Recent results on reciprocals of well-posed linear systems.

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I. Introduction

This paper proposes a novel approach to studying well-posed linear systems via their reciprocal systems. A well-posed linear system has generating operators A, B, C that are typically unbounded operators. Under the generic assumption that $0 \in \rho(A)$, this well-posed linear system possesses a reciprocal system with the corresponding generating operators $A^{-1}, A^{-1}B, -CA^{-1}$ that are all *bounded*. In Curtain [3], [4] and Curtain and Sasane [5] this connection has been exploited to solve certain control problems for well-posed linear systems by showing that they are equivalent to corresponding control problems for the reciprocal systems. Due to the bounded nature of the generators, the problems for the reciprocal system are easier to solve and these solutions can be translated back to solutions for the original well-posed linear system. This approach works perfectly for stable well-posed linear systems [3], [5], but for unstable systems one needs to impose some extra assumptions. In Opmeer and Curtain [12] it is shown that, under a certain condition (see (II.9)), optimal control problems (with a coercive cost functional) for (unstable) well-posed linear systems have a solution if and only if the corresponding reciprocal optimal control problem has a solution. In this case, the cost operator is the solution of a Riccati equation involving only *bounded* operators. Since it is known that for well-posed linear systems, the expected Riccati equation is not always well-defined, this result is significant. The aim of this paper is to obtain new sufficient conditions for (II.9) to hold and so ensure that the results of [12] apply.

II. Reciprocal systems of well-posed linear systems

First we review the concept of a reciprocal system that was introduced in Curtain [3], [4] for a well-posed linear system Σ with the generating operators A, B, C and transfer function $\mathbf{G}$ under the generic assumption that $0 \in \rho(A)$. A generates a strongly continuous semigroup $T(t)$ on a Hilbert space Z, U, Y are Hilbert spaces, $C \in \mathcal{L}(D(A), Y)$, $A^{-1}B \in \mathcal{L}(U, Z)$, and B and C are admissible control and observation operators with respect to $T(\cdot)$, i.e., given $\tau > 0$ there exists a $\gamma > 0$ such that

$$\int_0^\tau \|CT(t)z\|^2 \, \mathrm{d}t \leq \gamma \|z\|^2 \quad \text{for all } z \in D(A),$$

and for any $\tau > 0$ there exists a $\beta > 0$ such that for all $u \in \mathbf{L}_2(0, \tau; U)$

$$\| \int_0^\tau T(t)Bu(t) \, \mathrm{d}t \|^2 \leq \beta \int_0^\tau \|u(t)\|^2 \, \mathrm{d}t.$$

If B, C are admissible operators with respect to $T(\cdot)$, then we define the observability map $\mathcal{C}^\tau \in \mathcal{L}(Z, \mathbf{L}_2(0, \tau; Y))$ and the controllability map $\mathcal{B}^\tau \in \mathcal{L}(\mathbf{L}_2(0, \tau; U), Z)$ by

$$\mathcal{C}^\tau z(t) = CT(t)z \quad \text{for} \quad 0 \leq t \leq \tau, z \in D(A) \quad \text{(II.1)}$$

$$\mathcal{B}^\tau u = \int_0^\tau T(\tau - s)Bu(s) \, \mathrm{d}s \; u \in \mathbf{L}_2(0, \tau; U). \quad \text{(II.2)}$$

The transfer function is determined up to an arbitrary constant for s, β in some right-half plane by the following expression

$$\mathbf{G}(s) - \mathbf{G}(\beta) = (\beta - s)C(sI - A)^{-1}(\beta I - A)^{-1}B. \quad \text{(II.3)}$$

If the right-hand side is uniformly bounded in norm on some right half-plane, then $A, B, C, \mathbf{G}(\beta)$ define a *well-posed linear system* with transfer function $\mathbf{G}$. The transfer function is independent of the choice of β and (II.3) can be extended analytically to $\rho_\infty(A)$, the largest component of the resolvent set that contains an interval $[r, \infty)$. It also has an extension to all $s, \beta \in \rho(A)$ (see Staffans and Weiss [16]), but this extension need not equal the analytic extension of the transfer function outside $\rho_\infty(A)$. A simple example illustrating this is given in Curtain and Zwart [2, Example 4.3.8]. To avoid confusion we reserve the name *characteristic function* and the symbol $\mathfrak{G}$ for this extension.

A large subset of well-posed linear systems has a more familiar expression for the transfer function. First we need to define the Lambda-extension of C by $C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda I - A)^{-1}x$ for $x \in D(C_\Lambda)$, the subset of Z for which the limit exists. Σ is a *regular linear system* if for each $u \in U$, $\mathbf{G}(s)u$ has the limit Du as s approaches infinity along the positive real axis for some $D \in \mathcal{L}(U, Y)$. A, B, C, D are called the generating operators of Σ and the transfer function has the more familiar form $\mathbf{G}(s) = D + C_\Lambda(sI - A)^{-1}B$ for s in some right half-plane. Its characteristic function $\mathfrak{G}(s) = D + C_\Lambda(sI - A)^{-1}B$ for $s \in \rho(A)$.

We introduce the following stability notions.

Definition 2.1: The well-posed system Σ with generating operators A, B, C and transfer function $\mathbf{G}$ is *stable* if

- it is *input stable*: there exists a constant $\beta > 0$ such that for all $u \in \mathbf{L}_2(0, \infty; U)$
$$\| \int_0^\infty T(t)Bu(t) \, \mathrm{d}t \|^2 \leq \beta \int_0^\infty \|u(t)\|^2 \, \mathrm{d}t;$$
- it is *output stable*: there exists a constant $\gamma > 0$ such that for all $z \in D(A)$
$$\int_0^\infty \|CT(t)z\|^2 \, \mathrm{d}t \leq \gamma \|z\|^2;$$
- it is *input-output stable*: the transfer function $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(U, Y))$.

Remark 2.2: From the Paley-Wiener Theorem A.6.21 in [2] output stability implies for every $z \in Z$ the existence of the Laplace transform $\hat{\mathcal{C}}(\cdot)z \in \mathbf{H}_2(Y)$ of $CT(\cdot)z \in \mathbf{L}_2(0, \infty; Y)$. In other words, $C(sI - A)^{-1}z$ has an analytic extension to $\hat{\mathcal{C}}(s)z \in \mathbf{H}_2(Y)$ for all $z \in Z$. Moreover, this extension equals $C(sI - A)^{-1}z$ for $s \in \rho(A) \cap \mathbb{C}_0^+$, since for $z \in D(A)$ $\hat{\mathcal{C}}(s)(sI - A)z = Cz$ can be extended to $\mathbb{C}_0^+$, the domain of analyticity of $\hat{\mathcal{C}}$. This then can be extended to hold for all $z \in Z$. Similarly, output stability implies, for every $z \in Z, u \in U$, the existence of $\langle z, \hat{\mathcal{B}}(\cdot)u\rangle \in \mathbf{H}_2$ that satisfies $\langle z, \hat{\mathcal{B}}(s)u\rangle = \langle z, (sI-A)^{-1}Bu\rangle$ for all $s \in \mathbb{C}_0^+ \cap \rho(A)$.

Remark 2.3: For input stability the terminology B *is an infinite-time admissible control operator for* $T(\cdot)$ is often used and for output stability the terminology C *is an infinite-time admissible observation operator for* $T(\cdot)$ is in use. (We shall also use the terminology (A, B) is input stable and (A, C) is output stable.)

If Σ is output stable, then the map in (II.1) makes sense for $\tau = \infty$ and $\mathcal{C}^\infty \in \mathcal{L}(Z, \mathbf{L}_2(0,\infty;Y)$. Similarly, if Σ is input stable, $\mathcal{B}^\infty \in \mathcal{L}(\mathbf{L}_2(0,\infty;U), Z)$, where

$$\mathcal{B}^\infty u = \lim_{\tau\to\infty}\int_0^\tau T(s)Bu(s)\,\mathrm{d}s \quad \text{for} \quad u \in \mathbf{L}_2(0,\infty;U).$$

If B is an infinite-time admissible control operator, then its *controllability gramian* $L_B \in \mathcal{L}(Z)$ is defined by $L_B = \mathcal{B}^\infty(\mathcal{B}^\infty)^*$. If C is an infinite-time admissible observation operator, then its *observability gramian* $L_C \in \mathcal{L}(Z)$ is defined by $L_C = (\mathcal{C}^\infty)^*\mathcal{C}^\infty$.

The essential difference in Definition 2.1 to previous definitions is that we have made no stability assumptions on A and so it can have spectrum in $\mathbb{C}_0^+$. However, in Curtain [4, Lemma 2.3] it was shown that for a stable system (II.3) holds on $\mathbb{C}_0^+ \cap \rho(A)$; more precisely, the following is true.

Lemma 2.4: If the well-posed linear system Σ is either input stable or output stable, then the transfer function has an extension to an analytic function on $\mathbb{C}_0^+$, (II.3) holds for $s, \beta \in \mathbb{C}_0^+ \cap \rho(A)$ and the characteristic function equals the transfer function in this region. Moreover, if Σ is regular, then $\mathbf{G}(s) = D + C_\Lambda(sI-A)^{-1}B = \mathfrak{G}(s)$ for $s \in \mathbb{C}_0^+ \cap \rho(A)$.

In Grabowski [7] it is shown that C is an infinite-time admissible observation operator for $T(\cdot)$ if and only if the observation Lyapunov equation has a self-adjoint non-negative solution $L \in \mathcal{L}(Z)$

$$A^*Lz + LAz = -C^*Cz \quad \text{for all } z \in D(A). \tag{II.4}$$

Moreover, the observability gramian L_C is the smallest self-adjoint non-negative solution. The key step in the concept of a reciprocal system is to notice that if $0 \in \rho(A)$, then (II.4) has a solution if and only if the following equation does

$$A^{-*}L + LA^{-1} = -A^{-*}C^*CA^{-1}. \tag{II.5}$$

This is the observability Lyapunov equation for the pair (A^{-1}, CA^{-1}). Similarly, the control Lyapunov equation for the infinite-time admissible B operator has a solution if and only if the control Lyapunov equation for the pair $(A^{-1}, A^{-1}B)$ has (Hansen and Weiss [8]). Notice that the operators $A^{-1}, A^{-1}B, CA^{-1}$ are all bounded. If we substitute $\beta = 0$ in (II.3), we obtain

$$\begin{aligned}\mathfrak{G}(s) &= \mathfrak{G}(0) + sC(sI-A)^{-1}A^{-1}B && \text{(II.6)}\\ &= \mathfrak{G}(0) - CA^{-1}(\frac{1}{s}I - A^{-1})^{-1}A^{-1}B. && \text{(II.7)}\end{aligned}$$

This motivates the following definition.

Definition 2.5: Suppose that the well-posed linear system with generating operators A, B, C, D, transfer function $\mathbf{G}$ and characteristic function $\mathfrak{G}$ is such that $0 \in \rho(A)$. Its reciprocal system is the regular linear system Σ_- with the bounded generating operators $A^{-1}, A^{-1}B, -CA^{-1}, \mathfrak{G}(0)$.

Remark 2.6: (II.6) shows that the characteristic functions of the well-posed linear system and its reciprocal system are related by

$$\mathfrak{G}(s) = \mathfrak{G}_-(\frac{1}{s}) \quad \text{for} \quad s \in \rho(A). \tag{II.8}$$

As remarked earlier, such a relationship may not hold for the transfer functions.

In Opmeer and Curtain [12] it is shown that a necessary and sufficient condition for relating a control problem for a well-posed system with a control problem for its reciprocal system is that the following should hold at least for s in some right half-plane.

$$\mathbf{G}(s) = \mathbf{G}_-(\frac{1}{s}); \quad \hat{\mathcal{C}}(s) = \frac{1}{s}\hat{\mathcal{C}}_-(\frac{1}{s}). \tag{II.9}$$

In Curtain [4, Lemma 2.3] it was shown that this always holds for stable systems.

Theorem 2.7: Suppose that A, B, C are generating operators of a well-posed linear system with transfer function $\mathbf{G}$ and zero is in the resolvent set of A. Then

1) C is an infinite-time admissible observation operator for $T(t)$ if and only if CA^{-1} is one such for $T_-(t) = \exp A^{-1}t$. If they are infinite-time admissible, then their observability gramians are identical, and

$$\hat{\mathcal{C}}(s)z = \frac{1}{s}\hat{\mathcal{C}}_-(\frac{1}{s})z \quad \text{for} \quad s \in \mathbb{C}_0^+, z \in Z. \tag{II.10}$$

2) B is an infinite-time admissible control operator for $T(t)$ if and only of $A^{-1}B$ is one such for $T_-(t)$. If they are infinite-time admissible, then their controllability gramians are identical and

$$\langle z, \hat{\mathcal{B}}(s)u\rangle = \frac{1}{s}\langle z, \hat{\mathcal{B}}_-(\frac{1}{s})u\rangle \quad \text{for} \quad s \in \mathbb{C}_0^+, u \in U, z \in Z.$$

3) If Σ is input or output stable, then $\mathbf{G}(s) = \mathbf{G}_-(\frac{1}{s})$ for $s \in \mathbb{C}_0^+$, and so Σ is input-output stable if and only if Σ_- is.
4) Σ is a stable system if and only if Σ_- is a stable system.

III. Admissible feedbacks

The main aim of this section is to obtain sufficient conditions under which (II.9) holds. To achieve this we examine the effect of admissible feedbacks on the reciprocal relationship and, in particular, output stabilizing feedbacks.

Definition 3.1: The well-posed linear system Σ with generating operators A, B, C and transfer function $\mathbf{G}$ is *output stabilizable* if there exists $F \in \mathcal{L}(D(A), U)$ such that

- $A, B, [C; F]$ are generators of a well-posed linear system with transfer function $\mathbf{G}^F$ satisfying $\mathbf{G} = [I; 0]\mathbf{G}^F$;

- $[0, I]$ is an admissible feedback operator for $\mathbf{G}^F$, i.e., $(I - [0, I]\mathbf{G}^F)$ has a well-posed inverse, and the closed-loop system Σ^{cl} is well-posed with semigroup generator A_{cl} and transfer function given by

$$\mathbf{G}^{cl} = \mathbf{G}^F(I - [0, I]\mathbf{G}^F)^{-1}. \tag{III.11}$$

- Σ^{cl} is output stable.

Note that we have used the notation $[M; N]$ for a column block and $[M, N]$ for a row block.

Remark 3.2: Note that a regular linear system is output stabilizable if F exponentially stabilizes (A, B) in the sense of Rebarber [13]. Our definition of stabilizability is different from those in Staffans [15] and in Mikkola [10]. Ours does not assume uniform boundedness of the semigroup.

Remark 3.3: In the case that Σ^F has bounded generating operators and feedthrough operator $[D; D_F]$, $[0; I]$ is an admissible feedback operator for Σ^F if and only if $I - D^F$ has a bounded inverse. The closed-loop system Σ^{cl} has the generating operators

$$\begin{aligned}&A_{cl} = A + B(I - D_F)^{-1}C,\ B^{cl} = B(I - D^F)^{-1},\\ &[C^{cl}; F^{cl}] = [C + D(I - D^F)^{-1}F; (I - D^F)^{-1}F],\\ &D^{cl} = [D; D^F](I - D^F)^{-1}.\end{aligned}$$

Direct computations verify that the inverse of $I - [0; I]\mathbf{G}^F(s) = I - [0; I]\mathfrak{G}^F(s)$ is $(I - D^F)^{-1} - (I - D^F)^{-1}F(sI - A_{cl})^{-1}B(I - D^F)^{-1}$, which is always uniformly bounded on some right half-plane, since A_{cl} is bounded. Moreover, the inverse of $I-[0; I]\mathfrak{G}^F(s)$ exists for all $s \in \rho(A) \cap \rho(A_{cl})$. So, in addition to (III.11) direct algebraic computations show that

$$\mathfrak{G}^{cl} = \mathfrak{G}^F(I - [0, I]\mathfrak{G}^F)^{-1} \quad \text{for } s \in \rho(A) \cap \rho(A_{cl}). \tag{III.12}$$

For the closed-loop system to have a well-defined reciprocal system we need $0 \in \rho(A_{cl})$. The following result from Salamon [14, Lemma 4.4] shows that a sufficient condition for this to hold is that $1 \in \rho([0, I]\mathfrak{G}^F(0)$.

Lemma 3.4: Let Σ be a well-posed linear system with generating operators A, B, C and transfer function $\mathbf{G}$. Suppose that there exists an $F \in \mathcal{L}(D(A), U)$ such that $A, B, [C; F]$ are generators of a well-posed linear system with transfer function $\mathbf{G}^F$ with $\mathbf{G} = [I; 0]\mathbf{G^F}$ and $[0, I]$ is an admissible feedback operator for $\mathbf{G}^F$. If A_{cl} is the semigroup generator of the closed-loop system, then
$\lambda \in \rho(A)$ and $1 \in \rho([0, I]\mathfrak{G}^F(\lambda)) \Longrightarrow \lambda \in \rho(A_{cl})$.

Remark 3.5: A corollary of the above lemma to systems with bounded generating operators as in Remark 3.3 is that if $\lambda \in \rho(A)$, then
$1 \in \rho([0, I]\mathfrak{G}^F(\lambda))$ if and only if $\lambda \in \rho(A_{cl})$.

The following lemma shows that if $0 \in \rho(A)$, then the condition $1 \in \rho([0, I]\mathfrak{G}^F(0)$ is necessary for $[0; I]$ to be an admissible feedback operator for Σ^F.

Lemma 3.6: Let Σ^F be a well-posed linear system with generating operators $A, B, [C; F]$, transfer function $\mathbf{G}^F$ and characteristic function $\mathfrak{G}^F$ and suppose that $0 \in \rho(A)$. Then

1) $[0; I]$ is an admissible feedback for the reciprocal system Σ_-^F if and only if $\mathcal{D} = I - [0; I]\mathfrak{G}^F(0)$ has a bounded inverse.
2) If $\mathcal{D}$ is boundedly invertible, then $I - [0; I]\mathbf{G}^F(s)$ is invertible in $\mathcal{L}(U)$ for s in some right half-plane
3) If $[0; I]$ is an admissible feedback operator for Σ^F, then $0 \in \rho(A_{cl})$.

Proof (1) This follows from Remark 3.3 since Σ_-^F has bounded generating operators. Denote the closed-loop system by Σ^{new} and its generating operators accordingly.
(2) Since $0 \in \rho(A)$ we can substitute $s = 0$ in (II.3) to obtain

$$\begin{aligned}\mathfrak{G}^F(s) &= \mathfrak{G}^F(0) + s[C; F](sI - A)^{-1}A^{-1}B\\ &= \mathfrak{G}^F(0) - [C; F]A^{-1}(\frac{1}{s}I - A^{-1})^{-1}A^{-1}B\\ &= \mathfrak{G}_-^F(\frac{1}{s}) \quad \text{for all} \quad s \in \rho(A).\end{aligned}$$

This equals the transfer function on some right half-plane and so we have for s in some right half-plane

$$[0; I]\mathbf{G}^F(s) = [0; I]\mathfrak{G}^F(s) = [0, I]\mathfrak{G}_-^F(\frac{1}{s}). \tag{III.13}$$

From part (1) we know that $I - [0, I]\mathfrak{G}_-^F(\lambda)$ has a bounded inverse if and only if $\mathcal{D}$ does and this inverse exists for $s \in \rho(A^{-1}) \cap \rho(A^{new})$. Now by Remark 3.5 we know that $\lambda \in \rho(A^{-1})$ and $1 \in \rho([0; I]\mathfrak{G}_-^F(\lambda) \Longrightarrow \lambda \in \rho(A^{new})$. So $\rho(A^{-1}) \subset \rho(A^{new})$. Since A is an infinitesimal generator $1/\rho(A^{-1})$ contains a right half-plane and so the same holds for $1/\rho(A^{new})$. So from (III.13) we see that $I - [0; I]\mathbf{G}^F(s)$ has a bounded inverse for s in some right half-plane.
(3) If $[0; I]$ is an admissible feedback operator for Σ^F, then by part 2 $\mathcal{D}^{-1}$ has a bounded inverse. So by Lemma 3.4 with $\lambda = 0$ we obtain $0 \in \rho(A_{cl})$.

Remark 3.7: In [3] a system was called *r-output stabilizable* if it is output stabilizable and $0 \in \rho(A_{cl})$. Lemma 3.6 shows that this extra condition is superfluous.

We show that the reciprocal relationship is preserved under admissible feedbacks.

Lemma 3.8: Let Σ^F be a well-posed linear system with generating operators $A, B, [C; F]$, transfer function $\mathbf{G}^F$ and characteristic function $\mathfrak{G}^F$. Suppose that $[0; I]$ is an admissible feedback operator for Σ^F that produces the closed-loop system Σ^{cl} having the semigroup generator A_{cl}, transfer function $\mathbf{G}^{cl}$ and characteristic function $\mathfrak{G}^{cl}$. If $0 \in \rho(A)$, then the reciprocal systems $\Sigma_-^F, \Sigma_-^{cl}$ of Σ^F, Σ^{cl}, respectively, are well-defined and $[0; I]$ is an admissible feedback operator for Σ_-^F that produces the closed-loop system Σ_-^{cl}. Moreover, if Σ is output stabilizable, then so is Σ_-.

Proof Since the characteristic functions and transfer functions agree on some right half-plane, from (III.11) and (II.8) we deduce

$$\mathfrak{G}_-^{cl}(s) = \mathfrak{G}_-^F(s)(I - [0, I]\mathfrak{G}_-^F(s))^{-1} \tag{III.14}$$

for $1/s$ in some right half-plane. From Lemma 3.4 $0 \in \rho(A_{cl})$ and from Remark 3.3 Σ_-^{cl} has the generating operators $A_{cl}^{-1}, A_{cl}^{-1}B^{cl}, -[C^{cl}; F^{cl}]A_{cl}^{-1}, \mathfrak{G}^{cl}(0)$.

On the other hand, by parts 1 and 2 of Lemma 3.6, $[0; I]$ is an admissible feedback operator for Σ_-^F with generating operators $A^{-1}, A^{-1}B, -[C; F]A^{-1}, \mathfrak{G}^F(0)$. From Remark 3.3 the resulting closed-loop system Σ^{new} has the generating operators
$A^{new} = A^{-1} - A^{-1}B\mathcal{D}^{-1}FA^{-1},\ \mathcal{D} = I - [0; I]\mathfrak{G}^F(0),$
$B^{new} = A^{-1}B\mathcal{D}^{-1},\ D^{new} = \mathfrak{G}^F(0)\mathcal{D}^{-1},$
$C^{new} = -CA^{-1} + [I; 0]\mathfrak{G}^F(0)\mathcal{D}^{-1}FA^{-1},$
$F^{new} = -\mathcal{D}^{-1}FA^{-1}.$
Moreover, from(III.12), for $s \in \rho(A^{-1}) \cap \rho(A^{new})$ we have

$$\mathfrak{G}^{new}(s) = \mathfrak{G}_-^F(s)(I - [0, I]\mathfrak{G}_-^F(s))^{-1}. \qquad \text{(III.15)}$$

We now show that for $1/s$ in some right half-plane there holds

$$\mathfrak{G}^{new}(s) = \mathfrak{G}_-^{cl}(s). \qquad \text{(III.16)}$$

Comparing (III.14) with (III.15) we see that we need to show that $1/\rho(A^{new})$ contains a right half-plane. From Lemma 3.4 we have that if $\lambda \in \rho(A) = 1/\rho(A^{-1})$ and $I - [0; I]\mathfrak{G}_-^F(\frac{1}{\lambda})$ has a bounded inverse, then $\lambda \in 1/\rho(A^{new})$. But

$$I - [0; I]\mathfrak{G}_-^F(\frac{1}{\lambda}) = I - [0; I]\mathfrak{G}^F(\lambda)$$

and this has a bounded inverse for λ in some right half-plane. Moreover, since A is an infinitesimal generator, its resolvent set contains a right half-plane. So $1/\rho(A^{new})$ contains a right half-plane and we have shown (III.16).
Since both Σ_-^{cl} and Σ^{new} have bounded operators, we can conclude that their feedthrough operators are equal, i.e.,

$$\mathfrak{G}^{cl}(0) = D_-^{cl} = D^{new} = \mathfrak{G}^F(0)(I - [0; I]\mathfrak{G}^F(0))^{-1}.$$

So we have

$$I - [0; I]\mathfrak{G}^{cl}(0) = (I - [0; I]\mathfrak{G}^F(0))^{-1}, \qquad \text{(III.17)}$$

and $I - [0; I]\mathfrak{G}^{cl}(0)$ has the bounded inverse $(I - [0; I]\mathfrak{G}^F(0)$. But since $0 \in \rho(A) \cap \rho(A^{cl})$, both $\mathfrak{G}^F$ and $\mathfrak{G}^{cl}$ are analytic in a neighbourhood of the origin. So in this neighbourhood

$$(I - [0; I]\mathfrak{G}^{cl}(s))^{-1} = I - [0; I]\mathfrak{G}^F(s),$$

and the following equalities hold at least for s in some right half-plane

$$(I - [0; I]\mathfrak{G}_-^{cl}(s))^{-1} = I - [0; I]\mathfrak{G}_-^F(s)$$
$$\mathfrak{G}_-^{cl}(s) = \mathfrak{G}_-^F(s)(I - [0; I]\mathfrak{G}_-^F(s))^{-1}.$$

But (III.15) also holds at least in some right half-plane since A^{new} and A^{-1} are bounded operators. Thus $\mathfrak{G}^{new} = \mathfrak{G}_-^{cl}$ on some right half-plane and

$$\mathbf{G}^{new} = \mathbf{G}_-^{cl} \text{ on some right half-plane}. \qquad \text{(III.18)}$$

Next we show that the output maps are equal. For $s \in \rho(A^{new}) \cap \rho(A^{-1})$ the output map of Σ^{new} is given by

$$[C^{new}; F^{new}](sI - A^{new})^{-1} = \qquad \text{(III.19)}$$
$$-(I + \mathfrak{G}^{new}(s)[0; I])[CA^{-1}; FA^{-1}](sI - A)^{-1}.$$

Taking Laplace transforms of the perturbation formula for the closed-loop observation map (Weiss [18, (6.13)]) for s in some right half-plane we obtain

$$\hat{\mathcal{C}}^{cl}(s) = (I + \mathbf{G}^{cl}(s)[0; I])\mathcal{C}^F(s)$$

or equivalently

$$[C^{cl}; F^{cl}](sI - A_{cl})^{-1} = (I + \mathfrak{G}^{cl}(s)[0; I])[C; F](sI - A)^{-1}.$$

With some algebraic manipulations and substituting $1/s$ for s this gives

$$[C^{cl}A_{cl}^{-1}; F^{cl}A_{cl}^{-1}](sI - A_{cl}^{-1})^{-1} =$$
$$(I + \mathfrak{G}_-^{cl}(s)[0; I])[CA^{-1}; FA^{-1}](sI - A^{-1})^{-1}$$
$$= (I + \mathfrak{G}^{new}(s)[0; I])[CA^{-1}; FA^{-1}](sI - A^{-1})^{-1}$$

for $1/s$ in some right half-plane, where we have used (III.16) in the last step. So combining this last equation with (III.19), we obtain

$$[C^{new}; F^{new}](sI - A^{new})^{-1} = \qquad \text{(III.20)}$$
$$-[C^{cl}A_{cl}^{-1}; F^{cl}A_{cl}^{-1}](sI - A^{cl})^{-1} \qquad \text{(III.21)}$$

for $1/s$ in some right half-plane. Now, as $s \to 0$, $[C^{new}; F^{new}](sI - A^{new})^{-1}A^{new}z \to -[C^{new}; F^{new}]z$, and $[C^{cl}A_{cl}^{-1}; F^{cl}A_{cl}^{-1}]A^{new}z \to [C^{cl}; F^{cl}]A^{new}z$ as $s \to 0$. Thus (III.20) implies that

$$[C^{new}; F^{new}] = -[C^{cl}; F^{cl}]A^{new}. \qquad \text{(III.22)}$$

To show that $A^{new} = A_{cl}$ we consider the resolvent identity for $A^{new} = A^{-1} + A^{-1}BF^{new}$

$$(sI - A^{new})^{-1} - (sI - A^{-1})^{-1} =$$
$$sI - A^{-1})^{-1}A^{-1}BF^{new}(sI - A^{new})^{-1},$$

which holds for $s \in \rho(A^{new} \cap \rho(A^{-1})$. Now we substitute from (III.20) to obtain

$$(sI - A^{new})^{-1} - (sI - A^{-1})^{-1} = \qquad \text{(III.23)}$$
$$-(sI - A^{-1})^{-1}A^{-1}BF^{cl}A_{cl}^{-1}(sI - A_{cl}^{-1})^{-1},$$

which now holds for $1/s$ in some right half-plane. (Recall that $1/\rho(A^{new}) \cap 1/\rho(A^{-1})$ contains a right half-plane). Now we take Laplace transformations of the semigroup perturbation formula in Weiss [18] to obtain

$$(sI - A_{cl})^{-1} - (sI - A)^{-1} = -(sI - A)^{-1}BF^{cl}(sI - A_{cl})^{-1}$$

for s in some right half-plane. With some manipulations this implies

$$(sI - A^{-1})^{-1} - (sI - A_{cl}^{-1})^{-1} =$$
$$-(sI - A^{-1})^{-1}A^{-1}BF^{cl}A_{cl}^{-1}(sI - A_{cl}^{-1})^{-1},$$

for $1/s$ in some right half-plane. Comparing this with (III.23) yields $(sI - A_{cl}^{-1})^{-1} = (sI - A^{new})^{-1}$ and so the generators are equal and (III.22) yields

$$[C^{new}; F^{new}] = [C_-^{cl}; F_-^{cl}].$$

A similar type of argument shows that $(sI - A^{new})^{-1}B^{new} = (sI - A_{cl}^{-1})^{-1}B^{cl}$ for $1/s$ in some right half-plane and this implies that $B^{new} = A_{cl}^{-1}B^{cl}$ and $\Sigma^{new} = \Sigma^{cl}$.
Suppose now that Σ^{cl} is output stable. Then Theorem 2.7 part 3 shows that $\mathbf{G}^{cl}$ and $\mathcal{C}^{cl}$ are analytic on $\mathbb{C}_0^+$ and for $s \in \rho(A_{cl})$ using (II.8), we have $\mathbf{G}^{cl}(s) = \mathfrak{G}^{cl}(s)$. But

$$\begin{aligned}\hat{\mathcal{C}}^{cl}(s) &= [C^{cl}; F^{cl}](sI - A_{cl})^{-1} \\ &= -\frac{1}{s}[C^{cl}; F^{cl}]A_{cl}^{-1}(\frac{1}{s}I - A_{cl})^{-1},\end{aligned}$$

and so

$$\frac{1}{s}\hat{\mathcal{C}}^{cl}(\frac{1}{s}) = -[C^{cl}; F^{cl}]A_{cl}^{-1}(sI - A_{cl}^{-1})^{-1} = \hat{\mathcal{C}}_-^{cl}(s)$$

for $s \in \rho(A_{cl}^{-1})$. This contains some right half-plane, and so $\hat{\mathcal{C}}_-^{cl}$ has an extension to an $\mathbf{H}_2$ function.

In Curtain [3] a slightly different formula of the closed-loop generator was given. We show that it equals the one found here.

Corollary 3.9: Under the same assumptions and notation as in Lemma, if Σ^F is regular with zero feedthrough operator, then the generator of the closed-loop semigroup A_{cl} satisfies

$$A_{cl}^{-1} = A^{-1} - A^{-1}BF_\Lambda A_{cl}^{-1}. \tag{III.24}$$

Proof Σ_-^F has the generating operators $A^{-1}, A^{-1}B, -[CA^{-1}; FA^{-1}], -[C_\Lambda A^{-1}B; F_\Lambda A^{-1}B]$. From Lemma 3.8 we have

$$\begin{aligned} A_-^{cl} &= A^{new} = A^{-1} - A^{-1}B(I + F_\Lambda A^{-1}B)^{-1}FA^{-1} \\ &= A^{-1} - A^{-1}B(I - F_\Lambda A_{cl}^{-1}B)FA^{-1} \quad \text{by (III.17).} \end{aligned}$$

So it remains to show that

$$A^{-1}BF_\Lambda A_{cl}^{-1} = A^{-1}B(I - F_\Lambda A_{cl}^{-1}B)FA^{-1}.$$

Applying Lemmas 7.9 and 7.10 in Weiss [18] to our closed-loop system Σ^{cl} gives for all $x \in D(F_\Lambda)$.

$$A_{cl}x = (A + BF_\Lambda)x \quad \text{and} \quad Ax = (A_{cl} - BF_\Lambda)x.$$

But $D(F_\Lambda)$ contains $D(A)$ and so with $x = A^{-1}z$ and $z \in Z$, we obtain $z = (A_{cl} - BF_\Lambda)A^{-1}z$ and so

$$\begin{aligned} F_\Lambda A_{cl}^{-1}z &= F_\Lambda A_{cl}^{-1}(A_{cl} - BF_\Lambda)A^{-1}z \\ &= FA^{-1}z - F_\Lambda A_{cl}^{-1}BFA^{-1}z, \end{aligned}$$

which completes the proof.

Remark 3.10: If Σ_- is ouput stabilizable, Σ need not be, as the characteristic function $\mathfrak{G}_-^{cl}(\frac{1}{s})$ need not be uniformly bounded on some right half-plane. However, in Curtain [3] it is shown that if B is a bounded operator, then Σ is output stabilizable if and only if Σ_- is output stabilizable.

We now obtain sufficient conditions that ensure that $\mathbf{G}$ has a unique analytic extension to $\rho(A) \cap \mathbb{C}_0^+$ that agrees with its characteristic function.

Theorem 3.11: Let Σ be a well-posed linear system with generating operators A, B, C, $0 \in \rho(A)$, transfer function $\mathbf{G}$ and characteristic funtion $\mathfrak{G}$. If Σ is output stabilizable and U, Y are finite-dimensional, then $\mathbf{G}$ has a unique extension to a function that, except for countably many isolated points, is analytic on $\mathbb{C}_0^+$ and $\mathbf{G} = \mathfrak{G}$ in this region.

Proof Now Σ^{cl} is output stable and so by Lemma 2.4, $\mathbf{G}^{cl}(s) = \mathfrak{G}^{cl}(s)$ for $s \in \rho(A_{cl})$. From (III.11) it is readily deduced that on some right half-plane

$$\begin{aligned} (I + [0, I]\mathbf{G}^{cl})(I - [0, I]\mathfrak{G}^F) = I \\ = (I - [0, I]\mathfrak{G}^F)(I + [0, I]\mathbf{G}^{cl}). \end{aligned} \tag{III.25}$$

Since $I + [0, I]\mathbf{G}^{cl}$ is analytic on $\mathbb{C}_0^+$, it has only isolated zeros and its inverse has at most countable singularities, say the set S^{sing} . So $I + [0, I]\mathbf{G}^{cl}(s)$ has a unique inverse defined on $\mathbb{C}_0^+/S^{sing}$. Since $[0, I]$ is an admissible feedback operator for Σ^F, $-[0, I]$ is an admissible feedback operator for Σ^{cl} and the following holds on some right half-plane

$$\mathbf{G}^F = \mathbf{G}^{cl}(I + [0, I]\mathbf{G}^{cl})^{-1}. \tag{III.26}$$

But $I + [0, I]\mathbf{G}^{cl}$ has a unique inverse on $\mathbb{C}_0^+/S^{sing}$. Thus (III.26) shows that $\mathbf{G}^F$ has an extension to a function that is analytic on $\mathbb{C}_0^+/S^{sing}$. This extension is unique and it necessarily agrees with its characteristic function on $\mathbb{C}_0^+/S^{sing}$.

Remark 3.12: In fact, we have shown that if a system with finite-dimensional U and Y is output stabilizable, its transfer function has at most countable singularities in $\mathbb{C}_0^+$.

We now show that under the same assumptions as in Theorem 3.11 the reciprocal relationships (II.9) hold.

Lemma 3.13: Suppose that A, B, C are the generating operators of the well-posed linear system Σ with transfer function $\mathbf{G}$ and $0 \in \rho(A)$. If it is output stabilizable and U and Y are finite-dimensional, then (II.9) holds for all $s \in \rho(A) \cap \mathbb{C}_0^+$, where $\mathbf{G}_-$ is the transfer function and $\mathcal{C}_-$ is the observation map of the reciprocal system Σ_-.

Proof By Lemma 3.8 Σ_-^F is also ouput stabilizable. So from Theorem 3.11 we have that both $\mathbf{G}^F$ and $\mathbf{G}_-^F$ have extensions to functions that are analytic on $\mathbb{C}_0^+/S^{sing}$, where S^{sing} is a countable set and these extensions agree with their characteristic functions on $\mathbb{C}_0^+/S^{sing}$. Taking Laplace transforms of the perturbation formula in Weiss [18, (6.13)] gives for s in some right half-plane

$$\hat{\mathcal{C}}^F(s) = (I + \mathbf{G}^F(s)[0; I])\hat{\mathcal{C}}^{cl}(s).$$

The ouput stability of Σ^{cl} shows that $\hat{\mathcal{C}}^{cl}$ is analytic on $\mathbb{C}_0^+$ and so $\hat{\mathcal{C}}^F$ has an extension to a function that is analytic on $\mathbb{C}_0^+/S^{sing}$ and similarly for $\hat{\mathcal{C}}_-^F$. As in Remark 2.2 for $s \in \rho(A) \cap \mathbb{C}_0^+/S^{sing}$ we have

$$\begin{aligned} \hat{\mathcal{C}}^F(s) &= [C; F](sI - A)^{-1} \\ &= -\frac{1}{s}[C; F]A^{-1}(\frac{1}{s}I - A^{-1})^{-1} \\ &= \frac{1}{s}\hat{\mathcal{C}}_-^F(\frac{1}{s}). \end{aligned} \tag{III.27}$$

Note that S^{sing} is a countable set and that the singularities of $[C; F](sI - A)^{-1}$ and of $[C; F]A^{-1}(\frac{1}{s}I - A^{-1})^{-1}$ are contained in $\rho(A)$. So the above equalities extend to $s \in \rho(A) \cap \mathbb{C}_0^+$ and we have proven the second part of (II.9).
Since $\hat{\mathcal{C}}^F$ has an extension to a function that is analytic on $\mathbb{C}_0^+/S^{sing}$, (II.3) can be extended to obtain

$$\mathbf{G}^F(s) = \mathfrak{G}^F(0) + s\hat{\mathcal{C}}^F(s)A^{-1}B \quad \text{for} \quad s \in \mathbb{C}_0^+/S^{sing}.$$

Combining this with (III.27) and noting that the set S^{sing} is countable, shows that the singularities of $\mathbf{G}^F$ in $\mathbb{C}_0^+$ are contained in $\rho(A) \cap \mathbb{C}_0^+$ and so for $s \in \rho(A) \cap \mathbb{C}_0^+$ we have

$$\mathbf{G}^F(s) = \mathfrak{G}^F(0) + \hat{\mathcal{C}}_-^F(\frac{1}{s})A^{-1}B. \qquad \text{(III.28)}$$

Arguing in a similar fashion for Σ_-^F we obtain

$$\mathbf{G}_-^F(s) = \mathfrak{G}^F(0) + \hat{\mathcal{C}}_-^F(s)A^{-1}B \qquad \text{(III.29)}$$

for $s \in \rho(A^{-1}) \cap \mathbb{C}_0^+$. So using (II.8), (III.28) and (III.29) we obtain

$$\mathfrak{G}(s) = \mathbf{G}^F(s) = \mathbf{G}_-^F(\frac{1}{s}) = \mathfrak{G}_-^F(\frac{1}{s})$$

for $s \in \rho(A) \cap \mathbb{C}_0^+$, which completes the proof of (II.9).

The following example shows that both U and Y need to be finite-dimensional for (II.9) to hold.

Example 3.14: Let A be the right shift on $l_2(\mathbb{Z}) = Z$. So

$$(Az)_k = z_{k-1}, \qquad (A^{-1}z)_k = z_{k+1} \quad \text{and}$$

$$\sigma(A) = \sigma_c(A) = \{s \in \mathbb{C} : |s| = 1\} = \sigma(A^{-1}) = \sigma_c(A^{-1}).$$

Moreover,

$$\rho_\infty(A) = \{s \in \mathbb{C} : |s| > 1\} = \rho_\infty(A^{-1}).$$

For the input let $U = Z$ and $B = I$. For the output take $Y = \mathbb{R}$ and define $Cz = z_{-1}$, We consider the system with bounded generating operators $A, B, C, 0$. This system is clearly exponentially stabilizable and hence ouput stabilizable. For example, $F = -2I$ shifts the spectrum into $\{s : \mathrm{Re}s \leq 1\}$ and since $A - 2I$ is bounded the semigroup is exponentially stable. We shall show that (II.9) does not hold.

To make the calculations easier we calculate the transfer functions of the dual system which has generating operators $A^{-1}, C^* = A^{-1}B_0, I, 0$, where $(B_0u)_k = \delta_{0,k}u$. Its impulse response is given by

$$\begin{aligned}\left(e^{A^{-1}t}A^{-1}B_0\right)_k &= \sum_{n=0}^{\infty} A^{-n-1}\frac{t^n}{n!}\delta_{0,k} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta_{0,k+n+1} \\ &= \frac{t^{-k-1}}{(-k-1)!} \quad \text{for} \quad k < 0, \quad \text{otherwise} \quad 0.\end{aligned}$$

So taking Laplace transforms we obtain

$$\mathbf{G}^d(s)_k = s^k \quad \text{for} \quad k < 0, \quad \text{otherwise} \quad 0.$$

The reciprocal system has the generating operators $A^{-1}, A^{-1}, -CA^{-1}, -CA^{-1}$ and its dual system has the generating operators $A, -B_0, A, -B_0$.

The impulse response of the dual reciprocal system is

$$\begin{aligned}\left(-Ae^{At}B_0\right)_k &= -\sum_{n=0}^{\infty} A^{n+1}\frac{t^n}{n!}\delta_{0,k} \\ &= -\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta_{0,k-n-1} \\ &= -\frac{t^{k-1}}{(k-1)!} \quad \text{for} \quad k \geq 1, \quad \text{otherwise} \quad 0.\end{aligned}$$

Taking Laplace transforms we obtain the transfer function of the dual of the reciprocal system to be

$$\mathbf{G}_-^d(s) = -s^{-k} \quad \text{for} \quad k \geq 0, \quad \text{otherwise} \quad 0.$$

So comparing this with our calculations above for $\mathbf{G}$, we see that we never have $\mathbf{G}^d(s) = \mathbf{G}_-^d(\frac{1}{s})$.

IV. Historical Remarks

It appears that the concept of a reciprocal system first appeared in the Russian literature in the context of nodes; see, for example, the book by Livsic [9]. Partial relationships between the pairs A, B and A^{-1}, BA^{-1} have been used by Fattorini and Triggiani in the study of controllability for boundary control systems and by Grabowski and Callier in their work on the circle criterion for boundary control via Lyapunov stability and Lur'e equations (see [1]). However, the closest connection we are aware of is in the finite-dimensional papers [6] by Fernando and Nicholson and [11] by Muscato, Nunnari and Fortuna who used the concept of reciprocal systems in the context of stochastic balancing and model reduction.

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