Applying orthogonal rational signal representations in system change detection

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Abstract— **The main topic of the paper is the application of frequency domain signal representations in detection system changes. The change detection is based upon the representation coefficients belonging to rational orthogonal bases selected by using *a priori* knowledge belonging to the system dynamics. An FFT-based effective algorithm is described for estimating the coefficients on the basis of non-uniformly spaced frequency domain measurements. This algorithm offers a good basis for analyzing the random errors occurring on the estimated coefficients, that is necessary to use them in the decision making procedure upon changes.**

Keywords— **Change detection, signal representations, rational orthogonal bases.**

I. Introduction

A frequency domain representation of bounded energy stable causal discrete-time signals based upon the concept of orthogonal rational bases, generated during the past decade as a product of several research groups, see e.g. [1], [2], [3], has been presented in [4]. The topic of the current paper is an approach of detection changes on the system structure by using these representations.

The starting point of the change detection scheme is a spectral function belonging to the system under consideration, i.e. a function of a single variable defined on the frequency domain. Frequency responses, transfer functions belonging to input-output pair of signals, as well as spectral density functions can both be considered as spectral functions; different system types (e.g. deterministic or stochastic ones) and analysis tasks (observation of a single signal, or input-output examination with or without excitation) can result in different spectral function concepts. Since uniformly sampled discrete time systems conform to the Shannon rules are considered, the spectral functions involved are periodic, as well as its energy bounded nature and stability imply that they belong to the Hilbert space $\mathcal{H}^2$.

It is assumed, that the spectral function can be measured in some discrete frequency points, that means either direct frequency domain measurement, or computation of function values on the basis of time-domain data. Nowadays the latter alternative is more frequently used: discrete Fourier transform algorithms can efficiently be realized by digital computers (consider e.g. FFT — Fast Fourier Transform).

Orthogonal rational bases offer an excellent framework to produce function representations that utilize a priori knowledge belonging to the system under consideration, hence they can adequately be used in change detection schemes. That is, any difference on the given system relative to the a priori assumed characteristics result in drastic changes in the representation, hence high sensitivity and selectivity can be obtained.

In the present paper detection methods based upon the representation coefficients themselves are discussed. Detection methods that avoid the direct computation are also available, one of these type of methods has been discussed in [5]. However direct computation of representation coefficients is also advantageous method due to an algorithm using FFT on non-uniformly distributed frequency domain data [6].

In the following sections after a brief introduction to rational orthogonal bases, the concept of change detection through representation coefficients, the algorithm of estimating the coefficients, as well as an error analysis will be discussed.

The following notations will be used: $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc on the complex plane; if $a \in \mathbb{C}$ then $\overline{a}$ denotes its complex conjugate; if A is a complex matrix A^* denotes its conjugated transpose; $\mathbb{N}$ and $\mathbb{C}$ denotes the set of natural and complex numbers, respectively.

II. Rational orthogonal bases in $\mathcal{H}^2$

The general form of the orthogonal rational basis in $\mathcal{H}^2$ generated upon the set of poles $\mathbf{a} = \{a_k \in \mathbb{D} \mid k = 0, 1, \ldots, N-1 \ ; \ N \in \mathbb{N}\}$ is given by the following construction:

$$\phi_0(q) \doteq \frac{\sqrt{1-|a_0|^2}}{z-a_0}, \phi_k(q) \doteq \frac{\sqrt{1-|a_k|^2}}{z-a_k} \prod_{j=1}^{k-1} b_{a_j}(z) \quad (1)$$

for $k = 1, 2, \ldots, N-1$, where

$$b_{a_k}(z) = e^{\mathrm{i}\delta_k} \frac{1-\overline{a_k}z}{z-a_k}$$

is the Blaschke-function belonging to parameter a_k (δ_k-s are arbitrary constants), the elements orthogonal rational basis of index $n = N\ell + k$ ($\ell = 0, 1, 2, \ldots$) are defined as

$$\Phi_n(z) \doteq \phi_k(z) B_{\mathbf{a}}^{\ell}(z) \quad B_{\mathbf{a}}(q) \doteq \prod_{j=0}^{N-1} b_{a_j}(z). \quad (2)$$

It can be proved, that the system (1)–(2) constitutes an orthonormal basis in the space $\mathcal{H}^2(\mathbb{D})$ [7].

The Blaschke-function and product are inner functions in the space $\mathcal{H}^2$ (referred as stable all-pass functions in the

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technical literature), hence the Blaschke-product defined upon a finite set of poles can be considered as a shift operator, which produces only rotation on the unit circle, similarly to the standard shift operator z. Of course infinitely many shift operators can be created depending on the selection of poles, hence many different orthogonal rational bases can be generated. It is obvious, that this construction is a generalization of the standard basis, because the latter one is restricted to zero poles (as it is a polynomial system of $1/z$), on the other hand the rational bases generated upon the poles are specific constructions representing the characteristics connected to the selected pole structure. This feature of the rational orthogonal bases gives their significance in the representation of systems: the *a priori* knowledge available of the system structure (embodied partly in the pole structure) can be built in the representing set of functions, resulting in efficient representations sensitive to changes.

Any function $F(z)$ in $\mathcal{H}^2(\mathbb{D})$ can be represented in the orthonormal rational basis; the representation and the coefficients can be given as follows:

$$F(z) = \sum_{n=0}^{\infty} c_n \Phi_n(z), \quad c_n = \langle F, \Phi_n \rangle_z \tag{3}$$

where $\langle ., . \rangle_z$ denotes the inner product with respect to variable z. The representation of rational functions containing finite number of poles can be interpreted through their partial fraction form

$$X(z) = \sum_{k=0}^{N-1} \sum_{m=1}^{m_k} \frac{A_{km}}{(z-a_k)^m} \tag{4}$$

where $m_k \geq 1$ is the multiplicity of pole a_k.

The representation coefficients can be computed by evaluating the inner product in (II); for an estimate an effectively realizable method has been proposed in [6]. The method is based upon the FFT algorithm applied on a finite set of measurement points generated by nonuniform sampling of the frequency function.

Now an algorithm realizing this method will be described by using the 'z'-notation that rather conforms to the practically applicable forms. The basic idea of the realization is the introduction of the argument-function belonging to the Blaschke product. For a single pole $a \in \mathbb{D}$ — since the Blaschke function is an inner function —

$$|b_a(e^{\mathrm{i}t})| = e^{-\mathrm{i}\beta_a(t)}$$

the function of the real variable t $\beta_a(t)$ is called argument-function. Extending the notion to multiple poles: for $\mathbf{a} = \{a_k \in \mathbb{D} \mid k = 0, 1, \ldots, N-1 \; ; \; N \in \mathbb{N}\}$

$$\beta_{\mathbf{a}}(t) \doteq \frac{\beta_{a_0}(t) + \beta_{a_1}(t) + \cdots + \beta_{a_{N-1}}(t)}{N},$$

that is

$$B_{\mathbf{a}}(e^{\mathrm{i}t}) = e^{-\mathrm{i}N\beta_{\mathbf{a}}(t)}.$$

A detailed characterization of the argument function can be found in [7].

By using the notion of the argument-function the elements of the generalized orthogonal basis can be expressed on the unit circle as

$$\Phi_{N\ell+k}(e^{\mathrm{i}t}) = \phi_k(e^{\mathrm{i}t})\, e^{-\mathrm{i}N\ell\,\beta_{\mathbf{a}}(t)}.$$

Hence the coefficient of index $n = N\ell + k$ of the representation belonging to function $F \in \mathcal{H}^2(\mathbb{D})$ will be given as

$$c_n \doteq \langle F, \Phi_{N\ell+k} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathrm{i}t})\, \overline{\phi_k(e^{\mathrm{i}t})}\, e^{\mathrm{i}N\ell\,\beta_{\mathbf{a}}(t)}\, dt. \tag{5}$$

Let the substitution $t = \beta_{\mathbf{a}}^{-1}(s)$ be applied:

$$\langle F, \Phi_{N\ell+k} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s)})\, \overline{\phi_k(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s)})}\, e^{\mathrm{i}N\ell\, s}\, \beta_{\mathbf{a}}'(s)\, ds.$$

For the sake of simplification let the function

$$f_k(s) \doteq F(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s)})\, \overline{\phi_k(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s)})}\, \beta_{\mathbf{a}}'(s) \quad (k = 0, 1, \ldots, N-1), \tag{6}$$

be introduced; the above form can be expressed in the form of a special — N-times "dilated" — Fourier integral:

$$\langle F, \Phi_{N\ell+k} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_k(s)\, e^{\mathrm{i}N\ell\, s}\, ds.$$

The Fourier-integral can approximately be computed by using a discrete scale by dividing the interval $[-\pi, \pi]$ into M equal parts (let M be even), i.e.

$$\Delta s = \frac{2\pi}{M} \qquad s_m = -\pi + m\Delta s \quad (m = 0, 1, \ldots, M-1).$$

The integral can be approximated by the sum (let be $j = N\ell$)

$$\begin{aligned} \hat{c}_{j+k} &= \frac{\Delta s}{2\pi} \sum_{m=0}^{M-1} f_k(s_m)\, e^{\mathrm{i}j(-\pi+m\Delta s)} = \\ &= \frac{1}{M} \sum_{m=0}^{M-1} f_k(s_m)\, e^{\mathrm{i}2\pi \frac{j(m-M/2)}{M}} = \\ &= \left(\frac{1}{M} \sum_{m=0}^{M-1} f_k(s_m)\, e^{\mathrm{i}2\pi \frac{jm}{M}} \right) e^{-\mathrm{i}2\pi \frac{j(M/2)}{M}}. \end{aligned} \tag{7}$$

The expression in the parenthesis is a standard discrete inverse Fourier-transform, which can effectively be evaluated by the Fast Fourier Transform (FFT) algorithm. The multiplicative term

$$e^{-\mathrm{i}2\pi \frac{j(M/2)}{M}}$$

applied on the Fourier-transform corresponds to a translation of $M/2$ points on the discrete form of the function to be transformed, and π translation on the continuous scale. Hence the algorithm to obtain approximately the coefficients is given by

$$\hat{c}_{j+k} = \frac{1}{M} \sum_{m=0}^{M-1} f_k(s_{m+M/2})\, e^{\mathrm{i}2\pi \frac{jm}{M}} \tag{8}$$

that can simply be proved by applying the translation invariance of the discrete Fourier transform by M indices (corresponding the 2π).

Since f_k functions are periodic by 2π, the translation of π is equivalent to swapping the positive and negative halves of the function. Swapping is a frequently used action in the FFT analysis, usually is used a posteriori to shape the spectrum in the customary layout with zero frequency on the centrum; here swapping can be used before the transform is performed.

The algorithm represented by (8) needs the values of the function f_k by using a displacement of π on the variable s ($M/2$ on the indices), that, because of its periodicity — is equivalent with swapping the left and right half of the function in the interval $[-\pi, \pi]$ before executing the transform (swapping is a frequently used action in the FFT techniques).

The transform (8) results in N number of redundant sequences $\hat{c}_j^{(k)}$ ($k = 0, 1, \ldots, N-1$) — each belonging to a particular ϕ_k function — N-times denser than the sequence of the coefficients. Hence the coefficients can be obtained by decimating the sequences by N:

$$\begin{array}{llll}
\hat{c}_0 = \hat{c}_0^{(0)} & \hat{c}_1 = \hat{c}_0^{(1)} & \ldots & \hat{c}_{N-1} = \hat{c}_0^{(N-1)} \\
\hat{c}_N = \hat{c}_N^{(0)} & \hat{c}_{N+1} = \hat{c}_N^{(1)} & \ldots & \hat{c}_{2N-1} = \hat{c}_N^{(N-1)} \\
\hat{c}_{2N} = \hat{c}_{2N}^{(0)} & \hat{c}_{2N+1} = \hat{c}_{2N}^{(1)} & \ldots & \hat{c}_{3N-1} = \hat{c}_{2N}^{(N-1)} \\
\vdots & \vdots & & \vdots \\
\hat{c}_{\ell N} = \hat{c}_{\ell N}^{(0)} & \hat{c}_{\ell N+1} = \hat{c}_{\ell N}^{(1)} & \ldots & \hat{c}_{(\ell+1)N-1} = \hat{c}_{\ell N}^{(N-1)} \\
\vdots & \vdots & & \vdots
\end{array}$$

The values of the functions f_k ($k = 0, 1, \ldots, N-1$) applied in the transform (8) can be computed on the basis of measurement values belonging to the function F to be represented, as well as sample values of the elements of the rational orthogonal basis. The sample points are determined by the inverse of the argument function $\beta_{\mathbf{a}}$ belonging to the basis: for $k = 0, 1, \ldots, N-1$,

$$f_k(s_j) = F(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s_j)})\, \overline{\phi_k(e^{\mathrm{i}\beta_{\mathbf{a}}^{-1}(s_j)})}\, \beta'_{\mathbf{a}}(s_j)$$

i.e. the sample points can be determined by the transform

$$t_j = \beta_{\mathbf{a}}^{-1}(s_j) \quad (j = 0, 1, \ldots, M-1), \tag{9}$$

where $\{s_j\}$ is a uniformly spaced division of the interval $[-\pi, \pi]$. Since The inverse argument function is nonlinear, the resulting $\{t_j\}$ scale is non-uniformly spaced division of the same interval. The transform (9) is referred as *inverse argument-transform* belonging to the generalized orthogonal basis generated upon the poles $\mathbf{a}$.

The sample values of the basis elements $\phi_k(e^{\mathrm{i}t_j})$ can be obtained by computations, hence they can be evaluated in any point within the interval $[-\pi, \pi]$. However, the sample points of the function F can be obtained by observations — physically realizable measurements or computations — performed on the system to be analyzed. Only the interval $[0, \pi]$ has physical meaning: the parameter t can be considered as *normalized circular frequency*, and the interval $[0, \pi]$ is related to the frequency interval $[0, f_N]$, where f_N is the Nyquist-frequency connected to the applied time-domain sampling rule. The mapping between the sample points t_j and f_j is given by

$$t_j = \frac{f_j}{f_N}\,\pi \quad \text{i.e.} \quad f_j = \frac{f_N}{\pi}\,t_j.$$

The points of F corresponding to the negative half of the interval can be generated by the rule

$$F(e^{-\mathrm{i}t}) = \overline{F(e^{\mathrm{i}t})} \tag{10}$$

that is valid for all the existing or physically realizable real systems, that possess only real poles or conjugated complex pole-pairs.

Hence the sample points of function F that are suitable to use in the rational orthogonal representation can be obtained by applying the following procedure:
1. Determining a non-uniformly spaced $\{t_j\}_{j=0}^{M-1}$ scale in the interval $[-\pi, \pi]$ by using the inverse argument transform belonging to the GOB starting from a uniformly spaced scale $\{s_j\}_{j=0}^{M-1}$ in the same interval.
2. Measuring or computing the values of the frequency function in frequencies corresponding to the parameter values $t_j \geq 0$ ($j = 0, 1, \ldots, M/2-1$).
3. Completing the function by adding its negative half by applying

$$F(e^{-\mathrm{i}t_{-j}}) = \overline{F(e^{\mathrm{i}it_j})} \quad (j = 1, 2, \ldots, M/2-1)$$

according to the rule (10).
The values of the frequency function can be generated directly as transfer functions measurements, or by using spectral estimation methods on the basis of time-domain measurements.

III. Principles of change detection

The representations in rational orthogonal bases can efficiently be used to solve system change detection tasks. Observable changes of the system structure and/or the behavior can be detected by analyzing the changing characteristics of adequate output signals: by comparing the momentary ones to an priori assumed reference; the signal representations in specific rational orthogonal bases offer efficient means to do this. The basic idea is building a rational orthogonal basis upon a fixed set of the poles assumed to be valid for the nominal system; than representing the signals belonging to the momentary state in this basis. The coefficients of the representation can be used in the detection of changes based upon the following considerations:

- The representation of signals containing exactly the poles assumed in building the rational basis is finite, i.e. finite number of coefficients differ from zero, namely the first n ones if the number of the poles is n. The poles with multiplicity greater than one are repeated by the number of multiplicity.
- Any displacement of the poles or occurrence of new poles results in infinite representation: infinite number of coefficients differ from zero.

• Disappearance or change on the multiplicity of any existing results in the change of the finite representation: the corresponding coefficients change value.

The finite representation of the exact case follows directly from the orthogonality of the rational basis (2). The detection problem can be considered as a decision whether selected coefficients differ from zero or not. This of course can be done on statistical basis because of measurement and estimation errors of statistical nature.

It can be noticed that decision upon zero value of the coefficients is not the only alternative of the change detection. An approach not directly using the representation coefficients has been described in [5]; a discrete scalar product has been used to discriminate the changes, which is based upon the reproducing kernel belonging to the subspace generated by a finite orthogonal rational representation. Furthermore more delicate decision tasks can be solved on the basis of the behavior of the coefficients in the case of pole changes. A detailed analysis on the nature of this behavior can be found in [8]: changes result in infinite sequence with exponential decay of speed linearly dependent on the magnitude of change. However, the elaboration of exact detection methods on this basis requires further research.

The latter feature of these representations will be utilized in change detection of systems based upon representations in rational orthogonal bases. The model used is as follows: a nominal system is defined with a priori fixed set of poles; a rational orthogonal basis is constructed upon these poles; the representation of the real system in this basis is generated (more exactly *estimated*) based upon measurements; and finally the changes are detected by analyzing the representation coefficients computed. According to the orthogonality of the rational basis generated upon the finite set of poles $\{a_k\}_{k=0}^{N-1}$ it can simply be proved, that the representation of the rational functions possessing no other poles than these ones (with any finite multiplicity) contain a finite number of nonzero coefficients. Namely, the number of nonzero coefficients with index $n = N\ell + k$ for specific $k = 0, 1, \ldots, N-1$ is exactly m_k, and – due to the orthogonality – the coefficients $n = N\ell + j$ $(j \neq k)$ will be zero.

The current paper focuses on the zero-detection problem of the representation coefficients. Figures 1, 2, and 3 present an example generated by assuming a nominal system with poles

$$[0.9, 0.99e^{0.5\mathrm{i}}, 0.99e^{-0.5\mathrm{i}}, 0.98e^{0.8\mathrm{i}}, 0.98e^{-0.8\mathrm{i}}]$$

Figure 1 presents the real and imaginary parts of the coefficients for the nominal case. Figure 2 shows the cause of a displacement on a pole, namely on the real pole ($a_0 = 0.98$ instead of 0.9); an exponential decay has been indicated on the coefficients belonging to it. Figure 2 presents the case when a random noise is added to the original function values. The result is the appearance of a random noise on all the coefficients both on zero and nonzero ones. A relatively small scale noise on the coefficients can also be seen on Figures 1 and 3 as the cause of errors resulted from the estimation algorithm and the computations. It is clear

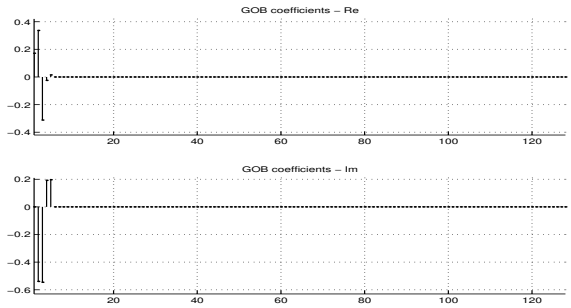

Fig. 1. Representation coefficients: nominal

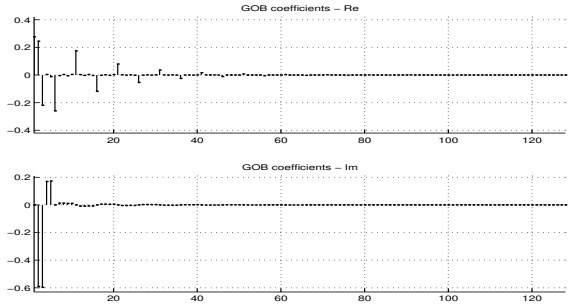

Fig. 2. Representation coefficients: displacement

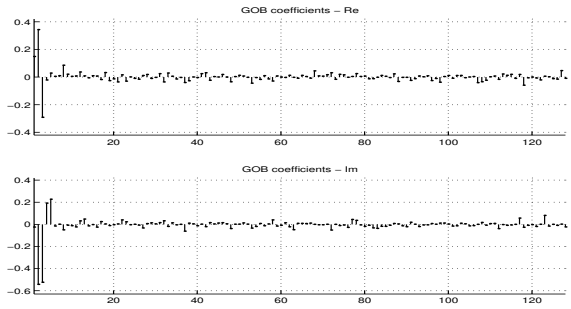

Fig. 3. Representation coefficients: noise

that reliable decision upon the coefficient values cannot be performed without some knowledge on the errors, and require statistical considerations in connection with random error components; the error analysis will be discussed in the next section.

The advantages of using orthogonal rational representations to detect changes arise from the fact that valuable a priori knowledge is built into the representation that is not done in classical spectral detection methods that are based upon the standard trigonometric basis. The linear model-based methods (AR, ARMA methods, detection filters) use also a priori knowledge, however they are restricted to one specific model, in contrast with the orthogonal rational representations, where a class of models is assumed, and a series expansion represents the valid model.

IV. Error analysis

The errors of estimating the coefficients of orthogonal rational representations are are originated from several sources and can be classified as deterministic and stochas-

tic ones, i.e. bias and variance type ones. Typical error sources are enumerated as follow:

- Error in selecting the basis of the nominal system.
- Error in approximate evaluation of the Fourier integral.
- Error in numeric inversion of the argument function.
- Rounding errors in computations.
- Measurement and computation errors in obtaining the original spectral function data.

The first three error components can be considered as bias errors on the coefficients. Rounding errors, nevertheless they are deterministic in nature, can successfully be modelled as random noise. Errors on the input data can have both bias and can contain random error components.

The basis selection error means that a difference can exist between the assumed nominal system and the real on in nominal state. This difference can be characterized as modelling error, i.e. it can be originated from unmodelled dynamics, e.g. omitted poles, or nonlinear behavior of the real system, furthermore from errors in estimation of the selected poles. The difference results in bias on the representation coefficients even in the nominal case. Since the real difference is not exactly known, and this type of error is mixed with the other error components, there is no way for deterministic correction, the best way is to minimize its influence by careful model selection.

The major part of the estimation error of coefficients is resulted from the approximate evaluation of the Fourier integrals. The discretized form of the Fourier integral can be considered as an approximation sum that converges to the integral with a successively refined division of the interval. Hence the approximation error can be minimized over any limit by with using denser sample scale in the input spectral data.

The algorithm of computing the representation coefficients contains the inversion of the argument function associated with the reference poles selected. This step can be performed numerically, since the argument function is a strictly increasing differentiable function, a simple successive approximation is sufficient to realize. Selecting a sufficiently small error bound to terminate the iteration is a significant constituent of the accuracy of the estimation. Extremely small error bound significantly increases the computing time, bigger one result in bias error on the coefficients.

The numeric inversion can be realized with sufficient accuracy during reasonable time by the digital computers commonly used today. More serious problem is whether the inverse function values that form a non-uniformly spaced scale on the frequency domain can be accurately applied in the spectral measurements or computation of the spectral functions. In the case of direct transfer function measurements the frequency resolution of the equipment used is usually finite, hence the desired frequency values can be adjusted with some error. In the case of using time domain measurement data, the frequency functions can be computed by some type of Fourier-transform method, e.g. direct Fourier transform or indirect methods by using the correlation functions. The FFT algorithms used commonly today assume uniform scale both on time and frequency domain, e.g. they cannot directly be used in our algorithm. Reasonable approach seems to be an FFT algorithm applied on uniform samples of number far greater than it is required for the non-uniform scale, and than selecting the function points nearest to the desired frequencies. However it has been verified in several cases, that this method cannot be produce the desired accuracy, especially when poles near to unit circle are present in the system. In most cases the nonuniform discrete Fourier transform proved to be adequate method, unfortunately these type of algorithms take considerably more computing time. Finding "fast" methods is subject of the further research.

The exact evaluation of rounding errors is an extremely difficult problem, and fall beyond to the scope of the current research. Rounding errors can cause serious problems in the cases when the system poles are placed extremely near to the unit circle. An error causing a pole to displaced virtually beyond to the unit circle can result in unpredictable errors in the estimation. In less drastic cases the rounding errors can successfully be modelled as a small size random uniform white noise on the coefficients computed.

The input data errors can contain both bias and variance type component depending on the measurement or computing algorithms applied. Direct transfer function measurements that can be performed in the frequency domain can usually be performed unbiased; the random errors can usually be modelled as normally distributed noise. The variance of the noise is usually unknown, however in practical cases it can be estimated on the basis of the specification of the measurement device, as well as the noise characteristics of the system tested.

The spectral methods that are based upon time-domain measurements result in biased estimates because of the window-effect occurring as a consequence of the finite record used. The bias can be decreased by applying special window functions, however it cannot be eliminate. The random error components are also dependent on the spectral estimation algorithms used. The time domain measurements are usually considered in statistical sense as independent normally distributed random variables. The simple discrete Fourier-transform, as being a linear operation, transforms them into random variables of joint normal distribution; and the mean and variance can be computed by linear and quadratic forms respectively. The correlation based indirect spectral estimation as well as the direct Fourier-transform based (Cooley-Tukey) method result in χ^2 distribution of degree of freedom equal to the number of data points, however due to the large number of data it can successfully be approximated with normal distribution. These results of the classical spectral estimation can be found in signal processing textbooks, e.g. see e.g. [9].

Based upon the above discussion the error analysis of the algorithm of computing the representation coefficients means a deterministic analysis to take in account the bias errors, as well as a statistical analysis to explore the random error components.

The deterministic analysis has been performed on the basis of displacement on the poles relatively to the reference ones. The conclusion is that error terms decaying exponentially with rate dependent on the difference occur on the coefficients. The detailed analysis can be found in [8].

The statistical analysis is based upon the characteristics of the estimation algorithm presented in Section II. A more general analysis can be performed on the basis of the definition form (5), or can be derived from general estimation theory results [10]; these approaches fall beyond the scope of the current paper. It can be verified, that from the point of view of the input data values $F_m = F(e^{\mathrm{i}t_m})$ the formula (8) — since the function f_k defined by formula (6) can be considered as multiplication of the data values by constant factors — is linear. That is the estimation formula (8) can be expressed as

$$\hat{c}_{j+k} = \frac{1}{M} \sum_{m=0}^{M-1} \alpha_{j,m}^{(k)} F_m \tag{11}$$

where

$$\alpha_{j,m}^{(k)} \doteq \overline{\phi_k(e^{\mathrm{i}t_{m+M/2}})}\, \beta'_{\mathbf{a}}(s_{m+M/2})\, e^{\mathrm{i}2\pi \frac{jm}{M}},$$

s_m, t_m $(m = 0, 1, \ldots, M-1)$ are the sample values of the uniform and non-uniform scale respectively, by applying the notations of Section II.

By assuming that the spectral sample points F_m $(m = 0, 1, \ldots, M-1)$ are random variables with joint normal distribution, according to the well known argument of the probability theory the coefficients computed by (11) are also random variables of joint normal distribution. The sets of the coefficients obtained for the several indices of k are decimated by N that results in the final set of the representation coefficients containing M elements. The decimated set, consisting of normally distributed random variables, is also jointly normally distributed. The mean values and the covariance matrix belonging to the coefficients are sufficient parameters to describe their statistical characteristics.

The mean values of the coefficients can be obtained by the same decimation procedure that performed on the sets of the coefficients themselves, i.e.

$$\mu_{N\ell+k}^{(c)} = \frac{1}{M} \sum_{m=0}^{M-1} \alpha_{N\ell,m}^{(k)} \mu_m^{(F)},$$

where where $\mu_i^{(c)}$ and $\mu_i^{(F)}$ denote the mean values of the coefficients c_i and the function values F_i respectively.

To derive the covariance matrix let the decimated coefficients centered to the means be considered

$$\tilde{c}_{\ell N}^{(k)} = \frac{1}{M} \sum_{m=0}^{M-1} \alpha_{N\ell,m}^{(k)} \tilde{F_m},$$

where $\tilde{F_m} = F_m - \mu_m^{(F)}$. The covariance matrix element of indices $(N\ell + p, N\kappa + q)$ belonging to the coefficients can be expressed as follows:

$$\sigma_{N\ell+p,N\kappa+q}^{c} = E[\tilde{c}_{N\ell}^{(p)} \overline{\tilde{c}_{N\kappa}^{(q)}}] = \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \alpha_{N\ell,m}^{(p)} \overline{\alpha_{N\kappa,n}^{(q)}} \sigma_{m,n}^{F}$$

where $\sigma_{m,n}^{F} = E[\tilde{F}_m \overline{\tilde{F}_n}]$, i.e. the elements of the covariance function of the measurement data.

By defining the matrix containing the parameters α that have been remained after the decimation in the following form:

$$[A]_{j,n} = \alpha_{N\ell,m}^{(k)} \quad j = N\ell + k$$

the covariance matrix of the coefficients can be expressed by the transformation

$$\Sigma^{(c)} = \frac{1}{M^2}\, A\, \Sigma^{(F)} A^*$$

where $\Sigma^{(F)}$ denotes the covariance matrix of the measurement data. According to the well-known argument of the probability theory in association with the linear transform of a normally distributed set of random variables, the matrix A should be nonsingular.

The statistics of estimating the representation coefficients, that have been derived here, can be used in setting up thresholds for detection to ensure a desired level of confidence, or can be used in more elaborated statistical (e.g. Bayes or likelihood type) decision schemes.

V. Conclusions

The algorithm described in Section II offers effective tool to estimating representation coefficients in rational orthogonal bases as well as serves as the basis of the error analysis including that of the random errors. The error analysis given in Section IV completes the requisites indispensable to apply the orthogonal rational representations in change detection tasks.

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