Finite time convergent observers for linear time-varying systems

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***Abstract*— This paper focuses on the design of observers with finite convergence time for continuous time linear time–varying systems. For this purpose we outline how recent results for finite time convergent observers for time-invariant systems can be expanded to the time–varying case. Besides deriving general conditions that guarantee finite time convergence we furthermore show how for systems that can be transformed to observer canonical form suitable observer parameters can be obtained. The obtained results are exemplified considering a simple example system.**

I. Introduction

Observing the state of a continuous time system is an important problem and many different observer designs exist by now (see for example[1]). However, most of the existing observer designs guarantee only that the estimation error tends to zero asymptotically. In principle it is desirable to have observers that allow to reconstruct (in the nominal case) the system state in finite time. Such observers are for example of interest for the application of state feedback controllers.

So far there exist only a couple of observers with a finite convergence time. Examples are sliding mode based observers [2], [3], [4], moving horizon based observers [5], [6], [7], [8] and the observer design presented in [9]. The Sliding mode based observers can be applied to linear and nonlinear systems and guarantee finite time convergence of the estimation error. However, the differential equations describing the estimation error can only be rendered semi–global stable in the sense that a bounded region of system states must be considered. Moving horizon observers are based on the online solution of a dynamic optimization problem involving the output measurements and the system model. They can be applied to linear as well as nonlinear systems. Under the assumption that the global solution to the dynamic optimization problem can be found and that a certain finite time observability assumption holds, moving horizon observers can in principle guarantee finite time convergence of the estimation error. The main drawback of moving horizon observers is that a dynamic optimization problem must be solved online. The observer design presented in [9] is, in comparison to the moving horizon and sliding mode observer designs rather simple to implement. However, it can only be applied to linear time–invariant systems. Basically two identity observers with different speeds of convergence are used and the state estimate is constructed based on delayed and current state estimates of the identity observers. One of the key advantages of this approach is that no bounded region of system states is required.

In this note we propose an extension of the design presented in [9] to linear time-varying systems. For this purpose we derive conditions for the (time–varying) observer feedback matrices and the delay used that lead to guaranteed finite time convergence of the observer error. In general, however, it is rather difficult to obtain suitable observer feedback matrices and a suitable delay that satisfy the derived conditions. However, as outlined, for systems that can be brought to observer canonical form it is trivial to pick suitable parameters, thus allowing the application of the derived observer design to a wide class of systems.

The overall paper is structured as follows: In Section II we present the proposed observer for time-varying continuous time MIMO systems and state conditions that guarantee the finite time convergence of the observer error. Since it is in general difficult to obtain suitable observer parameters, we show in Section III how for MISO systems that can be transformed to observer canonical form suitable parameters can be obtained. Note that while the results are only derived for MISO systems for notational simplicity, they can be trivially expanded to the MIMO case. Section IV contains a small example, showing the application of the derived observer to a fourth order system with two outputs. The paper finally concludes with a short discussion of the results in Section V.

II. General linear time-varying systems

Consider a continuous time linear time–varying MIMO system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \tag{1a}$$

$$y(t) = C(t)x(t) \tag{1b}$$

with the state $x(t) \in \mathbb{R}^n$, the output $y(t) \in \mathbb{R}^p$ and the input $u(t) \in \mathbb{R}^m$. We assume that $u(\cdot)$, $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are piecewise continuous and bounded over $[t_0, \infty)$ and are known exactly. This implies that the solution of (1) exists, is unique and bounded for all times [10], [11].

The observer proposed has basically the same structure as the observer introduced in [9] for the time-invariant case. It consists of two identity observers and the state estimate is obtained utilizing current and delayed state estimates of the observers involving the observers transition matrices.

The two used identity observers are of the following form:

$$\begin{aligned}\dot{z}_i(t) &= (A(t) - H_i(t)C(t))z_i + H_i(t)y(t)\\ &\quad +B(t)u(t), \quad i = 1,2, t \geq t_0,\end{aligned}$$

with $z_i \in \mathbb{R}^n$ and $H_i \in \mathbb{R}^{p\times n}$. In order to simplify the presentation we combine the observer states in the vector $z = [z_1^T, z_2^T]^T$. Thus we have

$$\dot{z}(t) = F(t)z(t) + G(t)u(t) + H(t)y(t), \tag{2}$$

where

$$F(t) := \begin{bmatrix} F_1(t) & 0 \\ 0 & F_2(t)\end{bmatrix}, \qquad G(t) := \begin{bmatrix} B(t) \\ B(t)\end{bmatrix},$$
$$F_i(t) := (A(t) - H_i(t)C(t)), \qquad H(t) := \begin{bmatrix} cH_1(t) \\ H_2(t)\end{bmatrix},$$

In the following let $\Phi(t_2,t_1)$ be the transition matrix related to $F(t)$. It is defined via the matrix differential equation

$$\frac{d\Phi}{dt} = F(t)\Phi(t,t_1), \quad \Phi(t_1,t_1) = I. \tag{3}$$

Remark 1 *If $F(t) = F$ is time–invariant then the transition matrix is given by $\Phi(t_2,t_1) = e^{F(t_2-t_1)}$.*

The state estimate in the proposed observer (1) is given by

$$\begin{aligned}\hat{x}(t) &= K(t)[z(t) - \Phi(t,t-D)z(t-D)], &\text{(4a)}\\ K(t) &:= [I_{n,n}, 0_{n,n}][T, \Phi(t,t-D)T]^{-1}, &\text{(4b)}\end{aligned}$$

where $I_{n,n}$ denotes the identity matrix of dimension $n\times n$, $T := \left[I_{n,n}, I_{n,n}\right]^T$, and where $D > 0$ denotes a (time) delay.

Remark 2 *For validity of equation* (4) *the observer states $z(t)$ must be defined for $t \in [t_0 - D, t_0]$, even so that the observer might be started at t_0. We assume, without loss of generality, that $z(t)$ is fixed to $z_0 = T\hat{x}_0$ for $t \in [t_0 - D, t_0]$, where $\hat{x}_0$ is arbitrary but bounded.*

As stated in the following theorem the estimate $\hat{x}(t)$ converges, under certain conditions, in finite time to the real state $x(t)$

Theorem 1 *If $H(t)$ and D are (chosen) such that*

1) *$H(t)$ is bounded and piecewise continuous,*
2) *the matrix $[T, \Phi(t,t-D)T]^{-1}$ exists for all $t \geq t_0$,*

then the observer error $e(t) = \hat{x}(t) - x(t)$ for the state estimate $\hat{x}(t)$ given by (2) *and* (4) *vanishes for $t \geq t_0 + D$. Furthermore, the estimation error $e(t)$ remains bounded during the convergence interval $t_0 \leq t < t_0 + D$.*

Proof: For $t \geq t_0$ we have that

$$\begin{aligned}\frac{d(z(t) - Tx(t))}{dt} &= F(t)z(t) + G(t)u(t) + H(t)y(t)\\ &\quad -T(A(t)x(t) + B(t)u(t))\\ &= F(t)(z(t) - Tx(t))\\ &\quad +(F(t)T - TA(t) + H(t)C(t))x(t)\\ &\quad +(G(t) - TB(t))u(t)\\ &= F(t)(z(t) - Tx(t)).\end{aligned}$$

Thus for any $t \geq t_0 + D$

$$z(t) = Tx(t) + \Phi(t,t_0)(z(t_0) - Tx(t_0)) \tag{5}$$

and

$$z(t-D) = Tx(t-D) + \Phi(t-D,t_0)(z(t_0) - Tx(t_0)). \tag{6}$$

Substituting (5) and (6) into (4) leads to

$$\begin{aligned}\hat{x}(t) &= K(t)[z(t) - \Phi(t,t-D)z(t-D)]\\ &= K(t)Tx(t) + K(t)\Phi(t,t_0)(z(t_0) - Tx(t_0)) \qquad (7)\\ &\quad -K(t)\Phi(t,t-D)Tx(t-D)\\ &\quad -K(t)\Phi(t,t-D)\Phi(t-D,t_0)(z(t_0) - Tx(t_0)).\end{aligned}$$

Furthermore, from the definition of $K(t)$ we obtain

$$\begin{aligned}K(t)\Phi(t,t-D)T &= 0_{n,n}\\ K(t)T &= I_{n,n}.\end{aligned}$$

Using this together with the standard property

$$\Phi(t,t-D)\Phi(t-D,t_0) = \Phi(t,t_0)$$

of transition matrices we can conclude from (7) that

$$\hat{x}(t) = x(t), \quad \forall\, t \geq t_0 + D.$$

It remains to show that $e(t)$ stays bounded during the convergence interval $t_0 \leq t < t_0 + D$. Substituting (5) into equation (4) and using that $z(t) = T\hat{x}_0$, $t \in [t_0 - D, t_0]$ leads to

$$\begin{aligned}\hat{x}(t) &= K(t)Tx(t) + K(t)\Phi(t,t_0)(T\hat{x}_0 - Tx(t_0))\\ &\quad -K(t)\Phi(t,t-D)T\hat{x}_0\\ &= x(t) + K(t)\Phi(t,t_0)(T\hat{x}_0 - Tx(t_0))\end{aligned}$$

for all $t \in [t_0 - D, t_0]$. Therefore

$$\hat{x}(t) - x(t) = K(t)\Phi(t,t_0)(T\hat{x}_0 - Tx(t_0)).$$

Thus $e(t)$ stays bounded since $K(t)$ exists. ■

So far we only outlined sufficient conditions on $H(t)$ and the delay D such that the proposed estimator is finite time convergent. For general time–varying systems, however, there is no guarantee that suitable values for $H(t)$ and D exists and how they should be chosen to satisfy the conditions.

Even so that in some cases it might be possible to calculate the transition matrix explicitly, thus allowing to obtain suitable values for $H(t)$ and D, this is in general not possible and limits the applicability of the derived results. To overcome this problem we outline in the next section that for systems which can be brought to observer canonical form suitable values always exist and can be easily obtained in a constructive way.

III. Finite time convergent observers for systems that can be transformed to observer canonical form

For notational simplicity we limit the following presentation to MISO systems, i.e. $y(t) \in \mathbb{R}$. Note, however, that the results can be straightforwardly expanded to MIMO systems.

In order to derive a constructive way to compute a finite time convergent estimator we assume that the system is uniformly observable (see for example [12]):

Definition 1 *The system* (1) *is called uniformly observable if the observability matrix*

$$Q(t) = \begin{bmatrix} c(t) \\ L_A c(t) \\ \vdots \\ L_A^{n-1} c(t) \end{bmatrix}$$

has rank n for all times t, where the differential operator L_A is defined as $L_A c(t) := \dot{c}(t) + c(t)A(t)$.

If the system is uniformly observable, then there exists a state transformation of the form

$$w(t) = S(t)x(t),\ x(t) = S^{-1}(t)w(t)$$

with $S \in \mathbb{R}^{n\times n}$ and $w \in \mathbb{R}^n$ which transforms the system (1) into observer canonical form [12], [13]:

$$\dot{w}(t) = E_n w(t) - a(t)w_n(t) + \bar{B}(t)u(t) \tag{8a}$$

$$\begin{aligned} y(t) &= w_n(t) \\ &= \bar{c}\, w(t), \end{aligned} \tag{8b}$$

where

$$E_n := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix},\ a(t) := \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix},\ \bar{c} := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The transformation to observer canonical form can be obtained from the inverse of the observability map Q^{-1} as outlined in [14]: Let

$$q(t) = Q^{-1}(t)[0 \quad \cdots \quad 0 \quad 1]^T$$

be the last column of the inverse observability matrix. Furthermore define the differential operator $\bar{L}_A$ as

$$\bar{L}_A q(t) := -\dot{q}(t) + A(t)q(t).$$

Then

$$S^{-1}(t) = [q(t), \bar{L}_A q(t), \cdots, \bar{L}_A^{n-1} q(t)].$$

The values of $a(t)$ and $\bar{B}(t)$ can be calculated straightforwardly from S [12], [13]. More precisely $a(t)$ is given as

$$-a(t) = S(t)\bar{L}_A^n q(t)$$

and $\bar{B}(t)$ as

$$\bar{B}(t) = S(t)B(t).$$

The key advantage of systems in observer canonical form is that the time-variant term $a(t)$ can be compensated by the observer correction term. The resulting observer error dynamics are time-invariant and thus simplify the observer design. The state estimates in the canonical observer coordinates can be transformed back to the original coordinates using the transformation matrix.

Guided by the derivations in Section II consider the following modified observer equations in observer canonical form

$$\dot{z}_i(t) = (E_n - \bar{h}_i\bar{c})z_i(t) - a(t)y(t) + \bar{B}(t)u(t) + \bar{h}_i y(t),$$

for $z_i \in \mathbb{R}^n$, $i = 1,2, t \geq t_0$ with $\bar{h}_i \in \mathbb{R}^n$. This can be shortly expressed as

$$\dot{z}(t) = \bar{F}z(t) + \bar{G}(t)u(t) - A(t)y(t) + \bar{h}y(t), \tag{9}$$

where

$$\bar{F} := \begin{bmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{bmatrix}, \qquad \bar{F}_i := E_n - \bar{h}_i\bar{c},$$

$$\bar{G}(t) := \begin{bmatrix} \bar{B}(t) \\ \bar{B}(t) \end{bmatrix}, \qquad \bar{h} := \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix}, \qquad A(t) := \begin{bmatrix} a(t) \\ a(t) \end{bmatrix}.$$

The state estimate is given identically to (4) by:

$$\hat{w}(t) \quad = \quad K[z(t) - e^{\bar{F}D}z(t-D)], \tag{10a}$$

$$K \quad := \quad [I_{n,n}, 0_{n,n}][T, e^{\bar{F}D}T]^{-1}. \tag{10b}$$

Given this setup the following theorem holds:

Theorem 2 *Assume that $\bar{h}$ and D are chosen such that $[T, e^{\bar{F}D}T]^{-1}$ exist. Then the estimation error $e(t) = \hat{w}(t) - w(t)$ of the state estimate $\hat{w}(t)$ given by* (9) *and* (10) *for the system* (8) *is zero for $t \geq t_0 + D$. Furthermore, the estimation error $e(t)$ remains bounded during the convergence interval $t_0 \leq t < t_0 + D$.*

Proof: From (9) it follows that

$$\begin{aligned} \frac{d(z(t)-Tw(t))}{dt} &= \bar{F}z(t) + \bar{G}(t)u(t) - A(t)y(t) + \bar{h}y(t) \\ &\quad -T(E_n w(t) - a(t)y(t) + \bar{B}(t)u(t)) \\ &= \bar{F}(z(t) - Tw(t)) \\ &\quad +(\bar{F}T - TE_n + \bar{h}\bar{c})w(t) \\ &= \bar{F}(z(t) - Tw(t)). \end{aligned}$$

The rest of the proof is identical to [9]. ■

In comparison to the general time-varying case it is easy to satisfy the assumptions of Theorem 2. In [9] it is shown that $[T, e^{\bar{F}D}T]^{-1}$ exists for almost any D if the real parts eigenvalues of $\bar{F}_1$ and $\bar{F}_2$ are separated:

$$\mathrm{Re}(\lambda_i(\bar{F}_2)) < \sigma < \mathrm{Re}(\lambda_j(\bar{F}_1)) < 0, \quad \forall\, i,j = 1,2,\ldots,n.$$

Note that the pair $(E_n, \bar{c})$ is observable. Therefor the eigenvalues of $\bar{F}_i$, $i = 1,2$ can be placed arbitrarily choosing $\bar{h}_i$, $i = 1,2$ appropriately.

We expand the results given in [9] slightly by showing that for any desired D there exist vectors $\bar{h}_i$ such that the inverse of $[T, e^{\bar{F}D}T]$ exists.

Lemma 1 *If the pair (A,c) is observable, then for any $D > 0$ two vectors h_i, $i = 1,2$ exist such that $[T, e^{FD}T]$ with*

$$e^{FD} := \begin{bmatrix} e^{F_1 D} & 0 \\ 0 & e^{F_1 D} \end{bmatrix}$$

where $F_i := A + h_i c$, $i = 1,2$, has full rank.

Proof: See Appendix. ■

Remark 3 *To obtain the states in original coordinates we can use*

$$\hat{x}(t) = S^{-1}(t)K[z(t) - e^{\bar{F}D}z(t-D)].$$

Thus if the system is uniformly observable and S is known, one can design an observer for the MISO system.

Summarizing we have the following corollary for linear time–varying single output systems:

Corollary 1 *Assume that the system* (1) *has one output and furthermore that it is uniformly observable. Then an observer for this system exists which estimation error converges to zero within any arbitrary finite time* $D > 0$.

Note that the results can be trivially expanded to the MIMO case. Furthermore, the proposed estimator is still applicable if the systems is only uniformly observable for a time interval $[t_0, t_1]$, $t_1 > t_0$. In this case, see Corollary 1, it is possible to construct a finite time convergent observer with arbitrary short convergence time. This observer could for example be used in combination with another observer that does not lack from the loss of uniform observability (e.g. a moving horizon observer). After the time t_1 one would switch to the second observer with the (already converged) estimates of the proposed observer.

Another alternative would be to stop the observer after the time t_1 up to a time t_2 at which the system is again uniformly observable. In between an open-loop simulation of the system could be performed to provide (uncorrected) state estimation.

Summarizing we have shown in this section that under the assumption of uniform observability it is always possible to design a finite time convergent observer for the time-varying system. Furthermore, we have outlined how suitable observer parameters can be obtained. In the next section we shortly present a simple example showing the application of the derived results.

IV. Example

As example we consider the MIMO system described in [12], Section 6.4:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -\cos t & \sin t \\ 0 & 0 & 1 & \cos t \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ \sin t & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} x(t)\end{aligned}$$

With the transformation $w(t) = S(t)x(t)$ where S is given by:

$$S(t) = \begin{bmatrix} -1 & 1 & 1 & \cos t \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

it is possible to transform the system to observer canonical form (see [12]):

$$\begin{aligned}\dot{w}(t) &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{:=\bar{A}} w(t) + \underbrace{\begin{bmatrix} \cos^2 t \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{:=a_1(t)} w_2(t) \\ &\quad + \underbrace{\begin{bmatrix} 1-\cos^2 t \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{:=a_2(t)} w_4(t) + \underbrace{\begin{bmatrix} 1+\sin t & 1-\cos t \\ 0 & -1 \\ \sin t & 1 \\ 0 & 0 \end{bmatrix}}_{:=\bar{B}(t)} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{:=\bar{C}} w(t).\end{aligned}$$

As observer equations we can for example choose

$$\dot{z}_i(t) = (\bar{A} - \bar{H}_i\bar{C})z_i(t) - a_1(t)y_2(t) - a_2(t)y_1(t) + \bar{H}_i y(t),$$

for $z_i \in \mathbb{R}^n$, $i = 1,2, t \geq t_0$ with

$$\bar{H}_i = \begin{bmatrix} 0 & d_1^i \\ 0 & d_2^i \\ e_1^i & 0 \\ e_2^i & 0 \end{bmatrix}.$$

The matrices $\bar{F}_i = \bar{A} - \bar{H}_i\bar{C}$, $i = 1,2$ then take the structure

$$F_i = \begin{bmatrix} 0 & -d_1^i & 0 & 0 \\ 1 & -d_2^i & 0 & 0 \\ 0 & 0 & 0 & -e_1^i \\ 0 & 0 & 1 & -e_2^i \end{bmatrix}.$$

Picking $D > 0$ and following the proof of Lemma 1 one can pick the parameters d_j^i, e_j^i, $i, j = 1,2$ to ensure that the inverse of $[T, e^{\bar{F}D}T]$ with

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{bmatrix}$$

exists. Then the matrix K is given by $K = [I_{n,n}, 0_{n,n}][T, e^{\bar{F}D}T]^{-1}$ and using

$$\hat{x}(t) = S^{-1}(t)K[z(t) - e^{\bar{F}D}z(t-D)]$$

we obtain an observer with finite time convergence of the estimation error $\hat{x}(t) - x(t)$ in original coordinates in the time D. Note that the matrix $S^{-1}(t)$ is given by:

$$S^{-1}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -\cos t & -1 & 1+\cos t \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

As can be seen from this example, the key obstacle for the application of the derived results is the transformation to observer canonical form. Once the system is transformed to the observer canonical form it is trivial to pick a suitable observer structure to cancel the (disturbing) time-varying part in the observer error dynamics and to pick suitable parameters to achieve finite time convergence.

V. CONCLUSION

Observing the state of a continuous time system is an important problem. In this paper we outlined how the finite-time convergent observer design for continuous time-invariant systems as outlined in [9] can be expanded to the linear time-varying case. After presenting the considered observer structure we stated conditions on the observer parameters that ensure the finite time convergence in the general case. One of the key drawbacks of the derived conditions is, that suitable values for the observer parameters are in general difficult to choose, and that the derived conditions are often difficult to check. To at least part wise overcome this problem we showed that for MISO systems that are uniformly observable it is always possible to design a finite-time convergent observer and that the necessary parameters can be easily obtained in observer canonical form. While the results are only derived for MISO systems they can be trivially expanded to MIMO systems. The proposed methods offer especially in the case that the transformation to observer canonical form is already available an efficient method for the design of finite time convergent observers. In comparison to the sliding-mode observer [2], [3], [4] and moving horizon based observers [5], [6], [7], [8] the resulting observer is simple to implement. However, one should also notice that the designed observer strongly depends on the system structure, whereas for example the moving horizon based observers are rather independent of the considered type of system.

We finally note that the results can be expanded to the general nonlinear time-varying case utilizing suitable transformations to normal forms.

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APPENDIX

Proof: (Proof of Lemma 1) Since the pair (A,c) is observable a transformation S exists such that

$$\begin{aligned} S^{-1}AS &= \bar{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix} \\ cS &= \bar{c} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned}$$

in transformed coordinates. Choose $n = dimA$ real, negative and distinct eigenvalues $\lambda_1 < \cdots < \lambda_n < 0$. Let b_k, $k = 1, \ldots, n$ fulfill the characteristic polynomial

$$\prod_{i=1}^{n}(s-\lambda_i) = s^n + \sum_{i=0}^{n-1} b_i s^i.$$

Pick

$$\begin{aligned} \bar{h}_i = & \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}^T + \\ & \begin{bmatrix} -\alpha_i^n b_0 & -\alpha_i^{n-1} b_1 & \cdots & -\alpha_i b_{n-1} \end{bmatrix}^T \end{aligned}$$

then the matrices

$$\bar{F}_i = \bar{A} + \bar{h}_i \bar{c} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -\alpha_i^n b_0 \\ 1 & 0 & 0 & \cdots & -\alpha_i^{n-1} b_1 \\ 0 & 1 & 0 & \cdots & -\alpha_i^{n-2} b_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -\alpha_i b_{n-1} \end{bmatrix}$$

have the eigenvalues $\alpha_i \lambda_j$, $j = 1, \ldots, n$, $i = 1,2$. The matrices $\bar{F}_i$, $i = 1,2$ can be expressed as

$$\bar{F}_i = K_{\alpha_i} \bar{T}(\alpha_i \Lambda) \bar{T}^{-1} K_{\alpha_i}^{-1}, i = 1,2$$

with

$$K_{\alpha_i} := \begin{bmatrix} \alpha_i^{n-1} & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \Lambda := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

The matrix $\bar{T}$ is the transformation matrix of the matrix $\bar{F}_i$ with α_i set to one to diagonal (Jordan) form Λ. With $h_i = S\bar{h}_i$, $i = 1,2$ we have in original coordinates

$$F_i = A + h_i c = A + S\bar{h}_i c = SK_{\alpha_i} \bar{T}(\alpha_i \Lambda) \bar{T}^{-1} K_{\alpha_i}^{-1} S^{-1}.$$

It is easy to see that $[T, e^{FD}T]^{-1}$ exists if $(e^{F_2 D} e^{-F_1 D} - I)^{-1}$ exists. This inverse exists for example if the maximal singular value of $e^{F_2 D} e^{-F_1 D}$ is lesser then one. Assume that $\alpha_2 > \alpha_1 \geq 1$ then the maximal singular value of $e^{F_2 D} e^{-F_1 D}$ can be bounded from above:

$$||e^{F_2 D} e^{-F_1 D}|| \leq \alpha_2^{n-1} ||S|| ||S^{-1}|| (||\bar{T}|| ||\bar{T}^{-1}||)^2 e^{(\alpha_2 \lambda_n - \alpha_1 \lambda_1) D}.$$

Therefore if α_1 and α_2 are chosen such that

$$\alpha_2^{n-1} ||S|| ||S^{-1}|| (||\bar{T}|| ||\bar{T}^{-1}||)^2 e^{(\alpha_2 \lambda_n - \alpha_1 \lambda_1) D} < 1,$$

we also have that $||e^{F_2 D} e^{-F_1 D}|| < 1$. This concludes the proof since this choice of α_1 and α_2 is always possible. ■