Stability of a Heat Process with Exponential Internal Source

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***Abstract* -- The paper investigates the stability of a 2nd order nonlinear parabolic partial differential equation (PDE). The PDE describes the temperature distribution of solids with exponential inner heat source. Due to the inner heat source the process may become unstable leading to a continuous increase in temperature. The stability of the thermal process is thus of prime interest. It is important to stress that even in case of ideal heat transfer instability may occur leading to a meltdown of the physical process. We develop the general solution of the PDE for different boundary conditions and investigate under which condition the process remains stable.**

***Index Terms* -- Stability test, distributed parameter systems, partial differential equations, heat equation.**

I. Introduction

A large class of parabolic partial differential equations (PDE) describes thermal processes arising in many industrial areas [2,3,12]. A typical problem is to determine the temperature distribution of solids for given boundary conditions or to control the temperature profile [6]. In case of an inner heat source one may encounter stability problems, i.e. more heat is generated that can be transferred to the surroundings. In many cases the heat source linearly depends on the temperature. In this case we can develop a Nyquist stability test for multilayered solids (based on the distributed transfer functions), as we described in a previous paper [13].

Now we consider a more general problem, a heat process with **exponential inner heat source**, i.e. the heat generated depends exponentially on the temperature. Typical examples are insulations (with dielectric loss), diffusion-reaction problems, vortex problems, electric space charge problems and nuclear processes. Since heat production rises exponentially with temperature, we may expect that beyond a critical inner heat gain the process becomes unstable, i.e. more heat is generated than can be transferred into the surroundings. So it is of vital importance to know the conditions of stability. Once we know the critical inner heat gain, we may lower the ambient temperatures or increase heat transfer to avoid instability - if it is possible at all.

The problem we shall consider is a very important one in many applications, among others in high-voltage cables and in certain type of nuclear processes. It is interesting from a historical point of view that one of the first attempt to solve the problem dates back to 1932. Copple, Hartree, Porter and Tyson used a differential analyzer of Vannevar Bush (MIT) to simulate the process [3]. The differential analyzer was a sort of analog computer, so they had to convert the PDE into a set of ordinary DE's. They also applied the simplest boundary conditions (namely zero). By simulation they could demonstrate that indeed for certain value of the source gain the process became unstable but they could not establish general conditions for stability.

With the advance of digital computers new simulation techniques were elaborated in the 70's [9,10,13]. One particular problem was how to approximate the exponential term in digital simulations. Angel and Bellman proposed the method of quasi linearization to overcome the problem and to simulate the steady-state solution [1]. However, no analytical solution has been developed and results concerning the stability have not been established.

In this paper we shall develop stability conditions under which the heat process remains stable. We also investigate the effect of different boundary conditions and provide a general analytic solution of the problem.

II. Problem Statement

Consider the following second order, parabolic partial differential equations which describes the temperature distribution in a solid [2,4,12]:

$$c\rho \frac{\partial \vartheta(t,\mathbf{x})}{\partial t} = div(\lambda\, grad\, \vartheta) + Q_s(t,\mathbf{x}) \qquad (1)$$

where $\vartheta(t,\mathbf{x})$ denotes the temperature distribution, c is specific heat in [Ws/kg K], ρ is the density in [kg/m^3], t denotes time in [sec], $\mathbf{x}\in\Omega$, Ω is a closed domain of p dimensional Euclidean space E^p, λ is the heat conductivity in [W/m K]. $Q_s(t,\mathbf{x})$ denotes the inner heat source. We shall assume that the source depends exponentially on the temperature distribution $\vartheta(t,\mathbf{x})$ as given by:

$$Q_s(t,\mathbf{x}) = K\, e^{b\{\vartheta(t,\mathbf{x})-\vartheta_0\}} \qquad (2)$$

where $Q_s(t,x)$ is the heat production in [W/m^3], K is the source gain [W/m^3], b is constant [1/K] and ϑ_0 is the reference temperature in [^{o}C]. Since we know that even in case of a linear heat source instability may occur [2,3,12], we certainly expect instability problems in this case. Unfortunately, the PDE is non-linear due to the form of (2) and investigating its stability also becomes more difficult [14]. We can not determine the eigenvalues of the system and thus has to choose another method. Our goal is to analyze the PDE with exponential term and establish the conditions for stability.

III. General Solution

Consider the following dimensionless heat equation with exponential source (without loosing generality we consider a one-dimensional solid):

$$\frac{\partial T(\tau,z)}{\partial \tau} = \frac{\partial^2 T(\tau,z)}{\partial z^2} + B e^{T(\tau,z)} \quad \text{in } (-1,1)\text{x}(0,\infty) \tag{3}$$

where the dimensionless variables are:

$$\begin{aligned} T &= b(\vartheta - \vartheta_0); \qquad & z &= \frac{x}{h}; \\ \tau &= \frac{\lambda}{c\rho h^2} t; \qquad & B &= \frac{\lambda}{c\rho h^2} K; \end{aligned} \tag{4}$$

We define the process to be stable if for a given source gain B the transient temperature reaches its steady-state and the steady-state is bounded:

$$\lim_{\tau\to\infty} T(\tau,z) = T_\infty(z); \qquad z \in [-1,1] \tag{5}$$

The critical gain B_c is defined as the largest gain possible still having a stable transient. Figure 1 demonstrates a stable- ($B < B_c$) and an unstable ($B > B_c$) transient.

NOTE: It is important to realize that a stable process may still exceed the maximum temperature permitted to a specific material.

To investigate the stability of (3) it suffices to study the steady-state solution of (3):

$$\frac{d^2 T(z)}{dz^2} + B e^{T(z)} = 0 \qquad z \in [-1,1] \tag{6}$$

Let us integrate both sides of (6) which lead to:

$$\left(\frac{dT(z)}{dz}\right)^2 = C_1 - 2Be^{T(z)} \tag{7}$$

where C_1 is the constant of integration. Rearranging and integrating again leads to the following inverse relation between the spatial variable z and the unknown temperature distribution $T(z)$:

$$z + C_2 = -\frac{2}{\sqrt{C_1}} \tanh^{-1}\left(\frac{\sqrt{C_1 - 2Be^{T(z)}}}{\sqrt{C_1}}\right) \tag{8}$$

We can now express the temperature distribution as:

$$T(z) = \ln\left\{\frac{C_1 - C_1 \tanh^2\left(\sqrt{C_1}(z+C_2)/2\right)}{2B}\right\} \tag{9}$$

where the unknown constants C_1 and C_2 can be determined from the actual boundary conditions. Rearranging again and introducing *cosh* we can finally express the temperature distribution $T(z)$ in the following form:

$$\boxed{T(z) = \ln\left\{2\beta^2/B\right\} - 2\ln\cosh\left(\beta(z+C_2)\right)} \tag{10}$$

where we introduced a new variable $\beta = \sqrt{C_1}/2$.

Note, that we have not yet defined the boundary conditions. The solution (10) is general and in its present form is independent of the boundary conditions. However, to establish conditions for stability, we need two boundary conditions [4,5].

In the following we shall consider stability for two boundary conditions: the Dirichlet- and mixed boundary conditions.

IV. Stability Condition for Dirichlet Boundary Condition

We consider now a heat process with Dirichlet (or 1st kind) boundary condition, i.e. the temperature is given at the boundary [2,4,14]. Assuming symmetrical geometry in $z\in[-1,1]$ the two BC's can be expressed as:

$$\underline{\text{BC-1:}} \qquad \left.\frac{dT(z)}{dz}\right|_{z=0} = 0 \tag{11}$$

$$\underline{\text{BC-2:}} \qquad T(z)\big|_{z=1} = T_1 \tag{12}$$

It follows from the first BC, that $C_2 = 0$ and the maximum temperature appears at the location $z = 0$:

$$T_m = T(z)\big|_{z=0} = \ln\left\{2\beta^2/B\right\} \tag{13}$$

Taking the BC's into account, we can define the following function between the source gain B, the unknown coefficient β, and the prescribed surface temperature T_1:

$$f_1(B,\beta,T_1) = \sqrt{\frac{2}{B}e^{-T_1}}\,\beta - \cosh(\beta) = 0 \tag{14}$$

Depending on the actual value of B, this equation may have two, one or no real solutions for β. To see that consider the first term on the right-hand side of (14) what defines a line. The second term is the $cosh(\beta)$ function which is positive in $z\in[-1,1]$. Clearly, the line may intersect the monotone increasing $cosh(\beta)$ function in two points or not at all, depending on $\sqrt{2e^{-T_1}/B}$. If the line is a tangent of $cosh(\beta)$, then there is only one solution.

We can determine the critical value of B (thus the maximum value at which a steady-state solution still exists) from the condition, that (14) has only one solution. This leads to the following equation:

$$\beta \tanh(\beta) = 0 \tag{15}$$

This equation has only one positive solution, namely

$$\beta_c = 1.1996 \tag{16}$$

This implies immediately that the critical value of B for the Dirichlet boundary condition can be expressed as:

$$\boxed{B_c = \frac{2\beta_c^2}{\cosh^2(\beta_c)}\, e^{-T_1}} \tag{17}$$

The result is very important for it shows that once we have chosen the boundary condition (value of T_1) the critical (or maximum) value of the inner heat gain B is determined. What is left is to calculate its value numerically. Note, that once the value of β_c is known, we can also determine the maximum temperature T_m:

$$T_m = T_1 + 2\ln\left(\cosh(\beta_c)\right) \tag{18}$$

Figure 2 shows the stable- and unstable regions of the process for the Dirichlet boundary condition. We can see that as T_1 increases, so decreases the critical value of B. This is understandable because at higher value of T_1 the heat generated increases exponentially, thus B must be decreased exponentially to keep the process stable.

V. Stability for Mixed Boundary Condition

We shall now consider the process with mixed (or 3rd kind) boundary condition, which is a combination of the Dirichlet and Neumann BC's. It links the outward heat-flux to the temperature difference on the surface. So the two boundary conditions can be expressed as (with dimensionless variables):

BC-1: $$\left.\frac{dT(z)}{dz}\right|_{z=0} = 0 \tag{19}$$

BC-2: $$-\left.\frac{dT(z)}{dz}\right|_{z=1} = \alpha_1\left\{\left.T(z)\right|_{z=1} - T_{amb}\right\} \tag{20}$$

where α_1 denotes the Biot number at $z = \pm 1$, T_{amb} is the ambient temperature. Without loosing generality we may assume the ambient temperature to be zero, thus $T_{amb} = 0$. The first BC leads again to $C_2 = 0$ and the maximum temperature appears at the location $z = 0$. The second boundary condition leads to the following equation:

$$2\beta\tanh(\beta) = \alpha_1\left\{T_m - 2\ln\cosh(\beta)\right\} \tag{21}$$

where T_m is defined by (13). Given B and α_1 we must determine the value of β which satisfies the boundary condition. So we define the following stability function by rearranging the boundary condition:

$$f_2(B,\beta,\alpha_1) = 2\beta\tanh(\beta) - \alpha_1\ln\left(\frac{2\beta^2}{B\cosh^2(\beta)}\right) = 0 \tag{22}$$

Figure 3 shows this function for a given α_1. We can see that for certain β values the function is negative (consider the cross-section with the zero-plane). In this region there are two real solutions (roots) in β and as B increases the two solutions are approaching each other[1]. The critical value of B can then be determined from the condition that the two real roots are equal. To locate the critical value of B we need an other condition as well, namely, the partial derivative of $f_2(B,\beta,\alpha_1)$ to β must be zero:

$$\begin{aligned}\frac{\partial f_2(B,\beta,\alpha_1)}{\partial\beta} = 2\tanh(\beta) + \frac{2\beta}{\cosh^2(\beta)} - \\ -2\alpha_1\left\{\frac{1}{\beta} - \tanh(\beta)\right\} = 0\end{aligned} \tag{23}$$

Note, that $\partial f_2(B,\beta,\alpha_1)/\partial\beta$ does not depend on the value of B! That implies that in case of the mixed boundary condition the value of α_1 determines completely the critical value of B_c.

To determine the region of stability we must find the real root of (23) for a given α_1. Table 1 gives the critical β_c and B_c values as a function of the boundary condition α_1.

Figure 4 shows the stable- and unstable region as a function of α_1.

Note, that although the critical value of B_c is monotone increasing as α_1 increases but it approaches a limit value. This is easy to see from (23) as:

[1] Recall that having two distinct solutions for β implies that our original differential equation has also two distinct solutions, i.e. there exist two different steady-state temperature distributions both satisfying the BC's. This is not possible from physical consideration. The higher value of β leads to a physically non realizable system where with increasing inner heat source the maximum temperature decreases.

$$\lim_{\alpha_1 \to \infty} \frac{\partial f_2(B,\beta,\alpha_1)}{\partial \beta} = \frac{1}{\beta} - \tanh(\beta) = 0 \tag{24}$$

The equation appears to be the same as (15), so the solution is:

$$\lim_{\alpha_1 \to \infty} \beta_c = \beta_\infty = 1{,}1996 \tag{25}$$

And so the critical values of B_c's are bounded and have a maximum value:

$$\lim_{\alpha_1 \to \infty} B_c = B_\infty = 0{,}878 \tag{26}$$

The fact that B_c is bounded implies that even in case of ideal heat transfer ($\alpha_1 \to \infty$) the process may become unstable! This is somewhat surprising but is of great practical importance for it shows the limit of cooling.

Remark 1: We can now also solve the inverse problem. Given a heat process with fixed inner heat gain, we can find the corresponding α_1 value which guarantees stability - if it exists. This if of great practical importance for in practice we can usually influence the heat transfer and not the heat source.

Remark 2: We must recognize that the result developed makes possible to analyze stability of similar PDE with different source term. If another source term is bounded by an exponential function for all values of z, than its stability stems from the stability of the later case.

VI. Conclusions

We have considered the 2nd order, non-linear PDE describing a heat process with exponential heat source. Depending on the heat source gain B and the actual boundary conditions the process may become unstable. We developed the general solution and then analyzed the stability of the process for two boundary conditions. In case of the Dirichlet BC's we give an explicit expression for the critical heat source gain B_c. For the mixed BC's we provide the stability region in graphical- and numerical form.

We can conclude that stability conditions for the PDE considered with Dirichlet- or mixed of boundary conditions have been established.

VII. References

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α_1	β_c	B_c
0	0	0
0,1	0,22174	0,0356
0,2	0,31067	0,0689
0,3	0,37687	0,1001
0,4	0,43097	0,1293
0,5	0,47686	0,1567
0,6	0,51694	0,1824
0,7	0,55246	0,2065
0,8	0,58429	0,2292
0,9	0,61304	0,2506
1,0	0,63923	0,2707
2,0	0,81289	0,4208
3,0	0,90546	0,5133
4,0	0,96269	0,5752
5,0	1,00145	0,6194
6,0	1,02937	0,6524
7,0	1,05053	0,6780
8,0	1,06681	0,6985
9,0	1,08003	0,7151
10,0	1,09085	0,7290
15,0	1,12479	0,7735
20,0	1,14259	0,7976
25,0	1,15356	0,8127
30,0	1,16099	0,8231
35,0	1,16636	0,8306
40,0	1,17043	0,8364
45,0	1,17360	0,8409
50,0	1,17616	0,8445
100,0	1,18781	0,8612
∞	1,1996	0,878

Table 1. The critical values of β_c and B_c as a function of the mixed boundary condition (α_1).

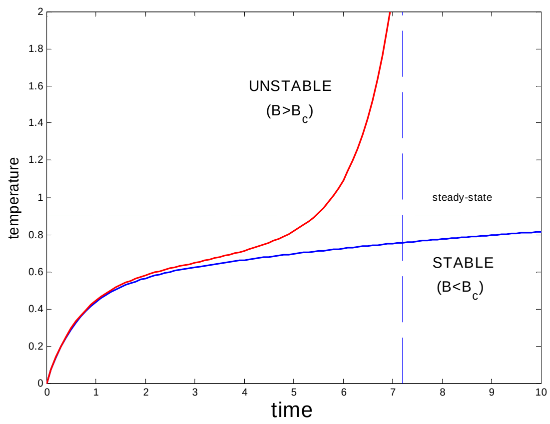

Figure 1. Transient of a stable- and unstable thermal process.

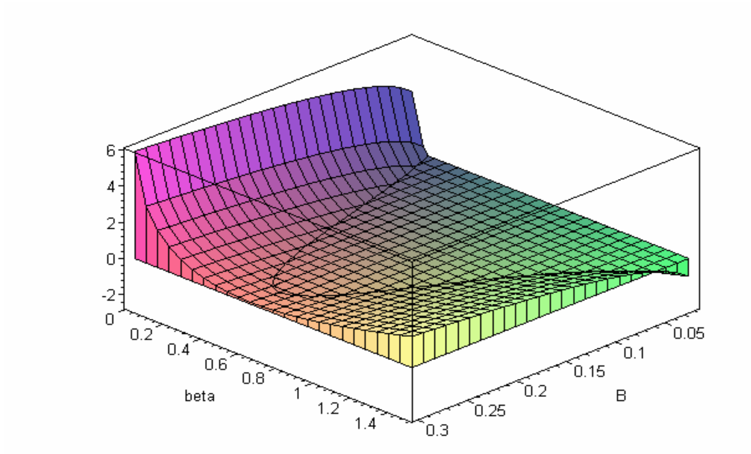

Figure 3. The stability function $f_2(B,\beta,\alpha_1)$ for $\alpha_1 = 1$. Stable- and unstable region of the process with mixed boundary condition.

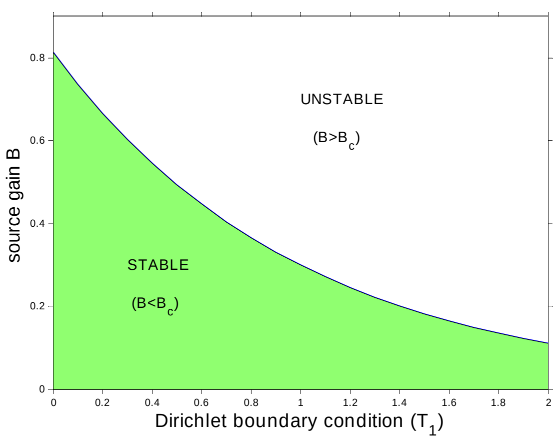

Figure 2. Stable- and unstable region of the process with Dirichlet boundary condition.

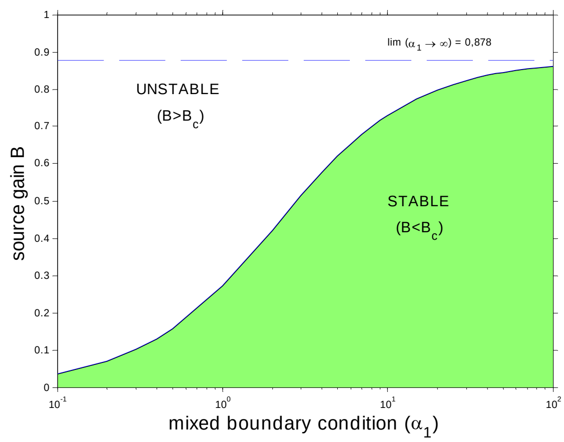

Figure 4. Stable- and unstable region of the process with mixed boundary condition.