1 of Lemma 1, these sets are nested (the level set "shrinks" as T increases). For $T \approx 2.0s$ the shrinking stops; the level sets for values $T \geq 2$ are all the same. The general shape of the level sets suggests that certain states (e.g. combining high altitude with high flight path angle, low speed with high flight path angle etc.) are unsafe and should be avoided. If the aircraft ever gets to such a state, then, whatever the FMS does from then on, it will sooner or later violate the flight envelope requirements. If the initial condition is inside the level set, however, unsafe states can be avoided by applying the optimal controls of Section III-B whenever the state trajectory hits the boundary of the level set (see [21] for practical problems associated with such a control strategy).

Better intuition about the unsafe states can be obtained if the level set for $T = 2.0s$ is projected along the three axes. The projection along the x_2 axis leads to the square $[V_{min}, V_{max}] \times [h_{min}, h_{max}]$ in the x_1–x_3 plane. This suggests that any combination of speed and altitude within these bounds is safe for some value of flight path angle. The projection along the x_3 axis leads to the set shown in Figure 4; the shape of the set is the same for all altitudes. Combinations of low speed with high flight path angle and high speed with low flight path angle are unsafe; the aircraft is bound to violate the speed restrictions for such combinations. The projection along the x_1 axis is shown in Figure 5. Combinations of high altitude with high flight path angle and low altitude with low flight path angle are unsafe for all speeds; the aircraft is bound to violate the flight level limitations for such combinations. The situation gets worse (i.e. the projection on the $x_2 - x_3$ coordinates gets smaller) as the speed increases.

IV. Concluding Remarks

The aim of this paper was to demonstrate how theoretical results on the characterisation of minimum cost optimal control problems can be useful in an air traffic control context. We were able to exploit the advantages of the characterisation of [13], [14] (namely, continuity of the value function, continuity of the Hamiltonian, standard Hamilton-Jacobi for an simple terminal conditions) to provide a numerical solution to the problem of preventing large deviations from a desired flight level, while at the same time satisfying constraints on speed and flight path angle.

Reachability and invariance can also be approached using tools from viability theory. Viability theory methods [1] have recently been extended from continuous systems to a broad class of hybrid systems known as impulse differential inclusions [2]. Current research concentrates on relating the results discussed here to the viability theory formulation. In terms of applications, current work concentrates on extending this approach to the problem of auto-rotative landing for helicopters.

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References

[1] J.-P. Aubin. *Viability Theory.* Birkhäuser, Boston, 1991.

[2] J.-P. Aubin, J. Lygeros, M. Quincampoix, S. Sastry, and N. Seube. Impulse differential inclusions: A viability approach to hybrid systems. *IEEE Transactions on Automatic Control*, 47(1):2–20, January 2002.

[3] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhäuser, 1997.

[4] E. Barron. Differential games with maximum cost. *Nonlinear Analysis: Theory, Methods & Applications*, 14(11):971–989, 1990.

[5] E. Barron and H. Ishii. The Bellman equation for minimizing the maximum cost. *Nonlinear Analysis: Theory, Methods & Applications*, 13(9):1067–1090, 1989.

[6] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Set-valued numerical analysis for optimal control and differential games. In M. Bardi, T. Raghavan, and T. Parthasarathy, editors, *Stochastic and Differential Games: Theory and Numerical Methods*, number 4 in Annals of the International Society of Dynamic Games, pages 177–247. Birkhäuser, Boston, 1999.

[7] L. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I., 1998.

[8] L. Evans and P. Souganides. Differential games and representation formulas for solutions of hamilton-jacobi-isaacs equations. *Indiana University Mathematics Journal*, 33(5):773–797, 1984.

[9] I. Fialho and T. Georgiou. Worst case analysis of nonlinear systems. *IEEE Transactions on Automatic Control*, 44(6):1180–1197, June 1999.

[10] W. Flemming and H. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York, 1993.

[11] C. Livadas, J. Lygeros, and N. Lynch. High-level modeling and analysis of the traffic alert and collision avoidance system (TCAS). *Proceedings of the IEEE*, 88(7):926–948, July 2000.

[12] C. Livadas and N. Lynch. Formal verification of safety-critical hybrid systems. In S. Sastry and T. Henzinger, editors, *Hybrid Systems: Computation and Control*, number 1386 in LNCS, pages 253–272. Springer Verlag, 1998.

[13] J. Lygeros. On reachability and minimum cost optimal control. In *IEEE Conference on Decision and Control*, 2002.

[14] J. Lygeros. Reachability, viability and invariance: An approach based on minimum cost optimal control. Technical Report CUED/F-INFENG/TR.444, Department of Engineering, University of Cambridge, 2002.

[15] J. Lygeros, D. N. Godbole, and S. Sastry. Verified hybrid controllers for automated vehicles. *IEEE Transactions on Automatic Control*, 43(4):522–539, April 1998.

[16] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, pages 349–370, March 1999.

[17] I. Mitchell, A. Bayen, and C. J. Tomlin. Validating a hamilton-jacobi approximation to hybrid system reachable sets. In M. Di Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control*, number 2034 in LNCS, pages 418–432. Springer Verlag, 2001.

[18] I. Mitchell, A. M. Bayen, and C. J. Tomlin. Computing reachable sets for continuous dynamic games using level set methods. Preprint.

[19] I. Mitchell and C. J. Tomlin. Level set methods for computation in hybrid systems. In N. Lynch and B. H. Krogh, editors, *Hybrid Systems: Computation and Control*, number 1790 in LNCS, pages 310–323. Springer Verlag, 2000.

[20] A. Nuic. User manual for the base of aircraft data (BADA) revision 3.3. Technical Report EEC Note no. 20/00, Eurocontrol Experimental Centre, December 2000.

[21] M. Oishi, C. Tomlin, V. Gopal, and D. Godbole. Addressing multiobjective control: Safety and performance through constrained optimization. In M. Di Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control*, number 2034 in LNCS, pages 459–472. Springer Verlag, 2001.

[22] S. Osher and J. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[23] J. A. Sethian. *Level Set Methods: Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision, and Materials Science.* Cambridge University Press, New York, 1996.

[24] C. Tomlin, J. Lygeros, and S. Sastry. Aerodynamic envelope protection using hybrid control. In *American Control Conference*, pages 1793–1796, Philadelphia, Pennsylvania, USA, June 24-26 1998.

[25] C. Tomlin, J. Lygeros, and S. Sastry. A game theoretic approach to controller design for hybrid systems. *Proceedings of the IEEE*, 88(7):949–969, July 2000.

[26] C. Tomlin, I. Mitchell, and R. Ghosh. Safety verification of conflict resolution manoeuvres. *IEEE Transactions on Intelligent Transportation Systems*, 2(2):110–120, June 2001.

[27] C. J. Tomlin. *Hybrid control of air traffic management systems.* PhD thesis, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, 1998.