Solutions of Stochastic Linear Distributed Parameter Equations with Multiplicative Fractional Gaussian Noise

T. E. Duncan, B. Maslowski, and B. Pasik-Duncan†

Abstract— **In this paper explicit solutions are given for a family of stochastic linear distributed parameter equations with a multiplicative fractional Gaussian noise. The solutions can be strong, weak or mild depending on the particular assumptions on the equation description. A fractional Gaussian noise is the formal derivative of a fractional Brownian motion which is determined in probability law by its Hurst parameter. In this paper, the Hurst parameter is restricted to the interval (1/2, 1). Only a limited number of results are available for the existence of solutions of stochastic differential equations with a fractinal Brownian motion. The solutions given here require the use of a stochastic calculus for a fractional Brownian motion. Some examples of stochastic partial differential equations with a fractional Brownian motion are given.**

I. Introduction

Stochastic linear distributed parameter equations with multiplicative noise are an important family of stochastic equations with both theoretical and practical applications. In this paper, explicit solutions are given for a family of stochastic linear distributed parameter equations with a multiplicative fractional Gaussian noise. A fractional Gaussian noise is the formal derivative of a fractional Brownian motion. The results for existence and uniqueness of solutions for stochastic differential equations with a fractional Brownian motion are incomplete, so it is necessary to consider special classes of stochastic differential equations. A stochastic equation where the diffusion coefficient is deterministic can be solved from the results for the corresponding deterministic equation. However, a stochastic equation where the diffusion coefficient is stochastic requires a nontrivial use of stochastic analysis. This analysis is lacking even for a scalar equation.

For stochastic linear distributed parameter equations, with multiplicative Brownian motion, Da Prato and Zabczyk [2] have given explicit solutions. The approach used here for stochastic linear distributed parameter systems with multiplicative fractional Brownian motion is motivated by [2], but the analysis for this case requires additional methods as contrasted with [2].

Fractional Brownian motion is a family of Gaussian processes that appears to have wide applicability to modeling physical systems based on the analysis of empirical data.

Research supported in part by NSF Grants DMS 0204669 and ANI 0125410 and GACR Grant 201/01/1197.

T. E. Duncan and B. Pasik-Duncan are with Department of Mathematics, University of Kansas, Lawrence, KS 66045, duncan@math.ukans.edu and bozenna@math.ukans.edu

B. Maslowski is with the Institute of Mathematics, Academy of Sciences of Czech Republic, Prague, Czech Republic, maslow@math.cos.cz.

II. Preliminaries and Results

A standard fractional Brownian motion $(\beta^H(t),\, t \geq 0)$ with Hurst parameter $H \in (0,1)$ is a Gaussian process with continuous sample paths such that $E[\beta^H(t)] = 0$ and

$$E[\beta^H(s)\beta^H(t)] = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}]$$

for $s, t \in \mathbb{R}_+$. It is clear that for $H = 1/2$, the process is a (standard) Brownian motion. In this paper, it is assumed that $H \in (1/2, 1)$. These processes have a long range dependence (e.g., [3]) in addition to a self similarity. The Hurst parameter $H \in (1/2, 1)$ has been estimated from empirical data from many applications.

To give some perspective on the solutions given in this paper, consider the following scalar linear equation with $H \in (1/2, 1)$

$$\begin{aligned} dX(t) &= aX(t)dt + bX(t)dB^H(t) \\ X(0) &= 1, \end{aligned} \tag{1}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$. A solution is

$$X(t) = \exp\left[at + bB^H(t) - \frac{1}{2}b^2 t^{2H}\right],$$

which can be easily verified by an Itô formula for fractional Brownian motion (e.g., [4]). Furthermore, a multidimensional version of (1) with commuting linear transformations can also be explicitly solved [3].

For the infinite dimensional (Hilbert space) case of distributed parameter equations, it is useful to recall the notion of a strongly continuous evolution system [7]. A family of bounded linear operators $(U(t,s),\, 0 \leq s \leq t \leq T)$ on a separable Hilbert space V is called a *strongly continuous evolution system* corresponding to the linear operators $(A(t),\, t \in [0,T])$ if U is a strongly continuous function on $0 \leq s \leq t \leq T$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$ and the following equalities are satisfied on suitable domains in V

$$\begin{aligned} \frac{\partial}{\partial t}U(t,s) &= A(t)U(t,s) \\ \frac{\partial}{\partial s}U(t,s) &= -U(t,s)A(s). \end{aligned}$$

The stochastic equation considered here is given by

$$dX(t) = A(t)X(t)dt + \sum_{j=1}^{k} B_j X(t) d\beta_j^H(t) \tag{2}$$

$$X(0) = x_0 \in V,$$

where $t \in [0,T]$, $X(t) \in V$, V is a separable Hilbert space, $(\beta_j^H(t),\, t \geq 0,\, j = 1,\ldots,k)$ is a family of independent, standard fractional Brownian motions with a fixed Hurst parameter $H \in (1/2, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(A(t),\, t \in [0,T])$ and $(B_1,\ldots,B_k)$ are typically linear, densely defined operators on V.

The following assumptions are used subsequently.

(A1) The family of closed operators $(A(t),\, t \in [0,T])$ defined on a common domain $D := \mathrm{Dom}(A(t))$ for $t \in [0,T]$ generates a strongly continuous evolution system $(U_0(t,s),\, 0 \leq s \leq t \leq T)$.

(A2) The linear operators $B_1,\ldots,B_k$ generate mutually commuting, strongly continuous groups denoted $S_1,\ldots,S_k$, respectively, which commute with $A(t)$ for each $t \in [0,T]$ on D. For $j, m \in \{1,\ldots,k\}$, $\mathrm{Dom}(B_j B_m) \supset D$, $\mathrm{Dom}(A^*(t)) = D^*$ for each $t \in [0,T]$, and $D^* \subset \bigcap_{j,m=1}^k \mathrm{Dom}(B_j^* B_m^*)$, where $*$ denotes the adjoint.

(A3) The family of linear operators $\tilde{A}(t) := A(t) - Ht^{2H-1} \times \sum_{j=1}^k B_j^2$ generates a strongly continuous evolution system on V and $\mathrm{Dom}(\tilde{A}(t)) = D$ for each $t \in [0,T]$.

Additionally some more specific conditions on $A(t)$ and $(B_1,\ldots,B_k)$ are made that ensure (A1) and (A3). These conditions are useful in applications to stochastic partial differential equations (SPDEs) of parabolic type, which are considered in some subsequent examples.

(H1) For each $t \in [0,T]$, the linear operator $A(t)$ is a closed densely defined operator in V whose resolvent set $\rho(A(t))$ contains the half-plane $\mathrm{Re}\,\lambda \geq \omega_0$ for some fixed $\omega_0 \in \mathbb{R}$ and

$$|R(\lambda, A(t))|_{\mathcal{L}(\mathcal{V})} \leq \frac{M}{1 + |\lambda + \omega_0|}, \qquad \mathrm{Re}\,\lambda \geq \omega_0$$

for some real number M that does not depend on $t \in [0,T]$.

The condition (H1) implies that $A(t)$ generates an analytic semigroup [6] for each $t \in [0,T]$ so it can be assumed by a translation that $\omega_0 = 0$.

(H2) For each $t \in [0,T]$, $\mathrm{Dom}(A(t)) = D = \mathrm{Dom}(A)$ where $A := -A(0)$ and $A(t)A^{-1}$ is a Hölder continuous function in $\mathcal{L}(\mathcal{V})$, or, equivalently, the following inequality is satisfied

$$|A(t) - A(s)|_{\mathcal{L}(\mathcal{D},\mathcal{V})} \leq K|t-s|^\gamma$$

for $s, t \in [0,T]$, $K \in \mathbb{R}_+$, and $\gamma \in (0,1]$.

It is known (e.g., [7], Theorem 5.2.1) that the assumptions (H1) and (H2) imply (A1) and, furthermore, $\mathrm{Range}(U_0(t,s)) \subset D$

$$\left| \frac{\partial}{\partial t} U_0(t,s) \right|_{\mathcal{L}(\mathcal{V})} = |A(t)U_0(t,s)|_{\mathcal{L}(\mathcal{V})} \leq \frac{c}{t-s}, \qquad 0 \leq s < t \leq T$$

$$|U_0(t,s)|_{\mathcal{L}(\mathcal{D})} \leq c.$$

The following proposition verifies that (H1) and (H2) imply (A3) under some conditions on $(B_j^*,\, j = 1,\ldots,k)$.

Proposition 1 *Let $(B_j^2,\, j = 1,\ldots,k)$ be a family of closed operators such that $\mathrm{Dom}(B_j^2) \supset \mathrm{Dom}(A^\alpha)$ for some $\alpha \in (0,1)$. If* (H1) *and* (H2) *are satisfied, then the family of operators $(\tilde{A}(t),\, t \in [0,T])$ with $\mathrm{Dom}(\tilde{A}(t)) = D$ generates a strongly continuous evolution system on V, that is,* (A3) *is satisfied.*

The notions of strong, weak and mild solutions of (2) are given now.

Definition 1 A $\mathcal{B}([\prime, \mathcal{T}]) \otimes \mathcal{F}$ V-valued stochastic process $(X(t),\, t \in [0,T])$ is said to be

(i) a strongly solution of (2) if $X(t) \in D$ a.s. $\mathbb{P}$ and

$$X(t) = x_0 + \int_0^t A(s)X(s)ds + \sum_{j=1}^k \int_0^t B_j X(s) d\beta_j^H(s) \quad \text{a.s. } \mathbb{P} \tag{3}$$

for $t \in [0,T]$

(ii) a weak solution of (2) if for each $z \in D^*$

$$\langle X(t), z\rangle = \langle x_0, z\rangle + \int_0^t \langle X(s), A^*(s)z\rangle ds + \sum_{j=1}^k \int_0^t \langle X(s), B_j^* z\rangle d\beta_j^H(s) \quad \text{a.s. } \mathbb{P} \tag{4}$$

for $t \in [0,T]$

(iii) a mild solution of (2) if

$$X(t) = U_0(t,s)x_0 + \sum_{j=1}^k \int_0^t U_0(t,s) B_j X(s) d\beta_j^H(s) \quad \text{a.s. } \mathbb{P} \tag{5}$$

for $t \in [0,T]$.

The definition of the stochastic integrals in (3,4,5) can be found in [1], [4], [5].

The following theorem is the main result in this paper. It demonstrates that (A1)–(A3) are sufficient for weak or strong solutions. Furthermore, the solution is given explicitly in terms of the fractional Brownian motions and the linear operators that characterize the linear equation (2).

Theorem 1 *In* (A1)–(A3) *are satisfied, then there is a weak solution of* (2)*. Moreover, if $x_0 \in D$, then there is a strong solution. If $B_j \in \mathcal{L}(\mathcal{V})$, $j = 1,\ldots,k$, then there is a mild solution. In each of these cases, the solution is given explicitly as*

$$X(t) = \prod_{j=1}^k S_j(\beta_j^H(t)) U(t,0) x_0 \tag{6}$$

for $t \in [0,T]$, where $U(\cdot,\cdot)$ is the evolution system for $\tilde{A}$.

Two applications of Theorem 1 to stochastic partial differential equations are given.

Example II.1 Consider the stochastic parabolic equation of $2m$th order as follows

$$\frac{\partial u}{\partial t} = L(t,\xi)u(t,\xi) + b\frac{d\beta^H}{dt}, \qquad (t,\xi) \in [0,T]\times\mathcal{D} \tag{7}$$

$$u(0,\xi) = x_0(\xi), \qquad \xi \in \mathcal{D}$$

$$\frac{\partial^\alpha}{\partial\xi^\alpha}u(t,\xi) = 0, \qquad (t,\xi)\in[0,T]\times\mathcal{D}$$

$$\alpha \text{ is a multi-index, } |\alpha| \le m-1,$$

where $m \in \mathbb{N}$, $\mathcal{D} \subset \mathbb{R}$ is a bounded domain of class C^m, $b \in \mathbb{R}\setminus\{0\}$ and

$$L(t,\xi) := \sum_{|\alpha|\le 2m} a_\alpha(t,\xi)D^\alpha.$$

It is assumed that L is a strongly elliptic operator on $\mathcal{D}$, uniformly in $(t,\xi)\in[0,T]\times\bar{\mathcal{D}}$ and $a_\alpha(t,\cdot)\in C^{2m}(\bar{\mathcal{D}})$, $|\alpha|\le 2m$ for each $t\in[0,T]$.

The equation (7) is rewritten in the form

$$dX(t) = A(t)X(t)dt + BX(t)d\beta^H(t) \tag{8}$$
$$X(0) = x_0 \in V$$

for $t\in[0,T]$ where $V = L^2(\mathcal{D})$, $(A(t)u)(\xi) = L(t,\xi)u(\xi)$, $\mathrm{Dom}(A(t)) = D = H^{2m}(\mathcal{D})\cap\mathcal{H}_l^{\Updownarrow}(\mathcal{D})$ and $B = bI \in \mathcal{L}(\mathcal{V})$. It is assumed that

$$\sup_{\xi\in\mathcal{D}} |a_\alpha(t,\xi) - a(s,\xi)| \le L|t-s|^\gamma$$

for $|\alpha| \le 2m$, $s,t\in[0,T]$ and for some $L\in\mathbb{R}_+$ and $\gamma\in(0,1]$. The assumptions (H1) and (H2) are satisfied (e.g., [7], Theorem 3.8.3) and by Proposition 1 the assumptions (A1) and (A3) are satisfied. The assumption (A2) is trivially satisfied. Note that $D^* = \mathrm{Dom}(A^*(t)) = \mathrm{Dom}(A(t)) = D$. By Theorem 1, there is a weak and a mild solution to (8). If $x_0\in D$, then there is a strong solution.

Example II.2 Consider the Cauchy problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \sum_{i,j=1}^d a_{ij}(t)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(t,\xi)\\ &+ \sum_{i=1}^d d_i(t)\frac{\partial u}{\partial\xi_i}(t,\xi) + c(t)u(t,\xi)\\ &+ \sum_{i=1}^d b_i\frac{\partial u}{\partial\xi_i}(t,\xi)\frac{d\beta_1^H(t)}{dt}\\ &+ ru(t,\xi)\frac{d\beta_2^H(t)}{dt}, \qquad (t,\xi)\in[0,T]\times\mathbb{R}\end{aligned} \tag{9}$$

$$u(0,\xi) = x_0(\xi),$$

where a_{ij}, d_i, c are Hölder continuous functions and $b_i\in\mathbb{R}$ for $i,j\in\{1,\dots,d\}$.

Assume that the differential operator

$$L(t) := \sum_{i,j} a_{ij}(t)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_i d_i(t)\frac{\partial}{\partial\xi_i} + c(t)I$$

is uniformly elliptic, that is,

$$\sum_{ij} a_{ij}(t)v_iv_j \ge \alpha|v|^2,$$

where $\alpha > 0$ and $v\in\mathbb{R}$.

The equation (9) is rewritten as

$$dX(t) = A(t)X(t)dt + B_1X(t)d\beta_1^H(t) + B_2X(t)d\beta_2^H(t) \tag{10}$$

$$X(0) = x_0 \in V,$$

where $V = L^2(\mathbb{R})$, $A(t) = L(t)$, $\mathrm{Dom}(A(t)) = \mathrm{Dom}(A^*(t)) = H^2(\mathbb{R})$, $B_1 = \sum_i b_i(\partial/\partial\xi_i)$, $\mathrm{Dom}(B_1) = H^1(\mathbb{R})$ and $B_2 = rI$.

It is well known that the family of linear operators $(A(t),\, t\in[0,T])$ generates a strongly continuous evolution system $U_0(\cdot,\cdot)$ on V and (A1) is satisfied (e.g., [7], Theorem 5.2.1). The operators B_1 and B_2 generate strongly continuous groups on V that are given as follows,

$$[S_1(t)x_0](\xi) = x_0(\xi_1 + b_1t,\dots,\xi_d + b_dt)$$

for $\xi\in\mathbb{R}$ and $t\in\mathbb{R}$ and

$$S_2(t)x_0 = (e^{rt}I)x_0, \qquad t\in\mathbb{R},$$

respectively, so (A2) is clearly satisfied. To verify (A3), note that $\tilde{A}(t)$ with $\mathrm{Dom}(\tilde{A}(t)) = D = H^2(\mathbb{R})$ is a second order differential operator with time dependent Hölder continuous coefficients. It is only necessary to ensure that the operator is uniformly elliptic on [0,T]. The highest order term of $\tilde{A}(t)$ is

$$L_0(t) = \left(a_{ij}(t) - Ht^{2H-1}\sum_{i,j=1}^d b_ib_j\right)\frac{\partial^2}{\partial\xi_i\partial\xi_j}$$

so the ellipticity condition is

$$\sum_{i,j=1}^d a_{ij}(t)v_iv_j > Ht^{2H-1}\sum_{i,j=1}^d b_ib_jv_iv_j \tag{11}$$

for $t\in[0,T]$.

If (11) is satisfied, then Theorem 1 can be applied to obtain a strong solution of (10) if $x_0\in D = H^2(\mathbb{R})$ or a weak solution if $x_0\in V$. Note that if $(a_{ij}(t)) = (a_{ij})$ is a constant positive definite matrix then the ellipticity condition is always satisfied for sufficiently small intervals $[0,T]$, but for T sufficiently large (11) is not satisfied. Thus there is a solution on time intervals $[0,T]$, with $T>0$ but bounded above to ensure the strong ellipticity (11).

To elucidate this phenomenon more explicitly, consider the special case of a one dimensional equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t,\xi) &= a\frac{\partial^2 u}{\partial \xi^2} + b\frac{\partial u}{\partial \xi}\frac{d\beta^H}{\partial t} \\ u(0,\xi) &= x_0(\xi)\end{aligned} \tag{12}$$

for $t > 0$, $\xi \in \mathbb{R}$, where $a > 0$ and $b \in \mathbb{R} \setminus \{0\}$. The ellipticity condition (11) reduces to

$$a > Ht^{2H-1}b^2. \tag{13}$$

From the preceding analysis, the solution is defined on intervals $[0,T]$ such that

$$T < T_1 = \left(\frac{a}{b^{2H}}\right)^{1/(2H-1)}.$$

In this case, more can be determined. The solution is given by

$$X(t) = S_1(\beta^H(t))U(t,0)x_0$$

for $t \in [0,T]$ where $[S_1(t)x](\xi) = x(\xi + bt)$ and U is the evolution system corresponding to the equation

$$\begin{aligned}\frac{\partial y}{\partial t} &= (a - Ht^{2H-1}b^2)\frac{\partial^2 y}{\partial \xi^2} \\ y(0) &= x_0\end{aligned}$$

$U(\cdot,\cdot)$ can be computed by a time composition. If S_Δ is the heat semigroup on $\mathbb{R}$, that is,

$$(S_\Delta x)(\xi) = \int_{\mathbb{R}} (4\pi t)^{-1/2} \exp\left[\frac{-(\xi-\eta)^2}{4t}\right] x(\eta)d\eta$$

then

$$X(t) = S_1(\beta^H(t))S_\Delta\left(at - \frac{1}{2}b^2t^{2H}\right)x_0 \tag{14}$$

which is well defined if $at - (1/2)b^2t^{2H} \geq 0$, that is, for $t \in [0,T_2]$ where

$$T_2 = \left(\frac{2a}{b^2}\right)^{1/(2H-1)}.$$

In fact, the transformed time in the semigroup S_Δ initially increases from zero, but at the time T_1 it begins to decrease so that at T_2 it returns to zero. In general, the solution cannot be extended beyond T_2 because the solution may leave the space $V = L^2(\mathbb{R})$ after T_2. However, for a suitably chosen initial datum x_0 it may be continued. This corresponds to an ill-posed parabolic problem with reversed time.

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