All Quadratically Stabilizing Controllers for Polytopic LPV Systems

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Abstract—**In this paper, we proposed one methodology to parameterize all quadratically stabilizing controllers for linear parameter varying (LPV) system based on coprime factorization conception. That is, conception of doubly coprime factorization and Youla parameterization of LTI system is extended to LPV system. Such numerical solutions of these stabilizing controllers parameterization problem can be transferred to finite vertex LMI optimal problems according to the assumption of polytope characteristic of these LPV systems.**

Index Terms— **Coprime factorization; Linear parameter varying; Polytopic system; Quadratic stability; Youla parameterization.**

I. Introduction

It is well known that doubly coprime factorizations of linear time invariant systems are very efficient algebraic tools for solving many relevant control problems. Youla-Kučera parameterization [1], [2] construct the whole set of output feedback stabilizing controllers by one stable system, is also based on the coprime factorizations. A very important property of this parameterization is that it transforms the feedback control problem to a much simple open loop model-matching problem.

In linear time varying case, until now, Youla parameterization conceptions of LTV system have not been explored systematically. There are almost few papers concerning how to use those conceptions to solve numerical solutions of all stabilizing controllers for linear time varying system.

On the other hand, Shamma & Athans [3], [4] formalized a certain type of nonlinear system or LTV systems as a linear parameter varying (LPV) system, and succeeded in developing a control strategy for this system based on classical gain scheduled methodology. In [5], for polytopic LPV systems a necessary and sufficient condition for quadratic stability can be formulated in terms of a finite of linear matrix inequalities. The underlying quadratic Lyapunov functions can be also used to derive bounds on robust performance measures. Recently, significant progress has been made in this area, and a unified H-infinity approach is being developed that is also reducible to a linear matrix inequality (LMI) optimization problem [6]–[8].

In the present paper, as to LPV system with on-line measurable dependent parameter, firstly, conception of coprime factorization is extended to applying to LPV systems with state space function expression. Then a systematic way of obtaining all quadratically stabilizing controllers like Youla parameterization is presented. Such numerical solution of quadratically stabilizing controllers' parameterization problem can be transferred to finite LMI optimal problems according to polytope characteristic of these LPV systems.

The paper is organized as follows. First, the definition of LPV system and some tools are introduced in preliminary section. Second, Youla parameterization of LTI systems is overviewed. Youla parameterization for LPV systems is derived concisely. Then all stabilizing controller for polytopic LPV system is obtained according to conception of Youla parameterization in section four. Finally, a numerical example is presented to illustrate the design method.

II. Preliminary

In this section, firstly the notation and some assumption regarding LPV system are introduced. Useful conceptions and several lemmas are recapped.

Definition 1: LPV system

Consider the generalized LPV plant: $G(\theta(t))$ described by state space equations as

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \left(\begin{array}{c|cc} A(\theta(t)) & B_w(\theta(t)) & B_u \\ \hline C_z(\theta(t)) & D_{wz}(\theta(t)) & D_{uz} \\ C_y & D_{wy} & 0_{g\times q} \end{array}\right) \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{1}$$

Here state-space matrices have compatible dimensions. Moreover we have the following assumptions. Notations follow the reference [6].

(1) The state-space matrices $A(\theta), B_w(\theta), C_z(\theta), D_{wz}(\theta)$ depend affinely on $\theta(t)$.

(2) The real parameter $\theta(t)$ is continuous real-time measurable in the polytope Θ of vertices $\omega_1, \omega_2, \cdots, \omega_N,$ $N = 2^r$; it can be expressed as:

$$\begin{aligned} &\theta(t) \in \Theta := Co\{\omega_1, \omega_2, \cdots, \omega_N\} \\ &= \left\{\sum_{i=1}^{N} \alpha_i(t)\omega_i : \alpha_i(t) \geq 0, \sum_{i=1}^{N} \alpha_i(t) = 1\right\} \end{aligned} \tag{2}$$

(3) The pairs $(A(\theta), B_u)$ and $(A(\theta), C_y)$ are is quadratically stabilizable and quadratically detectable over Θ respectively.
The original LPV plant $G_{22}(\theta)$ from u to y can be expressed as:

$$\begin{pmatrix} A(\theta) & B_u \\ C_y & 0 \end{pmatrix} = \sum_{i=1}^{N} \alpha_i(t) \begin{pmatrix} A_i & B_u \\ C_y & 0 \end{pmatrix} \quad (3)$$

with $\alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1$.
Here, $A_i = A(\omega_i)$ for $i = 1, \cdots, N$.

Remark 1

In the classical LPV gain scheduled H infinity approach, it is required that the matrices B_u, C_y, D_{uz}, D_{wy} of augmented plant $G(\theta)$ be time invariant [6],[10]. However, when they are time varying, a simple means to enforce these requirements consists of filtering the control input and the output through low-pass filters having sufficiently large bandwidth. By this trick, the parameter trajectory is shifted into the state matrix A.

Lemma 1 Schur complement

The block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$ is negative definite if and only if $A_{22} < 0$ and $A_{11} - A_{12} A_{22}^{-1} A_{12}^T < 0$, where A_{22} is a regular matrix.

Lemma 2 Quadratic stability

Considering polytopic LPV system $\dot{x}(t) = A(\theta)x(t)$, a necessary and sufficient condition for quadratic stability of above system is that there exists a quadratic Lyapunov function $V = x^T P x$ such that

$$P > 0,\ A_i^T P + P A_i < 0,\ i = 1, \cdots, N \quad (4)$$

Lemma 3

If matrices $A_{11}(\theta)$ and $A_{22}(\theta)$ are quadratically stable, every block triangle matrix whose diagonal matrices consist of $A_{11}(\theta)$ and $A_{22}(\theta)$ is quadratically stable.

Proof: Consider a lower triangle matrix:

$$A = \begin{bmatrix} A_{11}(\theta) & 0 \\ A_{21}(\theta) & A_{22}(\theta) \end{bmatrix} \in Co\left\{ \begin{bmatrix} A_{11i} & 0 \\ A_{21i} & A_{22i} \end{bmatrix}, 1 \leq i \leq N \right\}.$$

According to the assumption, there exists P_1 and P_2, respectively, which satisfy

$$A_{11i}^T P_1 + P_1 A_{11i} < 0,\ i = 1, \cdots, N$$
$$A_{22i}^T P_2 + P_2 A_{22i} < 0,\quad i = 1, \cdots, N. \quad (5)$$

Considering the following matrix $P = \begin{bmatrix} P_1 & 0 \\ 0 & \lambda P_2 \end{bmatrix}$, (4) can be rewritten as

$$\begin{bmatrix} P_1 A_{11i} + A_{11i}^T P_1 & \lambda A_{21i}^T P_2 \\ \lambda P_2 A_{21i} & \lambda P_2 A_{22i} + \lambda A_{22i}^T P_2 \end{bmatrix} < 0$$

for $i = 1, \cdots, N$. (6)

Therefore, using Lemma 1, inequality (6) is equivalent to the following inequalities as

$$\lambda(P_2 A_{22i} + A_{22i}^T P_2) < 0$$
$$P_1 A_{11i} + A_{11i}^T P_1 - \lambda(A_{21i}^T P_2)[P_2 A_{22i} + A_{22i}^T P_2]^{-1}(P_2 A_{21i}) < 0$$

for $i = 1, \cdots, N$ (7)

This condition (7) can always be satisfied for any $\lambda > 0$ such that $\mu_i - \lambda \nu_i < 0$ for $i = 1, \cdots, N$,
where $\mu_i = \delta_{\max}(A_{11i} P_1 + P_1 A_{11i}^T)$
and $\nu_i = \delta_{\min}((A_{21i}^T P_2)\left[A_{22} P_2 + P_2 A_{22}^T\right]^{-1}(P_2 A_{21i}^T))$
The similar proof is also used for upper triangle matrix.

III. YOULA PARAMETERIZATION OF LTI SYSTEM

In this section, firstly traditional coprime fraction and Youla parameterization of LTI system is overviewed. Using this conception, all quadratically stabilizing controller design for LPV systems is discussed in the next section.

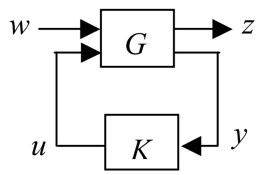


Fig.1 Standard feedback system

For LTI system (θ is frozen), original plant G_{22} is given as followed

$$G_{22}(s) := \left[\begin{array}{c|c} A & B_u \\ \hline C_y & 0 \end{array}\right] \quad (8)$$

where the pairs (A, B_u) and (A, C_y) are also stabilizable and detectable respectively.
The doubly coprime factorization is given by

$$\begin{bmatrix} D_r & X_l \\ N_r & Y_l \end{bmatrix} \cdot \begin{bmatrix} Y_r & -X_r \\ -N_l & D_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{9}$$

with each transfer functions in RH_∞, and where

$$\begin{bmatrix} D_r & X_l \\ N_r & Y_l \end{bmatrix} := \left[\begin{array}{c|cc} A+B_uF & B_u & -L \\ \hline F & I & 0 \\ C_y & 0 & I \end{array}\right] \tag{10}$$

$$\begin{bmatrix} Y_r & -X_r \\ -N_l & D_l \end{bmatrix} := \left[\begin{array}{c|cc} A+LC_y & -B_u & L \\ \hline F & I & 0 \\ C_y & 0 & I \end{array}\right] \tag{11}$$

All stabilizing controllers are parameterized from the central controller $Y_r^{-1}X_r = X_l Y_l$ as follows:

$$K = (X_l + D_r Q)(Y_l + N_r Q)^{-1} \quad Q \in RH_\infty \tag{12}$$

IV. Youla parameterization For LPV systems

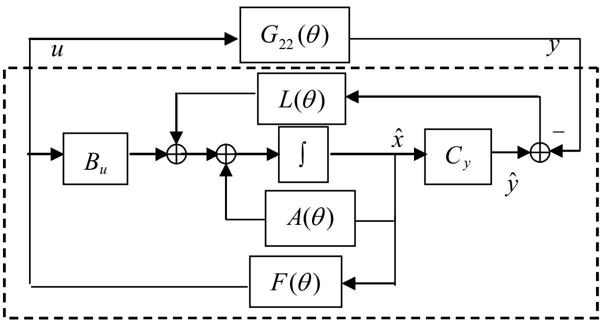
As to LPV systems, conception of doubly coprime factorization and Youla parameterization of LTI system is extended to polytopic linear parameter variant (LPV) system. A systematic way of obtaining all quadratically stabilizing controllers like Youla parameterization will be investigated in this section.

A. Quadratically stabilizing observed-based controller for LPV systems

The original LPV polytopic plant $G_{22}(\theta)$ can be expressed as:

$$\begin{aligned} \dot{x} &= A(\theta)x + B_u u \\ y &= C_y x \end{aligned} \tag{13}$$

where the plant is assumed to be quadratically stabilizable and quadratically detectable over Θ shown in (2).



Observer-based controller

Fig.2. Quadratically stabilizing observed-based controller

Lemma 4

A quadratically stabilizing observed-based controller for the plant (13) can be formulated as

$$\begin{aligned} \dot{\hat{x}} &= A(\theta)\hat{x} + B_u u + L(\theta)(C_y \hat{x} - y) \\ u &= F(\theta)\hat{x}, \end{aligned} \tag{14}$$

where $F(\theta)$ and $L(\theta)$ can be constructed as

$$F(\theta) = \sum_{i=1}^{N} \alpha_i(t) F_i \text{ and } L(\theta) = \sum_{i=1}^{N} \alpha_i(t) L_i \tag{15}$$

,where $F_i = V_i P_f^{-1}$ and $L_i = P_l^{-1} W_i$, that satisfy the following LMIs

$$\begin{aligned} &P_f > 0, \quad A_i P_f + P_f A_i^T + B_u V_i + V_i^T B_u^T < 0, \quad i = 1, \cdots, N \\ &P_l > 0, \quad A_i^T P_l + P_l A_i + W_i C_y + C_y^T W_i^T < 0, \quad i = 1, \cdots, N \end{aligned} \tag{16}$$

Proof: if the controller (14) is substituted into plant (13), the closed-loop state matrix can be expressed as

$$A_{cl}(\theta) = \begin{bmatrix} A(\theta) + B_u F(\theta) & B_u F(\theta) \\ 0 & A(\theta) + L(\theta) C_y \end{bmatrix} \tag{17}$$

Based on (15), (16) and lemma 2, we see that $A(\theta) + B_u F(\theta)$ and $A(\theta) + L(\theta) C_y$ are quadratically stable. Thus, using lemma 3, the system (17) is quadratically stable and consequently the controller (14) can be said to be quadratically stabilizing one for the LPV polytopic plant (13).

Remark 2

Using the elimination procedure of variable [8], conditions (16) can be replaced by the following equivalent condition (18) in which variables V_i and W_i do not appear.

$$\begin{aligned} &P_f > 0, \quad A_i P_f + P_f A_i^T < 2\delta B_u B_u^T, \quad i = 1, \cdots, N \\ &P_l > 0, \quad A_i^T P_l + P_l A_i < 2\mu C_y^T C_y, \quad i = 1, \cdots, N \\ &\delta > 0, \mu > 0 \end{aligned} \tag{18}$$

Substituting P_f, P_l satisfying (18) and also $V_i = F_i P_f$, $W_i = P_l L_i$ into (16), we obtain F_i and L_i. Finally $F(\theta)$ and $L(\theta)$ are also derived by (15).

Definition: Coprime factorization for LPV plant

Now we can define the coprime factorization of LPV plant that resembles the case of LTI system

$$\begin{bmatrix} D_r(\theta) & X_l(\theta) \\ N_r(\theta) & Y_l(\theta) \end{bmatrix} := \left[\begin{array}{c|cc} A(\theta)+B_uF(\theta) & B_u & -L(\theta) \\ \hline F(\theta) & I & 0 \\ C_y & 0 & I \end{array}\right]$$

$$\begin{bmatrix} Y_r(\theta) & -X_r(\theta) \\ -N_l(\theta) & D_l(\theta) \end{bmatrix} := \left[\begin{array}{c|cc} A(\theta)+L(\theta)C_y & -B_u & L(\theta) \\ \hline F(\theta) & I & 0 \\ C_y & 0 & I \end{array}\right] \tag{19}$$

Where $F(\theta), L(\theta)$ matrices satisfy the LMIs(16).

B. All quadratically stabilizing controller for LPV system

In this section, parameterization of all quadratcially stabilizing controller for LPV system is investigated.

Theorem

All quadratically stabilizing controller that make LPV plant (1) quadratically stable can be parameterized by the figure.3, where $M(\theta)$ is given by (20) and $Q(\theta)$ is arbitrary quadratically stable LPV system as (21)

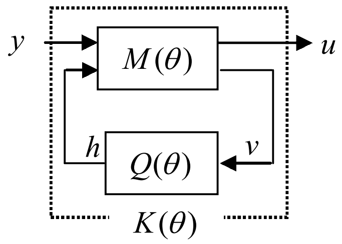


Fig.3 All quadratical stabilizing controller

$$M(\theta) = \left[\begin{array}{c|cc} A(\theta)+B_uF(\theta)+L(\theta)C_y & -L(\theta) & B_u \\ \hline F(\theta) & 0 & I \\ -C_y & I & 0 \end{array}\right] \tag{20}$$

$$Q(\theta) = \left[\begin{array}{c|c} A_Q(\theta) & B_Q(\theta) \\ \hline C_Q(\theta) & D_Q(\theta) \end{array}\right] \tag{21}$$

where there exists a positive definite matrix P_Q such that

$$A_Q^T(\theta)P_Q + P_QA_Q(\theta) < 0 . \tag{22}$$

Proof:

Sufficiency

Using (20) and (21), the controller $K(\theta)$ in Fig.3 is derived as

$K(\theta) =$

$$\left[\begin{array}{cc|c} A(\theta)+B_uF(\theta)+L(\theta)C_y-B_uD_Q(\theta)C_y & B_uC_Q & B_uD_Q-L(\theta) \\ -B_QC_y & A_Q(\theta) & B_Q(\theta) \\ \hline F(\theta)-D_Q(\theta)C_y & C_Q(\theta) & D_Q(\theta) \end{array}\right] \tag{23}$$

Connecting (23) to (13), the closed loop system can be expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{x}_Q \end{bmatrix} = \begin{bmatrix} A(\theta)+B_uF(\theta) & B_uF(\theta)-B_uD_Q(\theta)C_y & B_uC_Q(\theta) \\ 0 & A(\theta)+L(\theta)C_y & 0 \\ 0 & -B_Q(\theta)C_y & A_Q(\theta) \end{bmatrix} \begin{bmatrix} x \\ e \\ x_Q \end{bmatrix} \tag{24}$$

where $e = \hat{x} - x$ and x_Q is state variable of the $Q(\theta)$.

Using Lemma 3 twice, there exists a quadratic Lyapunov function $V = x^T P_{cl} x$ that P_{cl} is positive definite and symmetric matrix such that

$$V = x^T P_{cl} x = \begin{bmatrix} x_f \\ x_l \\ x_q \end{bmatrix}^T \begin{bmatrix} \lambda_f P_f & 0 & 0 \\ 0 & P_l & 0 \\ 0 & 0 & \lambda_q P_q \end{bmatrix} \begin{bmatrix} x_f \\ x_l \\ x_q \end{bmatrix} > 0$$

where λ_f, λ_q are positive definite numbers, this assures that above system is quadratically stable.

Necessity

We will show that any stabilizing controller $K(\theta)$ is expressed with a quadratically stable Q_0 as $K = F_l(M, Q_0)$, Considering a function $Q_0 = F_l(\hat{M}, K)$,where

$$\hat{M}(\theta) = \left[\begin{array}{c|cc} A(\theta) & -L(\theta) & B_u \\ \hline -F(\theta) & 0 & I \\ C_y & I & 0 \end{array}\right] \tag{25}$$

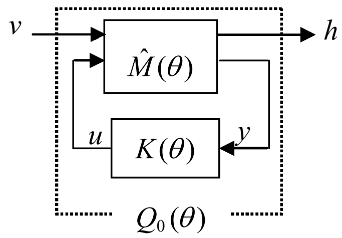


Fig.4 A function $Q_0 = F_l(\hat{M}, K)$

Since (2,2) block of $\hat{M}(\theta)$ is same as the original plant (13), we can get $Q_0(\theta)$ is a quadratically stable.

Now we substitute $Q_0(\theta)$ into $F_l(M(\theta), Q_0(\theta))$, we get

$$F_l(M(\theta), Q_0(\theta)) = F_l(M(\theta), F_l(\hat{M}(\theta), K(\theta)))$$
$$= F_l(J_{tmp}, K(\theta)),$$

where J_{tmp} can be obtained by using the sate space star product formula

$$J_{tmp} = \left[\begin{array}{cc|cc} A(\theta)+L(\theta)C_y+B_uF(\theta) & -B_uF(\theta) & -L(\theta) & B_u \\ L(\theta)C_y & A(\theta) & -L(\theta) & B_u \\ \hline F(\theta) & -F(\theta) & 0 & I \\ -C_y & -C_y & I & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} A(\theta)+L(\theta)C_y & -B_uF(\theta) & -L(\theta) & B_u \\ 0 & A+B_uF(\theta) & 0 & 0 \\ \hline 0 & -F(\theta) & 0 & I \\ 0 & C_y & I & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Hence $F_l(M(\theta), Q_0(\theta)) = F_l(J_{tmp}, K(\theta)) = K(\theta)$. This shows that any quadratically stabilizing controller can be expressed in the form as Fig.3 with quadratically stable $Q_0(\theta)$.

V. NUMERICAL EXAMPLE

A classical example of parameter-varying unstable plant that can be viewed as a mass-spring-damper system with time-varying spring stiffness is considered. The state space equation of this unstable un-weighted LPV plant is as follows

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -0.5-0.5\theta & -0.2 \end{bmatrix}, B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_y = \begin{bmatrix} -1 & 0 \end{bmatrix},\ D_{uy} = 0$$

Here the scope of nominal time-varying parameter $\theta(t)$ is in polytopic spaces $\Theta_1 := Co\{-1,1\}$.
According to the following LMIs

$$P_f > 0, \quad A_iP_f + P_fA_i^T + B_uV_i + V_i^TB_u^T < 0, \quad i = 1,2$$

$$P_l > 0, \quad A_i^TP_l + P_lA_i + W_iC_y + C_y^TW_i^T < 0, \quad i = 1,2$$

Using Matlab toolbox [9] then solving the six linear inequalities matrices, we got

$$P_f = \begin{bmatrix} 81.9 & -32.7 \\ -32.7 & 81.9 \end{bmatrix},\ F_1 = \begin{bmatrix} -0.43 & -0.87 \end{bmatrix} \text{ and}$$

$$F_2 = \begin{bmatrix} -1.43 & -0.87 \end{bmatrix}$$

$$P_f = \begin{bmatrix} 149.6 & -33.7 \\ -33.7 & 163.2 \end{bmatrix}, L_1 = \begin{bmatrix} 0.75 & 0.11 \end{bmatrix}^T \text{ and}$$

$$L_2 = \begin{bmatrix} 0.75 & 1.11 \end{bmatrix}^T$$

Finally we obtained all quadratically stabilizing controllers as Fig.3 with the following observer based controller;

$$M(\theta) = \left[\begin{array}{c|cc} A(\theta)+B_uF(\theta)+L(\theta)C_y & -L(\theta) & B_u \\ \hline F(\theta) & 0 & I \\ -C_y & I & 0 \end{array}\right]$$

by setting by setting $F(\theta)$ and $L(\theta)$ as $F(\theta) = \sum_{i=1}^{2} \alpha_i(t)F_i$ and $L(\theta) = \sum_{i=1}^{2} \alpha_i(t)L_i$, where $\alpha_1(t) = (1-\theta(t))/2$; and $\alpha_2(t) = (\theta(t)+1)/2$.

VI. CONCLUSION

A method to parameterize all quadratically stabilizing controllers for linear parameter variant (LPV) system has been proposed based on coprime factorization conception. This parameterization problem can be transferred to finite LMI optimal problems according to polytope characteristic of the LPV system. A numerical example is presented.

Based on proposed parameterization method, well known Q-parameter approach can be applied to variety of multi-objective control system design for LPV system.

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