Unknown inputs, robust state observation for parameter dependent linear systems

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***Abstract*— The problem of estimating the largest possible component of the state of a parameter dependent linear system in presence of an unknown input is considered. Using a geometric approach and suitable defined invariant subspaces that capture the concept of robustness with respect to parameter variation a procedure for detecting the largest component of the state that may possibly be estimated, a construction of the observer system and a technique for analyzing its asymptotic behavior is provided.**

***Index Terms*— parameter dependent linear systems, robust state estimation geometric methods.**

I. Introduction

The problem we consider in this paper is the state estimation of a linear system $\Sigma(q)$ depending on a parameter q, when unknown inputs are present. Unknown inputs may represent non accessible disturbances or may model limitations on the available information, as it happens e.g. in decentralized control schemes. Estimation of the state or, at least, of a component of the state, is useful for implementing control actions and/or for detecting and isolating faults.
Our interest, here, is in finding the largest possible component of the state x of $\Sigma(q)$, not depending on q, that can be asymptotically estimated by means of a suitable observer system. The independence of the estimated component from q motivates the choice, in the spirit of [2], of calling *robust* the state observation scheme we develop, although the observer system is itself a parameter dependent system.
The approach we follow is based on geometric tools and techniques (see [1], [2]) and it employs suitable notions of invariant subspaces which capture the concept of robustness, with respect to parameter variations, we have mentioned before. The results of the paper are a procedure for detecting the largest component of the state that may possibly be estimated, a construction of the observer system and a technique for analyzing its asymptotic behavior.

II. Preliminaries and notations

Let us consider a linear system $\Sigma(q)$, depending on a vector of real parameters $q \in Q \subseteq R^r$, where Q is an open set, defined by the the following set of equations

$$\Sigma(q) = \begin{cases} \dot{x}(t) & = & A(q)x(t) + B(q)u(t) \\ y(t) & = & C(q)x(t) \end{cases} \qquad (1)$$

where $x \in \mathcal{X} = R^n$ is the state, $u \in \mathcal{U} = R^m$ is the input, $y \in \mathcal{Y} = R^p$ is the output and $A(q)$, $B(q)$, $C(q)$ are matrices of suitable dimensions with entries depending on q.
Our goal is to investigate geometric conditions which guarantee the existence of an asymptotic observer, depending on the parameter $q \in Q$, for the state x, in case u is an unknown input. To this aim we introduce the notion of *robustness* for conditioned invariant subspaces.

Definition 1: Given a system $\Sigma(q)$ of the form (1), a subspace $\mathcal{S}$ of the state space $\mathcal{X}$ is said a *robust conditioned invariant subspace* if

$$A(q)(\mathcal{S} \cap \mathrm{Ker}\, C(q)) \subseteq \mathcal{S}, \ \ \forall q \in Q.$$

The notion of robust conditioned invariant subspace is dual to in that of robust controlled invariant subspace described in [2]. The same kind of invariant subspace has been introduced in [5].

Assumption 1: Given $\Sigma(q)$ as above, assume that the image of the input map $B(q)$ does not depend on q, namely $Im\, B(q) = \mathcal{B}$ for every $q \in Q$.

Under the above assumption it is possible to consider the set $\mathfrak{S}(A(q), C(q), \mathcal{B})$ of all the robust conditioned invariant subspaces containing $\mathcal{B}$. This set is a lower semilattice with respect to inclusion and intersection and thus it admits a minimum element denoted by $\mathcal{S}^*(\mathcal{B})$ or, shortly, by $\mathcal{S}^*$.
Remark that, for any fixed $\bar{q} \in Q$ there exists an output injection matrix $G(\bar{q})$ such that

$$(A(\bar{q}) + G(\bar{q})C(\bar{q}))\mathcal{S}^* \subseteq \mathcal{S}^*.$$

We will call the parameter depending matrix $G(q)$, for $q \in Q$, a friend of $\mathcal{S}^*$

III. Computation of $\mathcal{S}^*$

In the following we show that $\mathcal{S}^*$ can be computed in a finite number of steps as the limit of an ascending sequence of subspaces. Given a family $\mathcal{W}(q)$ of subspaces in R^n depending on a parameter $q \in Q$, we shortly denote by $\overline{\overline{\mathcal{W}(q)}}$ their sum, the minimum subspace containing all the $\mathcal{W}(q)$'s, namely

$$\overline{\overline{\mathcal{W}}} := \sum_{q \in Q} \mathcal{W}(q).$$

Given a system $\Sigma(q)$ as above, let us construct the following sequence of subspaces $\{\mathcal{S}_i\}_{i=1,\ldots}$:

$$\begin{aligned} \mathcal{S}_0 &:= \mathcal{B} \\ \mathcal{S}_i &= \mathcal{S}_{i-1} + \overline{\overline{A(q)(\mathcal{S}_{i-1} \sqcap \text{Ker } C(q))}} \end{aligned} \quad (2)$$

and let us prove the following proposition.

Proposition 1: The sequence $\{\mathcal{S}_i\}_{i=1,\ldots}$ converges to $\mathcal{S}^*$ in a finite number of steps.

Proof: $\{\mathcal{S}_i\}_{i=1,\ldots}$ is an ascending sequence, since at each step the dimension of $\mathcal{S}_i$ is greater or equal to the dimension of $\mathcal{S}_{i-1}$. Therefore, since $\mathcal{S}_i \sqsubseteq R^n$ for every i, there exists $k \geq 0$ such that $\mathcal{S}_k = \mathcal{S}_{k-1}$. It is easy to see that this implies that $\mathcal{S}_k = \mathcal{S}_{k-s}$ for all $s > 0$. The limit $\mathcal{S}_k$ is indeed a robust conditioned invariant subspace. In fact, from

$$\mathcal{S}_k = \mathcal{S}_{k-1} = \mathcal{S}_k + \overline{\overline{A(q)(\mathcal{S}_k \sqcap \text{Ker } C(q))}},$$

we have, for any $q \in Q$,

$$A^{-1}(q)(\mathcal{S}_k + \overline{\overline{A(q)(\mathcal{S}_k \sqcap \text{Ker } C(q))}}) = A^{-1}(q)(\mathcal{S}_k),$$

therefore

$$(\mathcal{S}_k \sqcap \text{Ker } C(q)) \sqsubseteq A^{-1}(q)(\overline{\overline{A(q)(\mathcal{S}_k \sqcap \text{Ker } C(q))}})$$

and

$$(\mathcal{S}_k \sqcap \text{Ker } C(q)) \sqsubseteq A^{-1}(q)\mathcal{S}_k.$$

Assume now that $\mathcal{S}$ is a robust conditioned invariant subspace containing $\mathcal{B}$. We have $\mathcal{S}_0 = \mathcal{B}$ and hence $\mathcal{S}_0 \sqsubseteq \mathcal{S}$.
Assume that $\mathcal{S}_j \sqsubseteq \mathcal{S}$: then, from

$$A(q)(\mathcal{S} \sqcap \text{Ker } C(q)) \sqsubseteq \mathcal{S}, \quad \forall q \in Q,$$

it follows

$$\mathcal{S}_{j-1} = \mathcal{S}_j + \overline{\overline{A(q)(\mathcal{S}_j \sqcap \text{Ker } C(q))}} \sqsubseteq \mathcal{S}.$$

Then, the limit $\mathcal{S}_k$ is contained in $\mathcal{S}$ and hence $\mathcal{S}_k = \mathcal{S}^*$. ■

The sequence (2) gives an explicit procedure for computing $\mathcal{S}^*$ for the systems $\Sigma(q)$ under the assumption $Im\, B(q) = \mathcal{B}$ for every $q \in Q$.
However, its implementation needs the computation of

$$\overline{\overline{A(q)(\mathcal{S}_{i-1} \sqcap \text{Ker } C(q))}} := \sum_{q \in Q} A(q)(\mathcal{S}_{i-1} \sqcap \text{Ker } C(q))$$

for several values of i.

In practice, this could reveal to be a difficult task, because of the presence of the parameters $q \in Q$. The kind of difficulties which one is likely to face are better focused through the following algorithm, which gives an explicit construction of $\overline{\overline{A(q)(\mathcal{S}_i \sqcap \text{Ker } C(q))}}$.

Summation algorithm

Step 0 Choose an arbitrary value $q' \in Q$ and define the subspace $\mathcal{W}_{i,0}$ as follows

$$\mathcal{W}_{i,0} := A(q')(\mathcal{S}_i \sqcap \text{Ker } C(q')).$$

If $dim\, \mathcal{W}_{i,0} = n$, then stop.

Step j For every $q \in Q$, check the condition

$$A(q)(\mathcal{S}_i \sqcap \text{Ker } C(q)) \sqsubseteq \mathcal{W}_{i,j-1}. \quad (3)$$

If condition (3) is satisfied for all $q \in Q$, then stop; otherwise, if condition (3) is not satisfied for $q'' \in Q$ define

$$\mathcal{W}_{i,j} := \mathcal{W}_{i,j-1} + A(q'')(\mathcal{S}_i \sqcap \text{Ker } C(q'')).$$

If $dim\, \mathcal{W}_{i,j} = n$, then stop.

As

$$dim\, \mathcal{W}_{i,j-1} \leq dim\, \mathcal{W}_{i,j} \leq n,$$

the above algorithm converges in a finite number of steps to

$$\overline{\overline{A(q)(\mathcal{S}_i \sqcap \text{Ker } C(q))}},$$

although it presents the problem of checking condition (3) for all $q \in Q$. However, as in [4], the following Assumption simplifies the situation.

Assumption 2: Let Q be an open subset in R and assume that the elements of $A(q)$ and $C(q)$ are polynomials in the scalar parameter $q \in Q$.

Under such assumption, it is computationally feasible to check condition (3). In fact, except for a finite number of values of q, that can be dealt with one by one, it is possible to chose a polynomial matrix $H_{i-1}(q)$ whose columns span the subspace $\mathcal{S}_{i-1} \sqcap \text{Ker } C(q)$.
Then, denoting by $W_{i,j-1}$ a basis matrix for the subspace $\mathcal{W}_{i,j-1}$, condition (3) is equivalent to

$$rank\, [A(q)H_{i-1}(q) \mid W_{i,j-1}] = rank\, W_{i,j-1}, \quad (4)$$

which, in turn, means that all the minors of $[A(q)H_{i-1}(q)|W_{i,j-1}]$ including columns of $W_{i,j-1}$ and one column of $A(q)H_{i-1}$, are the zero polynomial.

In case $Ker\, C(q)$ is constant, i.e.

$$Ker\, C(q) = \mathcal{C} \text{ for all q } \in Q,$$

the computation of $\overline{\overline{A(q)(\mathcal{S}_{i-1} \sqcap \mathcal{C})}} = \sum_{q \in Q} A(q)(\mathcal{S}_{i-1} \sqcap \mathcal{C})$ is still easier.

In this case, Assumption 2 can indeed be weakened, since what we need is simply the ability to check if linear combinations of the entries of $A(q)$ are zero for all $q \in Q$.

Example 1: Consider a system $\Sigma(q)$ of the form (1), where

$$A(q) = \begin{bmatrix} \sin(q) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} e^q \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad C = [0, 0, 1]$$

and $q \in Q = R^-$. We have

$$\mathcal{B} = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$Ker\ C = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

In this case Ker C does not depend on q. We have

$$\mathcal{S}_0 \sqcap \text{Ker } C = \mathcal{B}$$

and we can chose the vector $v = \{(1, 0, 0)^t\}$ as a basis. At Step 0 we can choose $q' = 0$ and define

$$\mathcal{W}_{0,0} := A(0)(\mathcal{S}_0 \sqcap \text{Ker } C).$$

We have

$$W_{0,0} := A(0)v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Step 1 For every q,

$$A(q)(\mathcal{S}_0 \sqcap \text{Ker } C) = span \left\{ \begin{pmatrix} \sin(q) \\ 1 \\ 0 \end{pmatrix} \right\},$$

then there are minors of dimension 2 in

$$[A(q)v | W_{0,0}] = \begin{bmatrix} \sin(q) & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which are different from zero, for instance for $q'' = \pi/2$.

We define

$$W_{0,1} := \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and then we look at the matrix

$$[A(q)v \mid W_{0,1}] = \begin{bmatrix} \sin(q) & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

that cannot have minors of rank greater than 2.
We conclude that

$$\overline{\overline{A(q)(\mathcal{S}_0 \sqcap \text{Ker } C)}} = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and that $\mathcal{S}_1 = \mathcal{S}_0 + \overline{\overline{A(q)(\mathcal{S}_0 \sqcap \text{Ker } C)}} = \mathcal{S}^*$, namely

$$\mathcal{S}^* = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Remark that the computation of $\mathcal{S}^*$ may be implemented in a very efficient way by means of symbolic computations tools, such as MapleV, Mathematica and CoCoA (see [3]).

IV. Unknown input observers

We can now consider the problem of estimating the state of the system $\Sigma(q)$, described by the equations (1), where u is an unknown input.

Assume that $Im\ B(q) = \mathcal{B}$ for every $q \in Q$ and denote by $\mathcal{S}^*$ the minimum robust conditioned invariant subspace containing $\mathcal{B}$.
Then, we can prove the following theorem.

Theorem 1: In the above hypothesis, given the systems $\Sigma(q)$, let $G(q)$ be such that

$$(A(q) + G(q)C(q))\mathcal{S}^* \subseteq \mathcal{S}^*$$

for all $q \in Q$ and consider the observer system $\Sigma_o(q)$ defined by

$$\dot{z}(t) = (A(q) + G(q)C(q))z(t) - G(q)y(t). \tag{5}$$

Then, if $z(0) = x(0)$, we have $z(t)_{\text{mod } \mathcal{S}^*} = x(t)_{\text{mod } \mathcal{S}^*}$ for all $t \geq 0$.

Proof: Let us consider the estimation error

$$e(t) := z(t) - x(t).$$

By subtracting the first equation in (1) from (5), we obtain

$$\dot{e}(t) = (A(q) + G(q)C(q))e(t) - B(q)u(t). \tag{6}$$

If $e(0) = z(0) - x(0) = 0$, the error evolves in $\mathcal{S}^*$, the minimum conditioned invariant subspace containing $Im\ B(q) = \mathcal{B}$. Therefore, on the quotient space $\mathcal{X}/\mathcal{S}^*$ the error is identically zero and $z(t)_{\text{mod } \mathcal{S}^*}$ coincides with the state $x(t)_{\text{mod } \mathcal{S}^*}$ for all $t \geq 0$. In other words, a component of the state $z(t)$ of the observer reproduces the component of $x(t)$, namely $x(t)_{\text{mod } \mathcal{S}^*}$, which does not depend on u. ■

As soon as the initial value $z(0)$ of the state of the observer is different from $x(0)$, a stability issue arises. A way to deal with the problem of assuring that the estimation error goes to 0 mod$\mathcal{S}^*$, is that of analyzing the poles of $\Sigma_o(q)|_{\text{mod } \mathcal{S}^*}$, for fixed q, that cannot be moved by any choice of $G(q)$ (compare with [1]). In order to characterize the possibility of assigning arbitrarily all the poles of $\Sigma_o(q)|_{\text{mod } \mathcal{S}^*}$ in a robust sense, that is for all $q \in Q$, we need to introduce another geometric object.
Let us assume that $C(q)$ does not depend on $q \in Q$, in particular $Ker\ C(q) = \mathcal{C}$ for all $q \in Q$, and let us consider, following [2], the following subspace.

Definition 2: Denoting by $\mathcal{V}(A(q), \mathcal{S}^*, Ker\ C)$ the family of subspaces $\mathcal{V} \subseteq Ker\ C \subseteq \mathbb{R}^n$ such that

$$A(q)\mathcal{V} \subseteq \mathcal{V} + \mathcal{S}^*, \ \ \forall q \in \mathcal{Q},$$

let $\mathcal{V}^*(\mathcal{S}^*)$ (or, for sake of brevity, $\mathcal{V}^*$) be the maximum element of such family.

In the terminology of [2], $\mathcal{V}^*$ is the maximum robust controlled invariant subspace, with respect to $\mathcal{S}^*$, that contains $\mathcal{B}$ (see [2] and [4] for properties and construction of this object). It can be proved that $\mathcal{V}^* + \mathcal{S}^*$ is the smallest unobservability subspace containing $\mathcal{S}^*$ considered in [5].

Choose now a basis matrix $\left[\begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array}\right]$ for $\mathcal{X}$, such that

$$\begin{aligned} Im[T_4] &= \mathcal{V}^* \cap \mathcal{S}^*; \\ Im[\ T_3 \ \ T_4\ ] &= \mathcal{S}^*; \\ Im[\ T_2 \ \ T_4\ ] &= \mathcal{V}^*; \\ Im[\ T_2 \ \ T_3 \ \ T_4\ ] &= \mathcal{S}^* + \mathcal{V}^*. \end{aligned}$$

In such basis, we write $A(q)$ and C as:

$$A(q) = \left[\begin{array}{cccc} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right] \tag{7}$$

and

$$C = \left[\begin{array}{cccc} C_1 & 0 & C_3 & 0 \end{array}\right],$$

where, in particular, $A_{ij} = A_{ij}(q)$.
We have $A_{14} = A_{24} = 0$ since

$$A(q)(\mathcal{S}^* \bigcap Ker\ C) \subseteq \mathcal{S}^*$$

and $A_{12} = 0$ since $A(q)\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{S}^*$.
Writing

$$\mathcal{Y} = \mathcal{W} \oplus C\mathcal{S}^*$$

for some direct summand $\mathcal{W}$, we have

$$dim\ C\mathcal{S}^* = dim\ Im\ T_3$$

since $Im\ T_3$ has no intersection with $Ker\ C$, and we can write, accordingly,

$$C = \left[\begin{array}{cccc} C_{11} & 0 & 0 & 0 \\ C_{21} & 0 & C_{23} & 0 \end{array}\right],$$

with C_{23} square and nonsingular.

Moreover, we can assume without loss of generality $C_{21} = 0$, since, otherwise, we can obtain this by applying the change of basis in $\mathcal{X}$ defined by $z = Tx$, with

$$T = \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ C_{23}^{-1}C_{21} & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right], \tag{8}$$

without modifying the structure of $A(q)$ and the location of its zero blocks.
In conclusion, we have

$$A(q) = \left[\begin{array}{cccc} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right],$$

$$C = \left[\begin{array}{cccc} C_{11} & 0 & 0 & 0 \\ 0 & 0 & C_{23} & 0 \end{array}\right].$$

Taking a friend $G(q)$ of $\mathcal{S}^*$ written as

$$G(q) = \left[\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \\ G_{41} & G_{42} \end{array}\right],$$

where $G_{ij} = G_{ij}(q)$, we have

$$A(q) + G(q)C(q) =$$

$$= \left[\begin{array}{cccc} A_{11} + G_{11}C_{11} & 0 & 0 & 0 \\ A_{21} + G_{21}C_{11} & A_{22} & 0 & 0 \\ A_{31} + G_{31}C_{11} & A_{32} & A_{33} + G_{32}C_{23} & A_{34} \\ A_{41} + G_{41}C_{11} & A_{42} & A_{43} + G_{42}C_{23} & A_{44} \end{array}\right].$$

If we now consider the quotient system $\Sigma_o(q)|_{mod\ \mathcal{S}^*}$, we can state the following result.

Theorem 2: With the above notations, for any fixed q, the poles of $\Sigma_o(q)|_{mod\ \mathcal{S}^*}$ coincide with the eigenvalues of

$$A_{11} + G_{11}C_{11} = A_{11}(q) + G_{11}(q)C_{11}$$

together with those of

$$A_{22} = A_{22}(q).$$

The eigenvalues of $A_{11}(q) + G_{11}(q)C_{11}$ can be assigned arbitrarily, for any $q \in \mathcal{Q}$, by a suitable choice of $G_{11} = G_{11}(q)$ and they form the largest subset of poles $\Sigma_o(q)|_{mod\mathcal{S}^*}$ that can be arbitrarily assigned for any $q \in \mathcal{Q}$.

Proof: It follows as in [1] from the properties of $\mathcal{V}^*$, which in particular guarantee that the pair $\{A_{11}(q), C_{11}\}$ is observable for any $q \in \mathcal{Q}$. ■

The consequences of the above Proposition are that the estimation error can be driven asymptotically to 0 by a suitable choice of $G(q)$ in a robust way, that is with the same observation scheme for all values of the parameter $q \in \mathcal{Q}$, if and only if the eigenvalues of $A_{22}(q)$ have negative real part for all values of q.

Example 2: Consider the system $\Sigma(q)$ given by

$$A(q) = \left[\begin{array}{cc} (q-1)^2 & 0 \\ -q(q-2) & 1 \end{array}\right], \quad B = \left[\begin{array}{c} q \\ -q \end{array}\right]$$

and

$$C = \left[\frac{1}{2q(5-3q)}, -q(q-2)\right],$$

with $q \in Q = R^{-}$. We have

$$Ker\ C = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : b = \frac{(5-3q)}{2(q-2)} a \right\}$$

and

$$Im\ B = span \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, $S_0 = span \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To proceed with algorithm (2), we need to compute

$$\overline{\overline{A(q)(S_0 \cap \text{Ker}\ C(q))}}.$$

Applying the Summation Algorithm, we choose $q' = 1$ and we get

$$\mathcal{W}_{0,0} = span \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is easy to check that condition (3) is satisfied for all $q \in Q$ and, therefore, we have that S^* coincides with S_0.
Note that, since $A(q)S_0 \subseteq S_0$, we can choose as friend of S^* the matrix $G(q) = 0_{2\times 1}$. An unknown input observer $\Sigma_o(q)$ for the system $\Sigma(q)$ is then given by

$$\dot{z}(t) = A(q)z(t).$$

Initializing $\Sigma_o(q)$ with $z(0) = x(0)$, we have that $z(t)_{mod\ S^*}$ equals $x(t)_{mod\ S^*}$ for all $q \in Q$.

Example 3: Consider the system $\Sigma(q)$ given by

$$A(q) = \begin{bmatrix} -q & 0 & 0 \\ q+1 & 0 & 0 \\ 0 & 1 & q \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ q \\ -q \end{bmatrix}, \quad C = [1, 0, 0],$$

with $q \in Q = R^{-}$. We have

$$\mathcal{B} = span \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

and

$$Ker\ C = span \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus, $S_0 = span \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. Using the procedure described in Section 2, we find $S_0 \cap \text{Ker}\ C = \mathcal{B}$ and then

$$\begin{aligned} S_1 &= S_0 + \overline{\overline{A(q)(S_0 \cap \text{Ker}\ C)}} \\ &= span \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = S^* \end{aligned}.$$

In this example, the computation of V^* is straightforward, since $Ker C$ is $A(q)$-invariant, namely

$$A(q)(Ker\ C) \subseteq Ker\ C,$$

hence $V^* = Ker\ C$.
Working as described at the end of the previous Section, we write

$$T_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, T_3 = \emptyset, T_2 = \emptyset, T_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

then $A_{22}(q)$ is empty, therefore all the poles of the observer can be arbitrarily assigned, so that we can make the estimation error go asymptotically to 0.

In fact,

$$A(q) = \begin{bmatrix} -q & 0 & 0 \\ q+1 & 0 & 0 \\ 0 & 1 & q \end{bmatrix} = \begin{bmatrix} A_{11} & A_{14} \\ A_{41} & A_{44} \end{bmatrix}$$

and, for instance,

$$G(q) = \begin{bmatrix} q-1 \\ 0 \\ 0 \end{bmatrix}$$

does the job.

References

[1] G. Basile, G. Marro, *Controlled and conditioned invariants in linear system theory*, Prentice Hall, New York, 1992.

[2] G. Basile, G. Marro, *On the robust controlled invariant*, *System and Control Letters*, **9**, 1987.

[3] A. Capani, G. Niesi, L. Robbiano, *CoCoA, a system for doing Computations in Commutative Algebra*. Available via anonymous ftp from cocoa.dima.unige.it, 1995.

[4] G. Conte, A. M. Perdon, G. Marro, *Computing the maximum robust controlled invariant subspace*, *System and Control Letters*, **17**, pp. 131-136, 1991.

[5] Z. Szabó, J. Bokor, G. Balas, *Detection filter design for LPV systems- A geometric approach*, *Proc. 15th IFAC World Congress* Barcelona, Spain, 2002.