Stability analysis of low-order combustion systems

Qing-Chang Zhong, Silviu-Iulian Niculescu and Anuradha M. Annaswamy

***Abstract*— In this paper, the stability conditions of second-order systems including a single-delay or two delays are considered. Such a model is frequently encountered in combustion systems. The stability analysis are vital for the safe operation of such systems. For the single-delay model, complete stability criteria are proposed; for the two-delay model, necessary conditions, sufficient conditions and necessary and sufficient conditions are obtained.**

Index terms: combustion system; delay; stability analysis; dual-locus diagram

I. INTRODUCTION

In recent years, there are more and more applications involving the stability analysis of low-order systems including delay(s), apart from the stability analysis of general time-delay systems using time-domain approaches [1], [2], which in general only offers sufficient conditions and hence no clear information about how far away from the necessity these conditions are. However, there is a great need from engineering of such information. For example, the control of combustion systems [3], [4] relies on the stability analysis of a second-order system including two delays and a mass-spring-damper system controlled over communication networks [5], [6] relies on the stability analysis of a second-order system including one delay. It is crucial to understand the stability region of such systems so that the instability can be avoided.

Complete stability criteria were given in [6] for the mass-spring-damper system controlled over communication networks, where the control gain is assumed to be negative. In this paper, the result in [6] is extended to a more general case which allows the control gain to be positive as well. Hence, it can be used to analyze the stability of combustion systems [4], which can be modeled (linearized) as the following general second-order linear systems including two delays:

$$H_0(s) = s^2 + 2\zeta\alpha s + \alpha^2 + k_1 e^{-s\tau_1} + k_2 e^{-s\tau_2} = 0 \quad (1)$$

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under the hypothesis $0 < \tau_1 < \tau_2$. Here, τ_1 represents the connective delay (due to the transport lag from the supply to the burning plane of the flame), and τ_2 represents the propagation delay (velocity perturbations). Our objective in this paper is to find the stability condition of (1).

II. THE SINGLE-DELAY CASE

The single-delay case, e.g. when $k_2 = 0$ without loss of generality, was considered in [6] for negative gain k_1 using dual-locus diagram. Here, we extend the results obtained there to the case of $k_1 \in (-\infty, +\infty)$.

When $k_2 = 0$, the system (1) becomes

$$s^2 + 2\zeta\alpha s + \alpha^2 + k_1 e^{-s\tau_1} = 0. \quad (2)$$

The stability of (2) is equivalent to that of

$$1 + G(s)e^{-s\tau_1} = 0,$$

where

$$G(s) = \frac{k_1}{s^2 + 2\zeta\alpha s + \alpha^2}.$$

The magnitude response of $G(s)$ is

$$|G(j\omega)| = \frac{|k_1|}{\sqrt{(\alpha^2 - \omega^2)^2 + 4\zeta^2\alpha^2\omega^2}}.$$

For $0 < \zeta < \frac{1}{\sqrt{2}}$, the peak of $|G(j\omega)|$ is

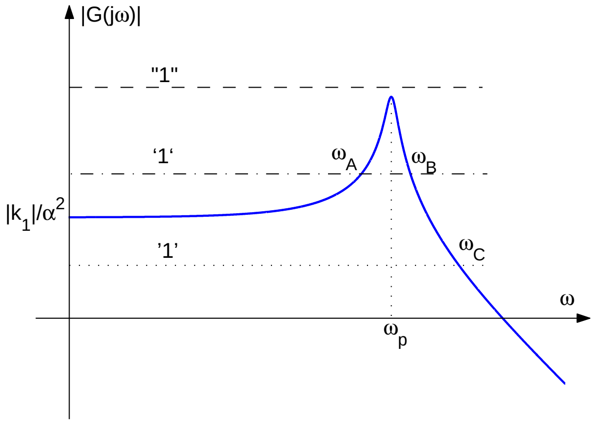
$$|G(j\omega)|_p = \frac{|k_1|}{2\zeta\alpha^2\sqrt{1-\zeta^2}}$$

at $\omega_p = \alpha\sqrt{1-2\zeta^2}$; for $\zeta \geq \frac{1}{\sqrt{2}}$, the peak of $|G(j\omega)|$ is

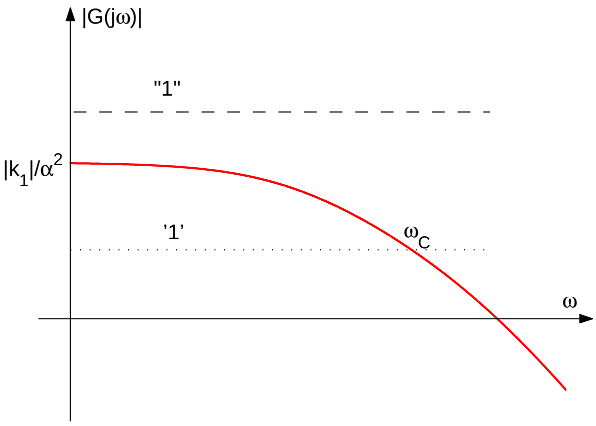
$$|G(j\omega)|_p = \frac{|k_1|}{\alpha^2}$$

at $\omega_p = 0$. In other words, the magnitude response has two kinds of shapes, as shown in Figure 1. In one case ($0 < \zeta < \frac{1}{\sqrt{2}}$), there is one peak and in the other case ($\zeta \geq \frac{1}{\sqrt{2}}$) the magnitude response is monotonically decreasing. It is worth noting that the step response has a peak if $\zeta < 1$, which should not be confused with the case of frequency responses.

According to the well-known small-gain theorem, the system (2) is delay-independently stable when $|G(j\omega)|_p < 1$. This provides two stability conditions: (i) $|k_1| < \alpha^2$ if $\zeta \geq \frac{1}{\sqrt{2}}$; (ii) $|k_1| < 2\zeta\alpha^2\sqrt{1-\zeta^2}$ if $0 < \zeta < \frac{1}{\sqrt{2}}$. In these cases, the magnitude curves do not intersect with the straight line "1" shown in Figure 1 (a) and (b) and



(a) $0 < \zeta < \frac{1}{\sqrt{2}}$



(b) $\zeta \geq \frac{1}{\sqrt{2}}$

Fig. 1. The magnitude response of $G(s)$

hence the system is delay-independently stable. Naturally, another two cases are: the magnitude curve intersects with the straight line twice at ω_A and then at ω_B (this only happens when $0 < \zeta < \frac{1}{\sqrt{2}}$) or only once at ω_C. The stability criteria for these two cases will be analyzed below:

Case 1: $|k_1| \geq \alpha^2$

The magnitude curve starts above $'1'$, see Figure 1 (a) and (b), and there is only one intersection at

$$\omega_C = \alpha\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + k_1^2/\alpha^4}}, \qquad (3)$$

which is the positive solution of $|G(j\omega)| = 1$, i.e. of

$$(\alpha^2 - \omega^2)^2 + 4\zeta^2\alpha^2\omega^2 = k_1^2.$$

If k_1 is positive, i.e. if $k_1 \geq \alpha^2$, then (2) is delay-dependently stable if

$$\omega_C\tau_1 + \arctan\frac{2\zeta\alpha\omega_C}{\alpha^2 - \omega_C^2} < \pi.$$

Since there is only one intersection ω_C, there is only one delay interval so that the system is stable. If k_1 is negative, i.e. if $k_1 \leq -\alpha^2$, then (2) is always unstable because there is no phase margin left for the delay term.

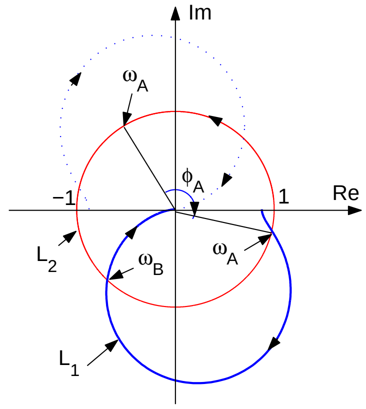


Fig. 2. The dual-locus diagram of the single-delay system

Case 2: $0 < \zeta < \frac{1}{\sqrt{2}}$ and $\alpha^2 > |k_1| \geq 2\zeta\alpha^2\sqrt{1 - \zeta^2}$

The magnitude curve starts above $'1'$ but the peak stays below $"1"$, see Figure 1(a). There are two intersections with the straight line '1' at

$$\omega_A = \alpha\sqrt{1 - 2\zeta^2 - \sqrt{4\zeta^4 - 4\zeta^2 + k_1^2/\alpha^4}}$$

and

$$\omega_B = \alpha\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + k_1^2/\alpha^4}},$$

which are the positive solutions of $|G(j\omega)| = 1$, i.e. of

$$(\alpha^2 - \omega^2)^2 + 4\zeta^2\alpha^2\omega^2 = k_1^2.$$

The dual-locus diagram in this case, $L_1(s) = G(s)$ and $L_2(s) = e^{\tau_1 s}$, is shown in Figure 2 for $\omega = 0 \sim +\infty$. A similar dual-locus diagram was used in [7], but here we can recover the phase angle (equal to the phase difference between the two loci) in the Nyquist plot. The locus L_1 starts inside the locus L_2 and moves outside of L_2 at ω_A then moves inside L_2 at ω_B. For positive k_1, L_1 (the solid one) starts on the positive real axis; for negative k_1, L_1 (the dashed one) starts on the negative real axis. The stability will be analyzed below according to the sign of k_1:

(i) k_1 is positive

When L_1 arrives at $\omega = \omega_A$, the phase difference ϕ_A (between L_1 and L_2) is

$$\phi_A = -\arctan\frac{2\zeta\alpha\omega_A}{\alpha^2 - \omega_A^2} - (\omega_A\tau_1).$$

There exists a unique nonnegative integer i such that the phase difference ϕ_A biased by $2i\pi$ belongs to $(-\pi, \pi)$, i.e.,

$$-\pi < \phi_A + 2i\pi < \pi. \qquad (4)$$

A different i is called a different cycle, which starts above the negative real axis and ends below the negative real axis (i.e. the negative real axis is split). When $i = 0$, the above inequality includes the principle half cycle $-\pi < \phi_A \leq 0$. According to the Nyquist criteria, the system (2) is stable if and only if the phase difference ϕ_B between L_1 and L_2, when L_1 arrives at $\omega = \omega_B$, is still in the same cycle, i.e.,

$$-\pi < -\arctan \frac{2\zeta\alpha\omega_B}{\alpha^2 - \omega_B^2} - (\omega_B\tau_1) + 2i\pi. \qquad (5)$$

Otherwise, the Nyquist plot encircles the point $(-1, 0)$, which implies the instability. The inequalities (4) and (5) together provide the following delay intervals to guarantee the system stability:

$$\frac{1}{\omega_A}(2i\pi - \pi - \arctan\frac{2\zeta\alpha\omega_A}{\alpha^2 - \omega_A^2}) < \tau_1 < \frac{1}{\omega_B}(2i\pi + \pi - \arctan\frac{2\zeta\alpha\omega_B}{\alpha^2 - \omega_B^2}), \quad (6)$$

where $i = 0, 1, 2, \cdots$ till the right side is no longer larger than the left.

(ii) k_1 is negative

In this case, due to the negativeness of k_1, there is an extra phase shift $-\pi$, i.e.,

$$\phi_A = -\pi - \arctan \frac{2\zeta\alpha\omega_A}{\alpha^2 - \omega_A^2} - (\omega_A\tau_1).$$

Due to this extra phase shift, the principle cycle (corresponding to $i = 0$) becomes $(-\pi, -3\pi)$ and hence there exists a unique nonnegative integer i such that the biased phase shift, $\phi_A + 2i\pi$, satisfies

$$-3\pi < \phi_A + 2i\pi < -\pi. \qquad (7)$$

The system (2) is stable if and only if the phase difference ϕ_B between L_1 and L_2, when L_1 arrives at $\omega = \omega_B$, is still in the same cycle, i.e,

$$-3\pi < -\pi - \arctan \frac{2\zeta\alpha\omega_B}{\alpha^2 - \omega_B^2} - (\omega_B\tau_1) + 2i\pi. \qquad (8)$$

The inequalities (7) and (8) together provide the following delay intervals to guarantee the system stability:

$$\frac{1}{\omega_A}(2i\pi - \arctan\frac{2\zeta\alpha\omega_A}{\alpha^2 - \omega_A^2}) < \tau_1 < \frac{1}{\omega_B}(2i\pi + 2\pi - \arctan\frac{2\zeta\alpha\omega_B}{\alpha^2 - \omega_B^2}), \quad (9)$$

where $i = 0, 1, 2, \cdots$ till the right side is no longer larger than the left. This condition, although in different form, is equivalent to that obtained in [6].

In summary, the following three lemmas hold:

Lemma 1. *The system (2) is always unstable if* $-\frac{k_1}{\alpha^2} \geq 1$. *See region* R_U *in Figure 3.*

Lemma 2. *The system (2) is delay-independently stable, see region* R_I *in Figure 3, if one of the following conditions hold:*

(i) $|k_1| < \alpha^2$ *if* $\zeta \geq \frac{1}{\sqrt{2}}$;

(ii) $|k_1| < 2\zeta\alpha^2\sqrt{1-\zeta^2}$ *if* $0 < \zeta < \frac{1}{\sqrt{2}}$.

Lemma 3. *The system (2) is delay-dependently stable if one of the following conditions hold:*

(i) $0 < \zeta < \frac{1}{\sqrt{2}}$ *and* $1 > -\frac{k_1}{\alpha^2} \geq 2\zeta\sqrt{1-\zeta^2}$. *See region* R_A *in Figure 3. The delay intervals guaranteeing the stability are given in (9);*

(ii) $0 < \zeta < \frac{1}{\sqrt{2}}$ *and* $-1 < -\frac{k_1}{\alpha^2} \leq -2\zeta\sqrt{1-\zeta^2}$. *See region* R_B *in Figure 3. The delay intervals guaranteeing the stability are given in (6);*

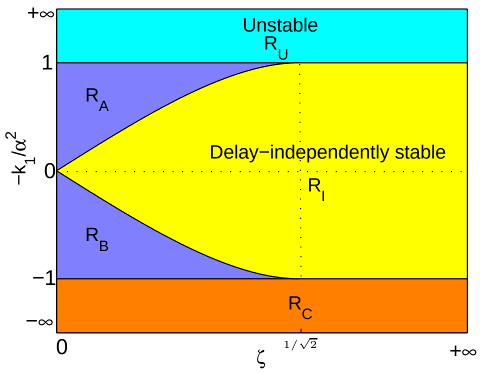


Fig. 3. The stability regions of the single-delay system

(iii) $-\frac{k_1}{\alpha^2} \leq -1$ *and* $0 \leq \tau_1 < \frac{1}{\omega_C}(\pi - \arctan\frac{2\zeta\alpha\omega_C}{\alpha^2-\omega_C^2})$ *with* ω_C *given in (3). See region* R_C *in Figure 3.*

III. Main Results

A. Sufficient conditions

The system (1) can be reformulated as

$$1 + M(s)e^{-s\tau_1} = 0$$

where

$$M(s) = \frac{k_1 + k_2 e^{-s(\tau_2 - \tau_1)}}{s^2 + 2\zeta\alpha s + \alpha^2}.$$

Since $\left|e^{-s(\tau_2-\tau_1)}\right| = 1$ for any $\omega \in (-\infty, +\infty)$, $M(s)$ has the following upper envelope

$$M_e(s) = \frac{|k_1| + |k_2|}{s^2 + 2\zeta\alpha s + \alpha^2}.$$

According to the result in the last section, $M_e(s)$ has a peak value of

$$M_{ep} = \frac{|k_1| + |k_2|}{2\zeta\alpha^2\sqrt{1-\zeta^2}}$$

for $0 < \zeta < \frac{1}{\sqrt{2}}$ or a peak value of

$$M_{ep} = \frac{|k_1| + |k_2|}{\alpha^2}$$

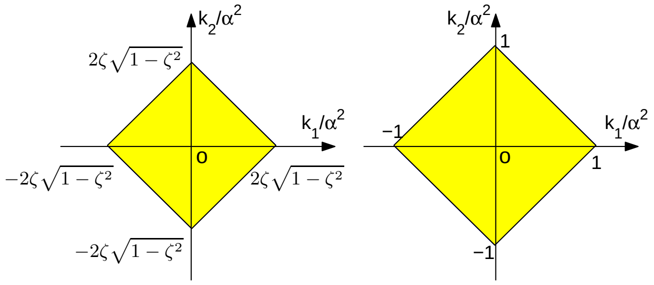
for $\zeta \geq \frac{1}{\sqrt{2}}$. According to the small-gain theorem, the following theorem holds:

Theorem 1. *The system (1) is delay-independently stable if one of the following conditions hold:*

(i) $|k_1| + |k_2| < \alpha^2$ *if* $\zeta \geq \frac{1}{\sqrt{2}}$;

(ii) $|k_1| + |k_2| < 2\zeta\alpha^2\sqrt{1-\zeta^2}$ *if* $0 < \zeta < \frac{1}{\sqrt{2}}$.

These conditions can be depicted as shown in Figure 4. The smaller the damping ratio, the smaller the stability region. When $\zeta \geq \frac{1}{\sqrt{2}}$, the stability region does not change.



(a) $0 < \zeta < \frac{1}{\sqrt{2}}$ (b) $\zeta \geq \frac{1}{\sqrt{2}}$

Fig. 4. The delay-independent stability region of (1)

B. Necessary conditions

Lemma 4. *[8], [9] Consider the following quasipolynomial*

$$H(s) = \sum_{l=1}^{r} \sum_{i=0}^{n} h_{il} s^{n-i} e^{\tau_l s},$$

where $\tau_1 < \tau_2 < \cdots < \tau_r$, $\tau_r + \tau_1 > 0$ and the main term $h_{0r} \neq 0$ (corresponding to the largest τ_l and the highest degree of s).

If $H(s)$ is stable, then the derivative of $H(s)$ w.r.t s, $H^{'}(s)$, is also a stable quasipolynomial.

This lemma will be used to derive a necessary condition for the combustion system (1). Denote

$$H_1(s) = (s^2 + 2\zeta\alpha s + \alpha^2)e^{s\tau_2} + k_1 e^{-s(\tau_1-\tau_2)} + k_2,$$

then the following result holds:

Theorem 2. *The system given in (1) is unstable if $-k_1(\tau_2 - \tau_1) \geq \alpha^2\tau_2 + 2\zeta\alpha$.*

The system given in (1) is delay-dependently stable only if *one of the following conditions hold:*

(i) $|k_1| < \frac{(2\zeta+\alpha\tau_2)\alpha}{\tau_2-\tau_1}$ *and* $\sqrt{2}(\tau_2\zeta\alpha + 1) \geq \sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}$ *;*

(ii) $|k_1| < \frac{2(\tau_2\zeta\alpha+1)}{\tau_2(\tau_2-\tau_1)} \cdot \sqrt{\alpha\tau_2(2\zeta + \alpha\tau_2) - (\tau_2\zeta\alpha + 1)^2}$ *and* $\sqrt{2}(\tau_2\zeta\alpha + 1) < \sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}$*;*

(iii) $\sqrt{2}(\tau_2\zeta\alpha + 1) < \sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}$ *and* $\frac{\alpha^2\tau_2+2\zeta\alpha}{\tau_2-\tau_1} > -k_1 \geq 2\frac{\zeta\alpha\tau_2+1}{\tau_2(\tau_2-\tau_1)}\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2 - (\zeta\alpha\tau_2 + 1)^2}$ *and* $\frac{1}{\hat{\omega}_A}(2i\pi - \arctan\frac{2(\zeta\alpha\tau_2+1)\hat{\omega}_A}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\hat{\omega}_A^2}) < \tau_1 < \frac{1}{\hat{\omega}_B}(2i\pi + 2\pi - \arctan\frac{2(\zeta\alpha\tau_2+1)\hat{\omega}_B}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\hat{\omega}_B^2})$*;*

(iv) $\sqrt{2}(\tau_2\zeta\alpha + 1) < \sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}$ *and* $-\frac{\alpha^2\tau_2+2\zeta\alpha}{\tau_2-\tau_1} < -k_1 \leq -2\frac{\zeta\alpha\tau_2+1}{\tau_2(\tau_2-\tau_1)}\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2 - (\zeta\alpha\tau_2 + 1)^2}$ *and* $\frac{1}{\hat{\omega}_A}(2i\pi - \pi - \arctan\frac{2(\zeta\alpha\tau_2+1)\hat{\omega}_A}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\hat{\omega}_A^2}) < \tau_1 < \frac{1}{\hat{\omega}_B}(2i\pi + \pi - \arctan\frac{2(\zeta\alpha\tau_2+1)\hat{\omega}_B}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\hat{\omega}_B^2})$*;*

(v) $-\frac{k_1(\tau_2-\tau_1)}{\alpha^2\tau_2+2\zeta\alpha} \leq -1$ *and* $0 \leq \tau_1 < \frac{1}{\hat{\omega}_C}(\pi - \arctan\frac{2(\zeta\alpha\tau_2+1)\hat{\omega}_C}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\hat{\omega}_C^2})$.

The above-mentioned $\hat{\omega}_A$, $\hat{\omega}_B$ and $\hat{\omega}_C$ are given below when the corresponding conditions hold:

$$\hat{\omega}_A = \frac{1}{\tau_2}\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2 - 2(\zeta\alpha\tau_2 + 1)^2(1 + \sqrt{1 - \frac{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}{(\zeta\alpha\tau_2 + 1)^2} + \frac{k_1^2\tau_2^2(\tau_2 - \tau_1)^2}{4(\zeta\alpha\tau_2 + 1)^4}}\ )},$$

$$\hat{\omega}_B = \frac{1}{\tau_2}\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2 - 2(\zeta\alpha\tau_2 + 1)^2(1 - \sqrt{1 - \frac{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}{(\zeta\alpha\tau_2 + 1)^2} + \frac{k_1^2\tau_2^2(\tau_2 - \tau_1)^2}{4(\zeta\alpha\tau_2 + 1)^4}}\ )},$$

$$\hat{\omega}_C = \frac{1}{\tau_2}\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2 - 2(\zeta\alpha\tau_2 + 1)^2(1 - \sqrt{1 - \frac{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}{(\zeta\alpha\tau_2 + 1)^2} + \frac{k_1^2\tau_2^2(\tau_2 - \tau_1)^2}{4(\zeta\alpha\tau_2 + 1)^4}}\ )}.$$

Proof. Obviously, $H_1(s)$ holds the same stability as $H_0(s)$. According to Lemma 4, if $H_1(s)$ is stable, then

$$H_1^{'}(s) = (2s + 2\zeta\alpha + \tau_2(s^2 + 2\zeta\alpha s + \alpha^2) + k_1(\tau_2 - \tau_1)e^{-s\tau_1})e^{s\tau_2}$$

is also stable, which means the following quasipolynomial is stable:

$$H_2(s) = s^2 + 2(\zeta\alpha + \frac{1}{\tau_2})s + (\frac{2\zeta}{\tau_2} + \alpha)\alpha + k_1(1 - \frac{\tau_1}{\tau_2})e^{-s\tau_1}.$$

This is the single-delay system of which the complete stability criteria was given in the last section. Directly using those results with the following substitutions completes the proof:

$$\alpha \longleftarrow: \sqrt{\alpha^2 + \frac{2\zeta\alpha}{\tau_2}}$$

$$\zeta \longleftarrow: \frac{\zeta\alpha\tau_2 + 1}{\sqrt{\alpha^2\tau_2^2 + 2\zeta\alpha\tau_2}}$$

$$k_1 \longleftarrow: k_1(1 - \frac{\tau_1}{\tau_2})$$

$$\frac{k_1}{\alpha^2} \longleftarrow: \frac{k_1(\tau_2-\tau_1)}{\alpha^2\tau_2+2\zeta\alpha}$$

$$\arctan\frac{2\zeta\alpha\omega_C}{\alpha^2-\omega_C^2} \longleftarrow: \arctan\frac{2(\zeta\alpha\tau_2+1)\omega_C}{\alpha^2\tau_2+2\zeta\alpha-\tau_2\omega_C^2}$$

□

C. Necessary and sufficient conditions

Lemma 5. *The argument of $k_1+k_2e^{-j\omega(\tau_2-\tau_1)}$ is monotonically decreasing with respect to ω : $0\to+\infty$ iff $|k_2| \geq |k_1|$. Hence, so is the argument of $M(j\omega)$ if $|k_2|\geq|k_1|$.*

Proof. Denote the argument of $k_1+k_2e^{-j\omega(\tau_2-\tau_1)}$ by ϕ_e and $\tau_2-\tau_1$ by Δ, then

$$\phi_e(\omega)=\arctan\frac{-k_2\sin\Delta\omega}{k_1+k_2\cos\Delta\omega}.$$

Differentiate ϕ_e with respect to ω, we have

$$\frac{\mathrm{d}\phi_e}{\mathrm{d}\omega}=-\Delta\cdot\frac{k_1k_2\cos\Delta\omega+k_2^2}{(k_1+k_2\cos\Delta\omega)^2+k_2^2\sin^2\Delta\omega}.$$

Since $\Delta=\tau_2-\tau_1>0$, $\frac{\mathrm{d}\phi_e}{\mathrm{d}\omega}\leq 0$ if and only if

$$k_1k_2\cos\Delta\omega+k_2^2\geq 0,$$

which is equivalent to

$$|k_2|\geq|k_1|\,.$$

It is easy to check that the other part of the argument of $M(j\omega)$ is always monotonically decreasing for ω : $0\to+\infty$. Hence, so is the argument of $M(j\omega)$. This completes the proof.

□

This condition can be depicted as the shadowed area shown in Figure 5. If this condition is satisfied, then the Nyquist curve of $k_1+k_2e^{-j\omega\Delta}$ encircles the origin for indefinite times. In the sequel, we will analyze the stability when this condition is met. Another important condition is that if $|k_1+k_2|<\alpha^2$ or not. Using these two conditions, the shadowed area is split into 4 regions ($\mathrm{R_U}$, $\mathrm{R_A}$, $\mathrm{R_B}$ and $\mathrm{R_C}$) as shown in Figure 5. The typical dual-locus diagrams with $L_1(s)=M(s)$ and $L_2(s)=e^{\tau_1 s}$are shown in Figure 6 for these regions. When $k_2>0$, the locus starts on the positive real axis; when $k_2<0$, the locus starts on the negative real axis. If $|k_1+k_2|<\alpha^2$, the locus starts inside of the unit cycle; if $|k_1+k_2|>\alpha^2$, the locus starts outside of the unit cycle. All locus approaches the origin when $\omega\to+\infty$. Hence, the locus intersects with the unit cycle for finite times: odd times if $|k_1+k_2|>\alpha^2$ or even times if $|k_1+k_2|<\alpha^2$. Denote the first outgoing (w.r.t. the unit cycle) crossing frequency by ω_1 and the rest ingoing and outgoing frequencies by $\omega_2,\cdots\omega_{2r-1}$ and ω_{2r}. The locus crosses the unit cycle for r rounds. If $|k_1+k_2|>\alpha^2$, the first crossing is ingoing and the corresponding crossing

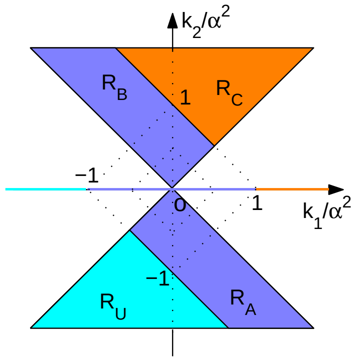


Fig. 5. The stability regions of (1). The dotted box is the delay-independent stability region (see Figure 4)

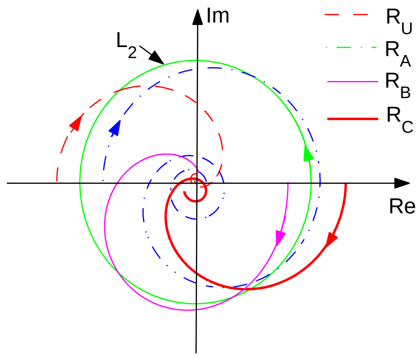


Fig. 6. Dual-locus diagrams of (1) when $|k_2|\geq|k_1|$

frequency is denoted by ω_0. Apparently, these frequencies are the ordered positive solutions of the equation

$$|M(j\omega)|=1,$$

i.e. of

$$k_1^2+2k_1k_2\cos\Delta\omega+k_2^2=(\alpha^2-\omega^2)^2+4\zeta^2\alpha^2\omega^2.$$

Bearing in mind, when the conditions given in Theorem 1 are satisfied there will be no crossing. The locus L_1 remains inside of the locus L_2.

If there is no delay τ_1 (i.e. $\tau_1=0$), then the system is stable if and only if the locus does not encircle the point $(-1,0)$. In other words, the crossings at ω_{2i-1} and ω_{2i} are in the same cycle c_i. When $\tau_1\neq 0$, the phase shift due to τ_1 should be considered. Using the similar arguments in the previous section, the following results can be obtained.

Theorem 3. *When $|k_2|\geq|k_1|$, the system (1) is unstable for any delays $\tau_1>0$ and $\tau_2>0$ if $k_1+k_2\leq-\alpha^2$. See region R_U in Figure 5.*

Theorem 4. *When $|k_2|\geq|k_1|$, the system (1) is (delay-dependently) stable if one of the following conditions hold:*

(i) $0>k_1+k_2>-\alpha^2$ and

there exists a $c_i \geq 0$ *for each* $i = 1, 2, \cdots, r$ *such that*

$$\frac{1}{\omega_{2i-1}}(2c_i\pi+\phi_e(\omega_{2i-1})-\arctan\frac{2\zeta\alpha\omega_{2i-1}}{\alpha^2-\omega_{2i-1}^2}) < \tau_1 < \frac{1}{\omega_{2i}}(2c_i\pi+2\pi+\phi_e(\omega_{2i})-\arctan\frac{2\zeta\alpha\omega_{2i}}{\alpha^2-\omega_{2i}^2}).$$

See region R_A *in Figure 5;*

(ii) $0 < k_1 + k_2 < \alpha^2$ *and*

there exists a $c_i \geq 0$ *for each* $i = 1, 2, \cdots, r$ *such that*

$$\frac{1}{\omega_{2i-1}}(2c_i\pi-\pi+\phi_e(\omega_{2i-1})-\arctan\frac{2\zeta\alpha\omega_{2i-1}}{\alpha^2-\omega_{2i-1}^2}) < \tau_1 < \frac{1}{\omega_{2i}}(2c_i\pi+\pi+\phi_e(\omega_{2i})-\arctan\frac{2\zeta\alpha\omega_{2i}}{\alpha^2-\omega_{2i}^2}).$$

See region R_B *in Figure 5;*

(iii) $k_1 + k_2 > \alpha^2$ *and*

$0 \leq \tau_1 < \frac{1}{\omega_0}(\pi + \phi_e(\omega_0) - \arctan\frac{2\zeta\alpha\omega_0}{\alpha^2-\omega_0^2})$ *and*

there exists a $c_i \geq 0$ *for each* $i = 1, 2, \cdots, r$ *such that*

$$\frac{1}{\omega_{2i-1}}(2c_i\pi-\pi+\phi_e(\omega_{2i-1})-\arctan\frac{2\zeta\alpha\omega_{2i-1}}{\alpha^2-\omega_{2i-1}^2}) < \tau_1 < \frac{1}{\omega_{2i}}(2c_i\pi+\pi+\phi_e(\omega_{2i})-\arctan\frac{2\zeta\alpha\omega_{2i}}{\alpha^2-\omega_{2i}^2}).$$

See region R_C *in Figure 5.*

The axis k_1/α^2 in Figure 5 has been cut into four pieces corresponding to the four regions. This is exactly the results given in lemmas 1-3. Hence, one might expect that the four regions shown in Figure 5 can be extended, at least, to the axis k_1/α^2. However, we haven't found solid evidence to support this.

IV. CONCLUSION

In this paper, the stability of combustion systems modeled as a second-order including one or two delays are analyzed. When there is only one delay, the complete stability criteria have been given; when there are two delays in the model, necessary conditions, sufficient conditions and necessary and sufficient conditions are obtained.

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