FACTORIZATION AND BIBO STABILITY OF CERTAIN DISCRETE VOLTERRA SYSTEMS

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Abstract

Throughout this paper we present some stability criterions for special non-linear discrete Volterra systems. Our method is based on a factorization algorithm which decomposes the original system as a star-product of a δ-operator and a linear series. Then the stability of the linear series guarantees the stability of the original non-linear system too. An extension in the case of Volterra systems containing products among inputs and outputs as well as some open-loop stability techniques are also provided.

Keywords: Algebraic approach, Computational approach, Factorization methods, Discrete Systems, Non-linear Systems, Volterra Systems.

1 Introduction

The Volterra/Wiener representation of non-linear systems, either continuous or discrete, are standard in the literature and have been studied a lot in the past [6]. One situation of interest was the stability behaviour of these systems. This notion has been addressed from several different angles. Series convergence, selection of proper coefficients bound [3], to mention but a few, are two classical methods used for the stability analysis of Discrete Volterra Systems.

In the present paper we are working with two kinds of Discrete Volterra systems:

$$y(t) = \sum_{n=1}^{k} \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} u(t-i_1) \cdots u(t-i_n) \tag{1}$$

$$y(t) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} y(t-i_1) \cdots y(t-i_n) u(t-j_1) \cdots u(t-j_m) \tag{2}$$

The first are discrete Volterra systems where the size of products is up to a certain number. It is nothing else than an autoregressive system. The second one contains products among inputs and output sequences. Both of these arise during the procedure of finding an Input/Output expression from a state-space representation, [6].

Our main objective is to establish some stability criterions , concerning systems (1) and (2) and then to provide algorithms which face the open-loop stability problem of the above systems. The open-loop stability for classical or Volterra systems has been a long-standing topic of research [6]. The major drawback of this method is the fact that it suffers from serious internal stability problems, specially when no Volterra systems are studying. Nevertheless when we are dealing with Volterra systems with zero initial conditions, the procedure is still ineresting and open. Certain approaches of this kind can be found under the name of tandem or cascade connection in [6].

The whole methodology is along the path of δ-operators and star-products. These are algebraic tools introduced for the description and study of the non-linear input-output discrete systems [2]. Throughout this paper N, Z and R will denote the sets of natural, integer and real numbers respectively.

2 The Algebraic Background

The algebraic notions that follow have been presented in details in [2], [3]. For the sake of completeness we have to repeat the main topics here, briefly. Let k be a positive integer. A subset of the set $\cup_{n=1}^{k} Z^n$ is called a set of indices and it is denoted by $\mathbf{I}$. We denote the elements of $\mathbf{I}$ by $\mathbf{i} = (i_1, i_2, \ldots, i_n)$. Given two indices $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_\lambda)$, the operation $\mathbf{i} \oplus \mathbf{j}$ is defined just juxtaposing $\mathbf{j}$ after $\mathbf{i}$. Let $y(t)$ be a real sequence defined over the set of integers Z and let F be the set of causal sequences. Thus if $y(t) \in F$ then $y(t) = 0$ for $t < 0$. Let i be an integer. We define the δ_i operator as the i-shift $\delta_i : F \mapsto F, \delta_i\{y(t)\} = \{y(t-i)\}$. The operator $\delta_{\mathbf{i}} : F \mapsto F$, $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ is defined as $\delta_{\mathbf{i}}\{y(t)\} = \delta_{i_1}\delta_{i_2}\delta_{i_3} \ldots \delta_{i_m}\{y(t)\}$ $= y(t-i_1)y(t-i_2)\cdots y(t-i_m)$. By convention we define $\delta_e\{y(t)\} = \{1\}$ for each $t \in Z$. The set of δ-operators is denoted by Δ. Given $\delta_{\mathbf{i}}, \delta_{\mathbf{j}} \in \Delta$ we define their sum as $\{\delta_{\mathbf{i}} + \delta_{\mathbf{j}}\}\{y(t)\} = \delta_{\mathbf{i}}\{y(t)\} + \delta_{\mathbf{j}}\{y(t)\}$ and their dot-product as the operator $\delta_{\mathbf{i}} \cdot \delta_{\mathbf{j}} = \delta_{\mathbf{i} \oplus \mathbf{j}}$.

In order to cope with systems including products among input and output signals we have to extend the δ-operators in a proper way. Let $\mathbf{i} = (i_1, i_2, \ldots, i_m), \mathbf{j} = (j_1, j_2, \ldots, j_n)$ be indices, $\delta_{\mathbf{i}}, \delta_{\mathbf{j}} \in \Delta$ and $(y, u) \in F \times F$. The operator $\delta_{\mathbf{i}} \times \delta_{\mathbf{j}} : F \times F \mapsto F$ is defined as $\delta_{\mathbf{i}} \times \delta_{\mathbf{j}}[y(t), u(t)] = y(t-i_1)y(t-i_2)\cdots y(t-i_m)u(t-j_1)u(t-j_2)\cdots u(t-j_n)$. We can immediately verify that $\delta_{\mathbf{i}} \times \delta_{\mathbf{j}}[y(t), u(t)] = [\delta_{\mathbf{i}} y(t)] \cdot [\delta_{\mathbf{j}} u(t)]$, where " $\cdot$ " the usual product among sequences. This means that the operator $\delta_{\mathbf{i}}$ acts exclusive on "outputs" and $\delta_{\mathbf{j}}$ exclusive on "inputs". Sometimes it is more convenient to use the notation $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ instead of $\delta_{\mathbf{i}} \times \delta_{\mathbf{j}}$. By means of the δ_e operator we can write and simple δ-operators or ϵ-operators in a $\delta\epsilon$-form.

An expression of the form $S = \sum_{\mathbf{i} \in \mathbf{I}} a_{\mathbf{i}} \delta_{\mathbf{i}}$ where $\mathbf{I}$ an infinite set of indices is called a δ-series. If $\mathbf{I}$ coincides with the set of positive integers then the expression $S = \sum_{i \in Z^+} a_i \delta_i = \sum_{i=0}^{\infty} a_i \delta_i$ is a linear series. A homogeneous δ-series of n-degree is the series $S = \sum_{\mathbf{i} \in I_n} a_{\mathbf{i}} \delta_{\mathbf{i}}$, where $\mathbf{I_n}$ is an infinite set of indices of the form $\mathbf{I_n} = \{(i_1, i_2, \ldots, i_n) \in Z^n\}$.

We can extend the above definitions in the case of $\delta\epsilon$-operators too. An expression of the form $\underline{S} = \sum_{\mathbf{i} \in \mathbf{I}} \sum_{\mathbf{j} \in \mathbf{J}} c_{\mathbf{ij}} \delta_{\mathbf{i}} \epsilon_{\mathbf{j}}$, $\mathbf{I}, \mathbf{J}$ infinite sets of indices, is called a $\delta\epsilon$-series.

A homogeneous $\delta\epsilon$-series of $n+m$-degree is the $\delta\epsilon$-series $\underline{S} = \sum_{\mathbf{i}\in I_n}\sum_{\mathbf{j}\in J_m} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ where $\mathbf{I_n} = \{(i_1, i_2, \ldots, i_n) \in Z^n\}$ and $\mathbf{J_m} = \{(j_1, j_2, \ldots, j_m) \in Z^m\}$. By $d(S)$ or $d(\underline{S})$ we denote the minimum delay appeared in S or $\underline{S}$. Using the above notation we can rewrite non-linear discrete Volterra systems of the form (1) and (2) shortly as follows $y(t) = Su(t)$, S a δ-series and $y(t) = \underline{S}[y(t), u(t)]$, $\underline{S}$ a $\delta\epsilon$-series.

Let A and B be δ-series. Their star-product is the composition operator $A * B = A \circ B$, where $\circ$ is the usual composition map among operators. If G is a δ-series and $\underline{S}$ a $\delta\epsilon$-series we define the star product of G and $\underline{S}$ as the composition operator $G * \underline{S} = G \circ \underline{S}$. If $A = \sum_{i=0}^{\infty} a_i\delta_i$ and $B = \sum_{j=0}^{\infty} b_j\delta_j$ are linear series then $A*B = \sum_{i=0}^{\infty}(\sum_{k=0}^{i} a_{i-k}b_k)\delta_i$. More complicated formulas, covering the general case can be found in [2], [3]. We can easily check that the distributive property $A*[B+C] = A*B + A*C$ is not valid, except in the case of A linear. This particular feature endows the set of δ-operators with some special properties.

3 Factorizations

A δ-series S is called δL-Factorizable if there are a δ-operator $\delta_{\mathbf{i}} = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}$ and a linear δ-series L such that $S = \delta_{\mathbf{i}} * L$. Certain theorems, characterized the class of δL-Factorizable series, can be found in [5] We provide now an algorithm for the δL-Factorization. We suppose that an algorithm, named "prod" for the factorization of a δ-series according to the dot-product, is available.

The δLF-Algorithm.

Input: A δ-series S.

Step 1: By means of the "prod"-algorithm we factorize S as follows: $S = S_1 \cdot S_2 \cdots S_n$, where S_i, $i = 1, \ldots, n$, linear series.

Step 2: For all S_i we set $S_i = \delta_{a_i} * L_i$, $a_i = d(L_i)$, $i = 1, \ldots, n$.

Step 3: **IF** $L_1 = L_2 = \cdots = L_n = L$ **THEN** give as **output** the quantities $\delta_{a_i}, i = 1, \ldots, n, L$ **ELSE** no solution.

Theorem 3.1 *[5] If the above algorithm gives as outputs the quantities δ_{a_i}, L, $i = 1, \ldots, n$, then $S = \delta_{a_1}\delta_{a_2}\cdots\delta_{a_n} * L$ is a δL-Factorization of the δ-series S.*

Let $\underline{S}$ be a $\delta\epsilon$-series, we say that $\underline{S}$ is $\delta\epsilon L$-Factorizable if we can write it in the form $\underline{S} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [L, M]$, where $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ a $\delta\epsilon$-operator, L a linear δ-series, and M a linear ϵ-series. One of the L and M series may be a linear δ-polynomial but not both of them. The following algorithm provides us with a method of $\delta\epsilon L$-Factorization. It is an extension of the δLF-algorithm.

The $\delta\epsilon$LF-Algorithm.

Input: A $\delta\epsilon$-series $\underline{S}$.

Step 1: By means of the "prod"-algorithm we factorize $\underline{S}$ as follows: $\underline{S} = L_1 \cdot M_1$, where L_1, a δ-series, M_1 an ϵ-series.

Step 2: By means of the δLF-Algorithm we factorize L_1 as follows $L_1 = \delta_{\mathbf{i}} * L$.

Step 3: By means of the δLF-Algorithm we factorize M_1 as follows $M_1 = \epsilon_{\mathbf{j}} * M$.

Step 4: Give as **output** the quantities $\delta_{\mathbf{i}}, \epsilon_{\mathbf{j}}, L, M$ **ELSE** no solution.

Theorem 3.2 *[5] If the above algorithm gives as outputs the quantities $\delta_{\mathbf{i}}, \epsilon_{\mathbf{j}}, L, M$, then $\underline{S} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [L, M]$ is a $\delta\epsilon L$-Factorization of the $\delta\epsilon$-series $\underline{S}$.*

4 The BIBO Stability Criterions.

We shall present now some BIBO conditions appropriate for the Volterra series we study here. Let us suppose that we have the system $y(t) = Lu(t)$, L a linear series of the form $L = \sum_{i=0}^{\infty} a_i\delta_i$. It is well known in the literature that this system is BIBO stable if and only if the series $\sum_{i=0}^{\infty} |a_i|$ converges. In this case we say that the series L is BIBO stable or just stable.

Theorem 4.1 *[5] We have the non-linear system $y(t) = Su(t)$. Let S be a homogeneous δ-series which is δL-Factorizable as follows: $S = \delta_{\mathbf{i}} * L$, $\delta_{\mathbf{i}} = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_n} \in \Delta$, L a linear series. The non-linear system is BIBO stable if and only if L is stable.*

Theorem 4.2 *[5] Let $y(t) = \underline{S}[y(t), u(t)]$ be a non-linear system containing products among input and output signals. The system is BIBO stable provided that:*
1) $S = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [L, U]$, $\delta_{\mathbf{i}} = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}$, $\epsilon_{\mathbf{j}} = \epsilon_{j1}\epsilon_{j2}\cdots\epsilon_{j_m}$, $L = \sum_{i=1}^{\sigma} l_i\delta_i$ a linear δ-polynomial, $U = \sum_{i=0}^{\infty} m_i\epsilon_i$ a linear ϵ-series.
2)The series $\sum_{i=0}^{\infty} |m_i|$ converges to the number ϕ.
3) The following inequality holds

$$\mathcal{L}^n K^n \phi^m M^m < K$$

$\mathcal{L} = \sum_{i=1}^{\sigma} |l_i|$, K is the bound of the initial values of the output and M is the bound of the input.

5 The Open-Loop Stability.

We say that a discrete Volterra system $y(t) = Su(t)$ is BIBO open-loop stabilized via a precompensator $u(t) = Fv(t)$, F a δ-series, if the cascade system $y(t) = S(Fv(t))$ is BIBO stable. We say that a precompensator $u(t) = Fv(t)$ open-loop stabilizes the discrete Volterra system $y(t) = Su(t)$ to the number ϕ, if the δ-series which corresponds to the system $y(t) = S(Fv(t))$ converges to the number ϕ. Obviously the second case contains the first.

Theorem 5.1 *[5] Any unstable linear system $y(t) = Lu(t)$ is always open-loop stabilized to a given number ϕ via a precompensator of the form $u(t) = Fv(t)$, F a linear δ-series.*

Theorem 5.2 *[5] Let $y(t) = Su(t)$ be a non-linear Volterra discrete system, where S is an n-degree homogeneous δ-series. If S is δL-Factorizable as follows $S = \delta_{\mathbf{i}} * L$, then any input of the form $u(t) = Fv(t)$, which open-loop stabilizes the linear series L, open-loop stabilizes the original non-linear system too.*

Theorem 5.3 *[5] Let $y(t) = \underline{S}[y(t), u(t)]$ be a non-linear system containing products among input and output sequences. If $\underline{S}$ is $\delta\epsilon L$-Factorizable, $\underline{S} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [L, U]$, L a linear δ-polynomial, then the following procedure provides us with an input, which stabilizes the original non-linear discrete system.*

Procedure.

Step 1: We decompose $\underline{S}$ as follows $\underline{S} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [L, U]$, $\delta_{\mathbf{i}} = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}$, $\epsilon_{\mathbf{j}} = \epsilon_{j_1}\epsilon_{j_2}\cdots\epsilon_{j_m}$, $L = \sum_{i=1}^{\sigma} l_i\delta_i$ a linear δ-polynomial, $U = \sum_{i=0}^{\infty}\mu_i\epsilon_i$ a linear ϵ-series.

Step 2: We choose a number ϕ such that

$$\mathcal{L}^n K^n \phi^m M^m < K$$

$\mathcal{L} = \sum_{i=1}^{\sigma}|l_i|$, K is the bound of the initial values of the output and M is the bound of the input,

Step 3: We choose a series $\sum_{i=0}^{\infty} a_i$ converging to the number ϕ.

Step 4: We construct the series $\sum_{i=0}^{\infty} f_i\delta_i$ with $f_i = \frac{a_i - \sum_{k=0}^{i-1}\mu_{i-k}f_k}{\mu_0}$

Step 5: The precompensator upon request has the form $u(t) = Fv(t)$ where $F = \sum_{i=0}^{\infty} f_i\delta_i$.

6 Examples

(1) We have the system

$$y(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j u(t-i)u(t-j-1) +$$

$$+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} a_i a_j a_l u(t-i)u(t-j-1)u(t-l-3) \tag{3}$$

with $a_i = i$. This is an unstable system. Indeed, using as input the random sequence $u(t) = rand(1)$ and zero initial conditions we take a constantly increasing output. We can see this behaviour in the first diagram of the figure 1. Using δ-operators we rewrite (3) in the form:

$$y(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j \delta_i\delta_{j+1} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} a_i a_j a_l \delta_i\delta_{j+1}\delta_{l+3} u(t) = Su(t)$$

Applying the algorithm δLF we get for the first tem of S:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j \delta_i\delta_{j+1} = a_1(\sum_{j=1}^{\infty} a_j\delta_{j+1})\delta_1 + a_2(\sum_{j=1}^{\infty} a_j\delta_{j+1})\delta_2 + \cdots =$$

$$= (\sum_{i=1}^{\infty} a_i\delta_i) \cdot (\sum_{j=1}^{\infty} a_j\delta_{j+1}) = [\delta_0 * (\sum_{i=1}^{\infty} a_i\delta_i)] \cdot [\delta_1 * (\sum_{j=1}^{\infty} a_j\delta_j)] = \delta_0\delta_1 * (\sum_{i=1}^{\infty} a_i\delta_i)$$

Working simultaneously with the second term too, we finally factorize S as follows: $S = (\delta_0\delta_1 + \delta_0\delta_1\delta_3) * (a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + \cdots) = P * L$. We seek now for a precompensator $u(t) = Fv(t)$ which stabilizes the linear series L. In other words we want the coefficients of the series $(\sum_{i=1}^{\infty} a_i\delta_i) * (\sum_{i=1}^{\infty} f_i\delta_i)$ to be identical equal with the coefficients of a stable linear series, let us say $(\sum_{i=1}^{\infty} r^i\delta_i)$, with $r = 0.9$. The equations presented in the proof of theorem 5.1 leads to the calculation of the coefficients f_i and thus to the relation $u(t) = Fv(t)$. As we stated before this input stabilizes the non-linear Volterra system too.

(2) We have the system

$$y(t) = \sum_{l=1}^{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_l a_i a_j y(t-l)u(t-i)u(t-j-1) \tag{4}$$

where $g_1 = 2, g_2 = 3$ and $a_i = i$. This is an unstable system. Following the steps of the procedure appeared in theorem 5.3 we shall find a precompensator, which stabilizes this system. By means of the $\delta\epsilon$-operators we rewrite (4) in the form $y(t) = \sum_{l=1}^{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_l a_i a_j \delta_l\epsilon_i\epsilon_j[y(t), u(t)] = \underline{S}[y(t), u(t)]$. Using the $\delta\epsilon LF$-algorithm we decompose it as follows: $\underline{S} = \delta_1\epsilon_1\epsilon_2 * [L, U]$, where $L = 2\delta_0 + 3\delta_1$ a δ-polynomial and $U = \epsilon_0 + 2\epsilon_1 + 3\epsilon_2 + \cdots$ an ϵ-series. Let us take as outputs bound the number $K = 1$. Then, since $\mathcal{L} = 2 + 3 = 5$, $K = 1$, $M = 1$,$n = 1$,$m = 2$ we seek for a number ϕ such that $\mathcal{L}K\phi^2M^2 < K$. The last inequality implies that $\phi < \frac{1}{\sqrt{5}} = 0.4472$. We choose as $\phi = 0.3$. Let $T = r^2\delta_2 + r^3\delta_3 + \cdots$ be a series converging to the number $\phi = 0.3$. This means that $r = 0.417891$. Finally, the system is stabilized via the law $u = Fv$ where $F = \sum_{i=0}^{\infty} f_i\delta_i$, $f_0 = r^2/1$, $f_i = r^{i+2} - \sum_{k=0}^{i-1}(i-k)f_k$ and $v(t) = rnd(1)$. If we repeat the same procedure with, let us say, $r = 0.7$ we take instability.

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