MATLAB ENVIRONMENT FOR CONTROL OF TIME-VARYING SYSTEMS

R. Prokop, R. Matušů, Z. Prokopová

Abstract – **Algebraic methods in control theory have brought many effective and efficient algorithms which are easily programmable and applicable. The contribution is focused on algebraic control design utilized for continuous-time systems with periodic parameters. Controllers are obtained via general solution of Diophantine equations in the ring of proper and stable rational functions. Various control problems are solved by conditions of divisibility in this ring. A Matlab+Simulink package was developed for automatic design and simulation. The program system cover generalized PI and PID controllers designed for first and second order systems. A scalar real parameter is introduced for tuning of proposed controllers. Robust properties of developed algorithms are studied through the infinity norm, Kharitonov`s theorem and visualized by Polynomial toolbox functions.**

Index terms - **Periodic Systems, Uncertainty, Robust Control, SISO Systems, PID Controllers**

I. INTRODUCTION

Control systems affected by bounded perturbations have been deeply studied in recent years [2], [8], [9]. The attention of researches concentrated on new robustness tools, such as H_∞, Kharitonov's theorem, μ synthesis, etc. Many industrial processes contain as at least two types of uncertainties, namely unstructured (nonparametric) and structured (parametric) [8]. The parametric uncertainty is more suitable and more realistic from the control engineer's point of view. A frequent case of such systems is the case with interval polynomials. It means that polynomials have coefficients lying within given intervals. Systems with periodically varying parameters can be considered as a special case.

The aim of this paper is to study the control design and behavior for a class of continuous-time systems with periodically varying parameters. The control design in the robust sense is outlined according to [5], [6], then tools for analysis are defined and the program environment for automatic design and simulation is described.

The paper is organized as follows. In Section 2 possible way of description of systems with periodical parameters is introduced. Section 3 gives basic outline of robust control design. In Section 4, some tools for robust analysis are studied. Program environment – software application of proposed algorithms – is described in Section 5. Further, in Section 6 an example for illustrating of possibilities of the toolbox is given. Finally, Section 7 offers some conclusion remarks.

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II. DESCRIPTION OF TIME-VARYING SYSTEMS

Time-varying continuous-time dynamic systems can be interpreted as transfer functions with harmonic parameters, for example the first order system is described by

$$G(s,t)=\frac{b_0(t)}{s+a_0(t)} \qquad (1)$$

The value of the parameter a_0 is responsible for stability or instability of the controlled system. Time varying parameters in (1) are described by equations:

$$\begin{aligned} b_0(t)&=\beta_0+\lambda_1\sin\omega_1 t\\ a_0(t)&=a_0+\lambda_2\sin\omega_2 t \end{aligned} \qquad (2)$$

where α_0,β_0 are real constants. The choice $\lambda_1=\lambda_2=0$ represents a time invariant linear system. It is clear that minimal and maximal values of parameters (2) are defined as:

$$\begin{aligned} b_0^{-}&=\beta_0-\lambda_1; \quad b_0^{+}=\beta_0+\lambda_1\\ a_0^{-}&=a_0-\lambda_2; \quad a_0^{+}=a_0+\lambda_2 \end{aligned} \qquad (3)$$

In the case of second or higher order system are parameters varied analogically.

III. OUTLINE OF CONTROL DESIGN

Let R_{PS} be a set of proper and Hurwitz stable rational functions. The fractional approach developed in Vidyasagar [10] and Kučera [4] supposes the description of linear systems in R_{PS} as a ratio of two rational fractions:

$$G(s)=\frac{b(s)}{a(s)}=\frac{\dfrac{b(s)}{(s+m)^n}}{\dfrac{a(s)}{(s+m)^n}}=\frac{B(s)}{A(s)} \qquad (4)$$
$$n=\max\{\deg(a),\deg(b)\},\quad m>0$$

The scalar positive parameter $m>0$ can be conveniently used as a tuning knob for control behavior.

A general feedback system is shown in Fig.1. It represents for $C(s)=\frac{Q(s)}{P(s)}$ a classical feedback one-degree-of freedom (1DOF) loop with the control law

$$P(s)u = Q(s)[w-y] \quad (5)$$

In a two-degree-of freedom (2DOF) control system, the controller $C(s)$ consists of two transfer functions $\frac{Q(s)}{P(s)}$ and $\frac{R(s)}{P(s)}$. The control law is governed by

$$P(s)u = R(s)w - Q(s)y \quad (6)$$

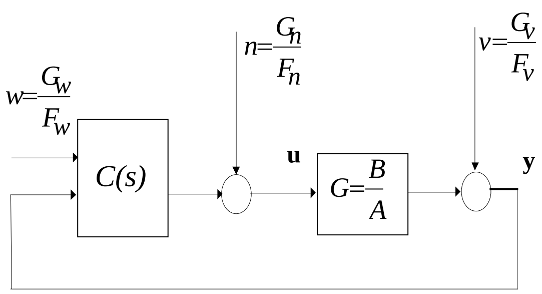

Fig. 1: General feedback system

Disturbances $v=\frac{G_v(s)}{F_v(s)}$ and $n=\frac{G_n(s)}{F_n(s)}$ corrupt the output and input, respectively and $w=\frac{G_w(s)}{F_w(s)}$ represents a reference. All transfer functions, $\frac{B}{A}, \frac{Q}{P}, \frac{R}{P}, \frac{G_v}{F_v}, \frac{G_n}{F_n}, \frac{G_w}{F_w}$ are supposed to be coprime in $R_{PS}(s)$. The disturbance v usually represents a harmonic signal while n can be modeled by a stepwise signal.

The objective is to design controller transfer functions $P(s)$, $Q(s)$, $R(s)$ such that the feedback system is internally BIBO stable, the reference error tends asymptotically to zero and the disturbances v and n are asymptotically eliminated from the plant output.

Commonly, it is desirable that the feedback system be internally BIBO stable in the sense that any bounded input produces a bounded output (e.g. [4], [10]). All transfer functions of the feedback system (Fig. 1.) have common denominator $AP+BQ$. One of the nice and convenient results of the algebraic philosophy is that this denominator should be a unit in the ring $R_{PS}(s)$. In other words, the term $(AP+BQ)^{-1}$ resides in $R_{PS}(s)$ and the feedback system is BIBO stable. If the elements A and B are coprime in $R_{PS}(s)$ then all stabilizing controllers are given through a solution of Diophantine (Bézout) equation:

$$AP_0 + BQ_0 = 1 \quad (7)$$

in a parametric form

$$\frac{Q_0 - AT}{P_0 + BT} \quad (8)$$

where T varies over $R_{PS}(s)$ while satisfying $P_0+BT\neq 0$. From the practical point of view, it is often desirable to ensure more than stability. Probably the most frequent problem of importance is that of reference tracking. Then the tracking error e tends to zero if

a) F_w divides P for 1DOF (9)
b) F_w divides 1-BR for 2DOF (10)

Another control problem of practical importance is a disturbance rejection and disturbance attenuation. In both cases, the effect of disturbances v and n should be asymptotically eliminated from the plant output. Since the both disturbances are external inputs into the feedback part of the system, the effect must be processed by a feedback controller. The algebraic approach enables to express these conditions by the notion of divisibility. The details can be found e.g. [4], [5] or [7]. More precisely, F_v must divide the multiple AP and F_n the multiple BP. When define relatively prime elements A_0, F_{v0} and B_0, F_{n0} in $R_{PS}(s)$

$$\frac{A}{F_v}=\frac{A_0}{F_{v0}}, \qquad \frac{B}{F_n}=\frac{B_0}{F_{n0}} \quad (11)$$

then the problem of disturbance rejection and attenuation is solvable if and only if the pairs F_v, B *and* F_n , B are relatively prime and the feedback controller is given by

$$C_b = \frac{Q}{P} = \frac{Q}{P_0 F_{v0} F_{n0}} \quad (12)$$

where P_0, Q are the solution of the equation

$$AF_{v0}F_{n0}P_0 + BQ = 1 \quad (13)$$

Moreover, for simultaneous reference tracking and disturbance rejection and attenuation F_w must divide P_0 for 1DOF or

$$F_w Z + BR = 1 \quad (14)$$

for 2DOF structure.

IV. TOOLS FOR ROBUST ANALYSIS

The H_∞ norm of the system (4) in $R_{PS}(s)$ is defined by

$$\|G\| = \sup_{\mathrm{Re}s\geq 0}|G(s)| = \sup_{\omega\in E}|G(j\omega)| \quad (15)$$

$$\|G_1 G_2\| = \sup_{\mathrm{Re}s \geq 0} \{|G_1(s)|^2 + |G_2(s)|^2\}^{\frac{1}{2}} \tag{16}$$

This (called infinity) norm is the radius of the smallest circle containing the Nyquist plot of the transfer function and it is a convenient tool for the evaluation of uncertainty. Almost all mathematical models differ from physical systems. Let $G(s) = \frac{B(s)}{A(s)}$ be a nominal plant and consider a family of perturbed systems $G'(s) = \frac{B'(s)}{A'(s)}$ where

$$\begin{aligned} &\|A - A'\| \leq \varepsilon_1, \quad \|B - B'\| \leq \varepsilon_2 \\ &or \quad \|A - A' \quad B - B'\| \leq \varepsilon \end{aligned} \tag{17}$$

where ε_1, ε_2, ε are positive constants.
For robust control, it is necessary to choose a part of all stabilizing controllers (7), (8) which stabilize perturbed plants. The answer can be found in [10]. For perturbed plants choose such P, Q in (7), (8) which fulfill the conditions

$$\varepsilon_1 \|P_0 + BT\| + \varepsilon_2 \|Q_0 - AT\| < 1 \tag{18}$$

$$\text{or} \quad \varepsilon \left\| \begin{matrix} P_0 + AT \\ Q_0 - AT \end{matrix} \right\| < 1 \tag{19}$$

For a deeper insight into robustness the notion of the sensitivity function:

$$\in = \frac{y}{v} = A(P_0 + BT) \tag{20}$$

can be used in the sense of [3]. For the mentioned SISO systems, sensitivity function $\in$ is a nonlinear function of $m > 0$ and it can be minimized by a simple scalar optimization method. In this way, the “most robust” controller of given structure can be obtained.
Another insight for the robust stability analysis can be performed through inspecting of open-loop Nyquist plots. The class of transfer functions (1), (2) can be confined by a two dimensional convex polygon. The vertices of this polygon are characterized by the following values of parameters $a_0(t), b_0(t)$:

$$\begin{aligned} V_1: \quad & b_0^{+} = \beta_0 + \lambda_1, \quad a_0^{+} = \alpha_0 + \lambda_2 \\ V_2: \quad & b_0^{-} = \beta_0 - \lambda_1, \quad a_0^{+} = \alpha_0 + \lambda_2 \\ V_3: \quad & b_0^{+} = \beta_0 + \lambda_1, \quad a_0^{-} = \alpha_0 - \lambda_2 \\ V_4: \quad & b_0^{-} = \beta_0 - \lambda_1, \quad a_0^{-} = \alpha_0 - \lambda_2 \end{aligned} \tag{21}$$

Then, for a designed feedback controller $\frac{Q}{P}$ the open loop Nyquist plots $\frac{BQ}{AP}$ for all vertices can be studied. A natural way for the stability test is the Nyquist criterion. If all four vertices exhibit the stable result then the designed controller $\frac{Q}{P}$ achieves the robust stability for all transfer functions (1), (2).
Another tool for robust stability analysis can facilitated via interval polynomials and Kharitonov’s theorem. This theorem can be used for testing stability of the closed-loop characteristic polynomial; it means the numerator of the rational function $AP+BQ$:

$$c(s,t) = num(AP + BQ) \tag{22}$$

The closed-loop characteristic polynomial (22) is then supposed as an interval one according to minimal and maximal values of parameters $a_0(t), b_0(t)$. In this case, Mikhailov-Leonhard test can be applied. Four stable Kharitonov’s polynomials means stability of the characteristic polynomial (22) with all possible combinations of parameters $a_0(t), b_0(t)$.

V. Program Implementation

A MATLAB-package with simulation support in SIMULINK was developed for plants with periodic parameters of the first and second orders with and without time-delay. The program for automatic control design was created in Matlab R12, Simulink and Polynomial Toolbox 2.5. It covers two control design strategies and both structures of the closed loop as described and outlined in Section III. The first strategy generates the controller by a user defined value of the tuning parameter m. The second one minimizes the sensitivity function (20) in the sense of H_∞ norm and without any knowledge of the perturbed plant tries to find the “most robust” controller for the nominal plant. The main menu window of the program is shown in Fig. 2.

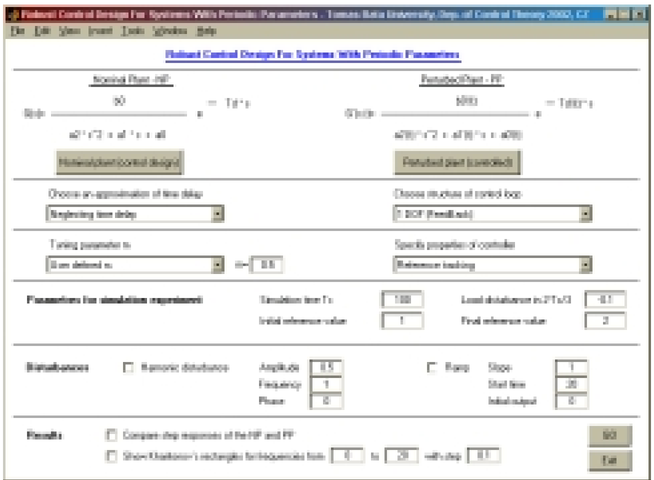

Fig. 2: The main menu window

The program system enables design and simulation of a wide spectrum of robust control problems. It is desirable for robust control that the nominal and perturbed (really controlled) plant is different. The program is user friendly as much as possible. A user can set up two different plants with different degrees, time constants and delays. First, a nominal plant of a desired structure (first or second order) with its transfer function and dead-time has to be defined. Then, a transfer function of the given perturbed plant can be set up. All parameters in this transfer function (including time-delay) can be time-varying according to (2). The situation is outlined in Fig. 3 and Fig. 4.

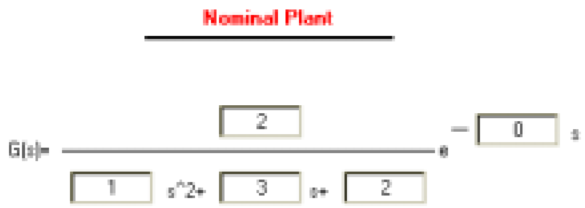

Fig. 3: Definition of nominal plant

Fig. 4: Definition of perturbed plant

If the nominal plant contains transport delay then the important step is to choose an approximation method of the time delay term. There are four possibilities of approximation as it is illustrated in Fig. 5.

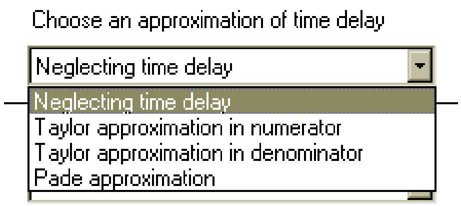

Fig. 5: Approximation of time delay

The program is not able to support Pade or Taylor approximation in denominator with 2nd order nominal plant since the resulting controller would be too high order.

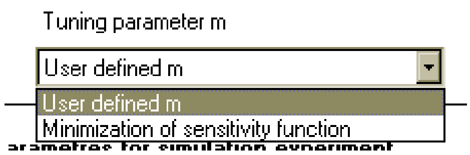

Fig. 6: Choices of control strategies

The program also covers both well-known control structures: feedback (1DOF) and feedback-feedforward (2DOF) one. Further, there are two options for the control loop structure and two mentioned options for the control design. These choices are defined in the main menu according to Fig. 6 and Fig. 7.

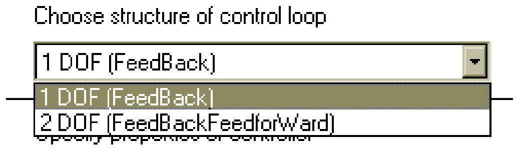

Fig. 7: Structure of the control scheme

The mentioned algebraic approach is able to design controllers which are prepared for simultaneous tracking and disturbance rejection. There are two possibilities which can be defined in the subwindow according to Fig. 8.

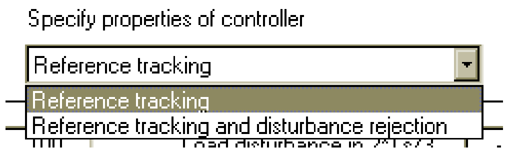

Fig. 8: Ability of the controller

Then a simulation run of the designed controller and the perturbed plant can follow. In some cases it is necessary to adjust simulation parameters: a simulation horizon, initial and final reference values, time of load disturbances and so on. It can be performed very simply in the part of the main window in Fig. 9.

Fig. 9: Simulation parameters

The software tools also enable to corrupt the measured output of the controlled plant by a harmonic or ramp signal. The user can define the presence and type of the disturbance according to Fig. 10.

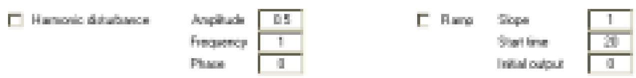

Fig. 10: Harmonic and ramp disturbances

A very important part of the simulation is the display of obtained results. The program environment enables comparison between step responses of nominal and perturbed plant and also it shows Kharitonov's rectangles for closed-loop characteristic polynomial in the selected range of frequencies – see Fig. 11.

Fig. 11: Selection of the output display

The plots of Kharitonov's rectangles are computed and displayed with help of the Polynomial Toolbox. The simulation of the perturbed plant (with harmonically perturbed parameters) with the controller designed for the nominal transfer function is performed in the standard Simulink environment. All simulation variables can be stored and transferred out of the Matlab workspace and ITAE, IAE or IE criteria can be calculated as a tool for comparison and quality evaluation of the control behavior.

VI. ILLUSTRATIVE EXAMPLE

A controlled time-varying system is given by a second order transfer function:

$$G(s,t)=\frac{2+0.5\sin(0.2t)}{s^2+[3+0.9\sin(0.5t)]s+[2+0.2\sin(0.9t)]} \tag{23}$$

It means a nominal system was supposed in the form:

$$G(s)=\frac{2}{s^2+3s+2} \tag{24}$$

The comparison of step responses of systems (23) and (24) is shown in Fig. 12.

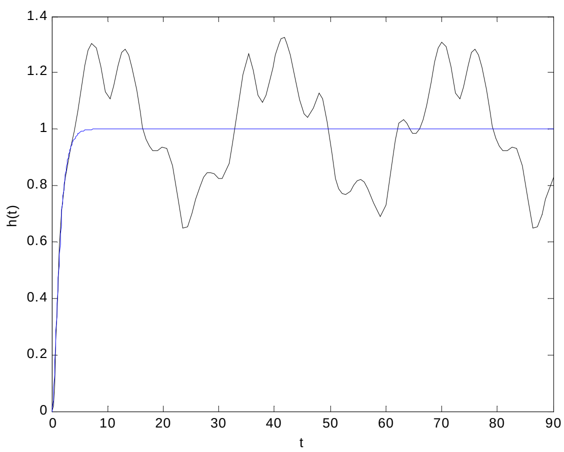

Fig. 12: Step responses of (22) and (23)

Further, it was supposed the 1DOF and 2DOF control structure with a reference tracking controller, load disturbance -1 in 2/3 of a simulation time and user defined tuning parameter m=4. The feedback controller then takes a form:

$$C_b(s)=\frac{27.5s^2+115s+128}{s^2+13s} \tag{25}$$

which gives the closed-loop characteristic interval polynomial according to minimal and maximal values of parameters in transfer function (23) in the sense of (3):

$$\begin{aligned} c(s,t)&=s^4+[15.1;16.9]s^3+[70.35;121.65]s^2+\\ &+[195.9;316.1]s+[192;320] \end{aligned} \tag{26}$$

Kharitonov's rectangles for interval polynomial (26) are shown in Fig. 13. Fig. 14 then gives detailed view on the area near the point [0; 0j]. As can be seen, this interval characteristic polynomial is stable in the sense of Mikhailov criterion for all possible combinations of periodic parameters.

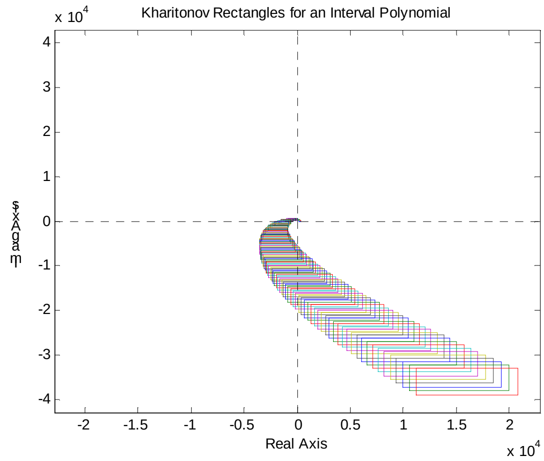

Fig. 13: Kharitonov's rectangles for (25)

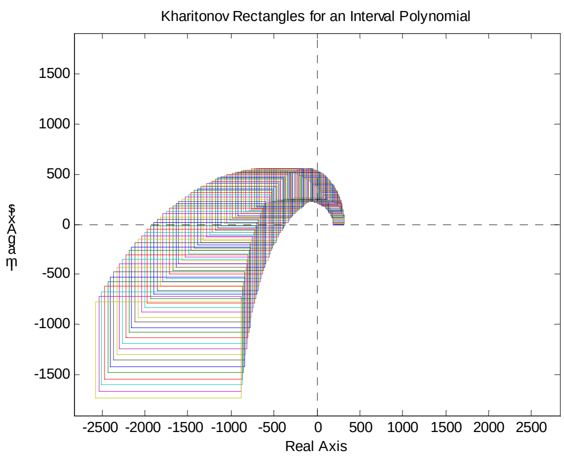

Fig. 14: Kharitonov's rectangles for (25) – detail for lower frequencies

The feedforward controller (for the 2DOF control structure) has a form:

$$C_f(s) = \frac{8s^2 + 64s + 128}{s^2 + 13s} \qquad (27)$$

The finally closed-loop control behavior of the system (23) for the 1DOF and 2DOF control structure is given in Fig. 15. These control responses represent a very acceptable behavior.

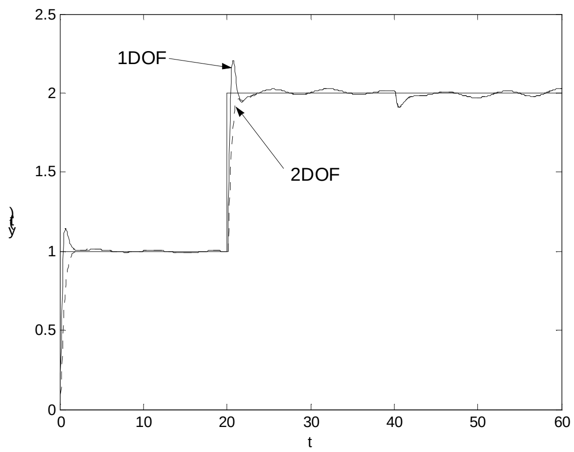

Fig. 15: The 1DOF and 2DOF control behavior of the system (22)

VII. CONCLUSION

A design method based on fractional representation was developed for SISO continuous-time systems with periodic parameters, generally with time delay. A time delay term can be approximated in various ways. Resulting control laws for first and second order systems give a class of generalized PI and PID structures. The algebraic methodology enables to derive controllers rejecting also disturbances. The robustness and control behavior can be tuned and influenced by a single scalar parameter $m>0$. The proposed methodology is supported by a Matlab + Simulink program system for automatic design and simulation.

VIII. ACKNOWLEDGEMENTS

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