A New Sufficient Condition for Stability of Decentralized Control: An LMI Approach

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***Abstract*— In this paper, the problem of achieving stability for large-scale systems composed of a number of subsystems by decentralized control output feedback structure is considered. For this problem, a new sufficient condition is obtained under which a collection of local controllers designed for individual subsystems achieves stability for the overall system. More specifically, our sufficient condition is in terms of the H_∞ norm of a transfer function matrix of each closed-loop individual subsystems and the Hermitian part of the interaction matrix. In addition, this condition is expressed as linear matrix inequalities. Furthermore, by an example, our sufficient condition is compared with the ones proposed in the previous researches.**

***Index Terms*— Decentralized control, large-scale systems, linear matrix inequality.**

I. INTRODUCTION

There has been continuing interest in the study of large-scale systems consisting of a number of interconnected subsystems [5], [7]-[10]. The reason for this interest follows since many control problems of modern industrial society are associated with the control of complex interconnected systems, e.g., electric power systems, transportation systems, chemical process control systems, socioeconomic systems, network flow problems, etc. In the study of such large-scale systems, an important notion, that of decentralized control, plays an important role. In decentralized control, the large-scale system has several local controllers, of which, each local controller observes only local subsystem outputs and controls only local inputs; all of the local controllers, however, are involved in controlling the same large-scale system. A decentralized control system exhibits several advantages over a centralized control system, i.e., a single controller which observes all outputs of the system to control all inputs of the system. In the ideal case these advantages include: flexibility in operation, failure tolerance, simplified design, and simplified tuning. The requirement that the control system be decentralized introduces the overall stability problem. A major handicap, however, is the fact that when the decentralized controller is applied to the overall system, the stability of the closed-loop system are not preserved. As a result, the stability achieved with the diagonal system is not guaranteed, and the overall stability is lost in most cases. This illustrates the need for a sufficient condition to examine the overall stability and alternative ways to design the decentralized controllers while they guarantee the overall stability.

In [4], the structured singular value interaction measure as a tool for the design of decentralized control was proposed. This approach provides a sufficient condition for stability of decentralized control, in terms of the subsystem design constraints, under which an aggregation of stable subsystem designs yields an overall stable design. However in [4], it is assumed that the initial system is square and also it requires very complicated computations when the dimensionality of the initial system is high. In [1], the sufficient condition for stability is in terms of the H_∞ norm of the closed-loop diagonal transfer function matrix and the structured singular value of the interaction matrix. A different approach was also presented in [6], where the sufficient condition for stability is in terms of the maximum eigenvalue of the Hermitian part of the state matrices of each closed-loop individual subsystems and the interaction matrix.

In this paper, the sufficient condition for stability is stated in terms of the H_∞ norm of a transfer function matrix of each closed-loop individual subsystems and the Hermitian part of the interaction matrix. In addition, this condition is expressed as linear matrix inequalities (LMIs) which can be considered during the local controllers designing procedure to enforce the overall stability. Furthermore, by an example, our sufficient condition is compared with the ones proposed in [1], [4], [6].

This paper is organized as follows. Section II is devoted to the formulation of our control problem and statement of preliminary definitions used throughout the paper. Section III presents the main results of this paper. In Section IV, a comparison example is presented. Finally, Section V concludes the paper.

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II. PROBLEM FORMULATION

Consider an input-output decentralized large-scale system $G(s)$, with state-space equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

composed of N subsystems $G_i(s)$, described by

$$\begin{cases} \dot{x}_i = A_{ii}x_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} A_{ij}x_j + B_{ii}u_i \\ y_i = C_{ii}x_i, \qquad i = 1,2,\dots,N \end{cases} \tag{2}$$

where $x_i \in R^{ni}$ is the state, $u_i \in R^{mi}$ is the control input, $y_i \in R^{pi}$ is the measured output of the ith subsystem. The matrices A_{ii}, B_{ii}, C_{ii} are constant and of appropriate dimensions, which represent the ith subsystem. The subsystems interact each other through the interconnections A_{ij}'s, where A_{ij}'s are constant matrices. In this note, we assume that the triple (A_{ii}, B_{ii}, C_{ii}) is stabilizable and detectable.

For each isolated subsystem of large-scale system (1), we consider a local output feedback controller $K_{ii}(s)$, described by

$$\begin{cases} \dot{x}_{ki} = A_{ki}x_{ki} + B_{ki}y_i \\ u_i = C_{ki}x_{ki} + D_{ki}y_i, \qquad i = 1,2,\dots,N \end{cases} \tag{3}$$

where $x_{ki} \in R^{nki}$ is the state of local controller and A_{ki}, B_{ki}, C_{ki}, D_{ki} are constant matrices to be determined. The resulting decentralized diagonal controller for the overall system $G(s)$ is given by

$$K(s) = diag\{K_{ii}(s)\}, \qquad i = 1,2,\dots,N \tag{4}$$

with state-space equations

$$\begin{cases} \dot{x}_k = A_k x_k + B_k y \\ u = C_k x_k + D_k y \,. \end{cases} \tag{5}$$

The overall closed-loop system obtained by applying the decentralized controller (5) to the large-scale system (1), is described as

$$\begin{cases} \dot{x}_c = \begin{bmatrix} A_d - BD_kC & -BC_k \\ B_kC & A_k \end{bmatrix} x_c + \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} x_c \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x_c \end{cases} \tag{6}$$

where

$$\begin{aligned} &A_d = diag\{A_{ii}\}, \quad i = 1,2,\dots,N \\ &H = A - A_d, \;\; x_c = \begin{bmatrix} x & x_k \end{bmatrix}^T. \end{aligned} \tag{7}$$

Now, define the matrix

$$\widetilde{K} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \tag{8}$$

which collects the representation for $K(s)$ into one matrix. It is simple to show that the closed-loop state-space equations can be represented in terms of the controller matrix $\widetilde{K}$ as

$$\begin{cases} \dot{x}_c = \widetilde{A}_c x_c \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x_c \end{cases} \tag{9}$$

where $\widetilde{A}_c = \widetilde{A}_d - \widetilde{B}\widetilde{K}\widetilde{C} + \widetilde{H}$ and

$$\widetilde{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \widetilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \widetilde{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, \widetilde{H} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

Then, the problem of achieving stability for the original system by a dynamic output feedback controller, can be reduced to the problem of achieving stability for the augmented system by a static output feedback controller.

III. STABILITY CONDITION

A. Mathematical Background

Lemma 1: For a square matrix $M \in R^{n\times n}$, we have

$$Re[\lambda_i(M)] \le \lambda_{max}(\frac{M+M^T}{2}), \qquad i = 1,2,\dots,n \tag{11}$$

where $\lambda_i(M)$ is the ith eigenvalue of M.

Proof: Assuming that v_i is the ith eigenvector of M then we have

$$v_i^H(\frac{M+M^T}{2})v_i = Re[\lambda_i(M)]v_i^H v_i, i = 1,2,\dots,n. \tag{12}$$

Since for a symmetric matrix for example $(M+M^T)/2$ and every vector x we have

$$x^H(\frac{M+M^T}{2})x \le \lambda_{max}(\frac{M+M^T}{2})x^H x \tag{13}$$

[3], therefore

$$Re[\lambda_i(M)] \le \lambda_{max}(\frac{M+M^T}{2}), \quad i=1,2,\dots,n. \tag{14}$$

B. LMI Statement of the Stability Condition

Linear matrix inequalities have emerged as a powerful formulation and design technique for a variety of linear control problems [2]. Since solving LMIs is a convex optimization problem, such formulations offer a numerically tractable means of attacking problems that lack an analytical solution. In addition, a variety of efficient algorithms are now available to solve the generic LMI problems. Consequently, reducing an overall stability problem to an LMI can be considered as a practical solution to this problem.

Theorem 1: The decentralized controller $K(s)$ stabilizes the overall system $G(s)$, if $K(s)$ stabilizes the diagonal system $\widetilde{G}(s)$ and there exists a positive definite matrix $P = P^T \in R^{n\times n}$ such that

$$\begin{bmatrix} (\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})P + P(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2}) + (\frac{\widetilde{H}+\widetilde{H}^T}{2})^2 & P \\ P & -I \end{bmatrix} \prec 0 \tag{15}$$

where $\widetilde{A}_{cd} = \widetilde{A}_d - \widetilde{B}\widetilde{K}\widetilde{C}$.

Proof: It is well known from system theory [2] that all the eigenvalues of matrix $(\widetilde{A}_c + \widetilde{A}_c^{\,T})/2$ are negative if and only if there exists a positive definite matrix $P = P^T \in R^{n\times n}$ such that

$$(\frac{\widetilde{A}_c+\widetilde{A}_c^{\,T}}{2})P + P(\frac{\widetilde{A}_c+\widetilde{A}_c^{\,T}}{2}) \prec 0. \tag{16}$$

For $\widetilde{A}_c = \widetilde{A}_{cd} + \widetilde{H}$, we have

$$(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})P + P(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2}) + (\frac{\widetilde{H}+\widetilde{H}^T}{2})P + P(\frac{\widetilde{H}+\widetilde{H}^T}{2}) \prec 0. \tag{17}$$

Using the inequality

$$(\frac{\widetilde{H}+\widetilde{H}^T}{2} - P)^T(\frac{\widetilde{H}+\widetilde{H}^T}{2} - P) \ge 0 \tag{18}$$

or equivalently,

$$(\frac{\widetilde{H}+\widetilde{H}^T}{2})^2 + PP \ge (\frac{\widetilde{H}+\widetilde{H}^T}{2})P + P(\frac{\widetilde{H}+\widetilde{H}^T}{2}) \tag{19}$$

one can see that a solution of the following LMI is also a solution of the LMI in (17)

$$(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})P + P(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2}) + (\frac{\widetilde{H}+\widetilde{H}^T}{2})^2 + PP \prec 0. \tag{20}$$

The Schur complement formula [2] implies the above LMI is equivalent to

$$\begin{bmatrix} (\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})P + P(\frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2}) + (\frac{\widetilde{H}+\widetilde{H}^T}{2})^2 & P \\ P & -I \end{bmatrix} \prec 0. \tag{21}$$

Now, by invoking Lemma 1, if the LMI in (21) be feasible, then all the eigenvalues of $\widetilde{A}_c$ will have negative real part, or equivalently, the overall stability will be guaranteed. Since the overall stability condition can be expressed as an LMI, one can include this condition as a design objective in a decentralized controller design problem via LMI optimization to enforce the overall stability.

C. H_∞ Statement of the Stability Condition

The following theorem presents the stability condition via the H_∞ norm.

Theorem 2: The decentralized controller $K(s)$ stabilizes the overall system $G(s)$, if $K(s)$ stabilizes the diagonal system $\widetilde{G}(s)$ and

$$\alpha \prec \left(\left\| \frac{\widetilde{H}+\widetilde{H}^T}{2} \right\|_\infty \right)^{-1} \tag{22}$$

where

$$\alpha = \left\| (sI - \frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})^{-1} \right\|_\infty . \tag{23}$$

Proof: By invoking Bounded Real Lemma [2], the LMI in (21) can be expressed as

$$\left\| (\frac{\widetilde{H}+\widetilde{H}^T}{2})(sI - \frac{\widetilde{A}_{cd}+\widetilde{A}_{cd}^{\,T}}{2})^{-1} \right\|_\infty \prec 1. \tag{24}$$

The H_∞ norm submultiplicative inequality implies the following inequality is a sufficient condition for satisfying the above inequality

$$\alpha \prec \left(\left\| \frac{\widetilde{H}+\widetilde{H}^T}{2} \right\|_\infty \right)^{-1} \tag{25}$$

where

$$\alpha = \left\| (sI - \frac{\tilde{A}_{cd} + \tilde{A}_{cd}^{\,T}}{2})^{-1} \right\|_{\infty} . \tag{26}$$

Remark 1: It can easily be shown that

$$\alpha = \max_{i} \{\alpha_i\} \qquad i = 1,2,\dots,N \tag{27}$$

where

$$\alpha_i = \left\| (sI - \frac{\tilde{A}_{cd\,i} + \tilde{A}_{cd\,i}^{\,T}}{2})^{-1} \right\|_{\infty} \tag{28}$$

and

$$\tilde{A}_{cd\,i} = \tilde{A}_{ii} - \tilde{B}_{ii}\tilde{K}_i\tilde{C}_{ii}, \quad \tilde{K}_i = \begin{bmatrix} A_{ki} & B_{ki} \\ C_{ki} & D_{ki} \end{bmatrix}$$

$$\tilde{A}_{ii} = \begin{bmatrix} A_{ii} & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{B}_{ii} = \begin{bmatrix} 0 & B_{ii} \\ I & 0 \end{bmatrix}, \ \tilde{C}_{ii} = \begin{bmatrix} 0 & I \\ C_{ii} & 0 \end{bmatrix}. \tag{29}$$

Therefore, the condition in (22) is straightforward to examine; $\tilde{H}$ is a constant matrix and $\|(\tilde{H} + \tilde{H}^T)/2\|_\infty$ is easily computable and α_i is the maximum value of α_i's, where α_i is the H_∞ norm of a transfer function matrix of each closed-loop individual subsystems with its local controller.

IV. Comparison Example

In this section, the stability condition proposed in the previous section is compared with the conditions of [1], [4], [6].

In [4], the stability of the overall system $G(s)$, by a decentralized controller is guaranteed if

$$\bar{\sigma}[\tilde{T}(j\omega)] \prec \mu^{-1}[E(j\omega)], \qquad \forall \omega \in R \tag{30}$$

where

$$E(j\omega) = [G(j\omega) - \tilde{G}(j\omega)]\tilde{G}^{-1}(j\omega). \tag{31}$$

$\tilde{T}(j\omega)$ represents the complementary sensitivity function of the diagonal system $\tilde{G}(j\omega)$ and $\bar{\sigma}[\tilde{T}(j\omega)]$ denotes the maximum singular value of $\tilde{T}(j\omega)$.

In [1], it is proposed that the decentralized controller $K(s)$ stabilizes the overall system $G(s)$, if $K(s)$ stabilizes the diagonal system $\tilde{G}(s)$ and

$$\rho_{max} \prec \mu^{-1}(H) \tag{32}$$

where

$$\rho_{max} = \max_{i} \left\{ \left\| (sI - A_{ii} + B_{ii}K_{ii}(s)C_{ii})^{-1} \right\|_{\infty} \right\} \tag{33}$$

and

$$\mu(H) = (\min_{\Delta}\{\bar{\sigma}(\Delta) \mid \det(I + \Delta H) = 0\})^{-1} . \tag{34}$$

Note that $\mu(H) = 0$, if no structured Δ exists such that $\det(I + \Delta H) = 0$.

In [6], it is proposed that the decentralized controller $K(s)$ stabilizes the overall system $G(s)$, if $K(s)$ stabilizes the diagonal system $\tilde{G}(s)$ and

$$\max_{i} \left[\lambda_{max}(\frac{\tilde{A}_{cd\,i} + \tilde{A}_{cd\,i}^{\,T}}{2}) \right] \prec -\max\left\{0, \lambda_{max}(\frac{\tilde{H} + \tilde{H}^T}{2})\right\}, \quad i = 1,2,\dots,N. \tag{35}$$

The following example illustrates that our stability condition is less conservative than the stability conditions of [4], [1] and is equivalent to the stability condition of [6]. Consider the system (A,B,C) and the decentralized controller $K(s)$ where

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \ B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ K(s) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}. \tag{36}$$

It is simple to show that

$$\bar{\sigma}[\tilde{T}(j0)] = \frac{3}{7}, \quad \mu^{-1}[E(j0)] = \frac{1}{3} \tag{37}$$

$$\rho_{max} = \frac{1}{2}, \quad \mu^{-1}(H) = \frac{1}{3}. \tag{38}$$

Since $\bar{\sigma}[\tilde{T}(j0)] \succ \mu^{-1}[E(j0)]$ and $\rho_{max} \succ \mu^{-1}(H)$, the conditions in (30) and (32), respectively, fail to be satisfied for this example. Also, it is straightforward to obtain that

$$\max_{i} \left[\lambda_{max}(\frac{\tilde{A}_{cd\,i} + \tilde{A}_{cd\,i}^{\,T}}{2}) \right] = -2, \qquad i = 1,2$$

$$-\max\left\{0, \lambda_{max}(\frac{\tilde{H} + \tilde{H}^T}{2})\right\} = 0 \tag{39}$$

$$\alpha = \frac{1}{2}, \quad \left(\left\| \frac{\widetilde{H} + \widetilde{H}^T}{2} \right\|_\infty \right)^{-1} = \infty . \tag{40}$$

It is clear that the stability conditions in (35) and (22) are satisfied for this case. Therefore, it is concluded that the decentralized controller stabilizes the overall system.

V. Conclusion

In this paper, a new sufficient condition for the overall stability of decentralized control systems has been obtained. This condition is straightforward to examine and is also useful to enforce the overall stability. Furthermore, by an example, it has been illustrated that our sufficient condition is less conservative than the other conditions available for examining the overall stability.

VI. References

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