About Optimality of Discrete-Time LQ Observer Based Regulator

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***Abstract*— It is shown that if the number of plant outputs is equal to at least half of the number of plant states then the observer may be designed which in one step detects the true value of the state. It is also shown that in multi-variable discrete-time (DT) systems the LQ regulator based on the reduced order Luenberger observer is optimal for adequate initial conditions of the observer. The case of non-adequate initial conditions of the observer is considered, too. Further on, the properties of this regulator applied in the closed loop system with nonzero excitations are analyzed. It this case the DT regulator based on the observer which in one step detects the true value of the plant state is optimal for transients, starting from the next step after appearance of the excitations.**

***Index Terms*— Linear-quadratic regulator; discrete-time; output regulator; multi-variable systems; observers.**

I. Introduction

Linear-Quadratic Regulator (LQR) problem, in infinite horizon, has usually the solution in the form of a static state feedback control law and may be implemented when all the state variables are available [3]. This observation concerns continuous- and discrete-time, as well as single- and multi-variable plants. When only the outputs of the plant are measured, the state feedback LQ control law may be implemented, if an appropriate state observer is included in the system [1].

There is almost common conviction that LQ control problem with output feedback may be solved by more modern H_2 approach [5]. This conviction does not concern the problem considered in the present paper. In the H_2 approach applied to the deterministic case there is the assumption about zero initial conditions of the regulator, while in our considerations nonzero initial conditions of the regulator play an essential role.

In the present paper, first the observer which in one step detects the true value of the plant state is considered. Then the following question is researched: whether and in what sense the observer based LQ regulator is optimal? The case of linear-quadratic regulator (LQR) problem with output feedback, for multi-variable discrete-time (DT) systems is considered.

It is shown that the LQ DT regulators based on the reduced order Luenberger observer are optimal for adequate initial conditions of the observer. The properties of these regulators in the case of non-adequate initial conditions are also researched. In this case the regulators with the observer detecting in one step the true value of the plant state are optimal starting from the next step after appearance of the excitations. It is also shown that the properties of these regulators remain unchanged for transients generated in the closed loop (CL) system with nonzero excitations belonging to a determined general class.

The contribution of the paper is partly in deriving the observer which in one step detects the true value of the plant state, partly in showing that the regulator based on the reduced order Luenberger observer with adequate initial condition is optimal and partly in proving that the regulator based on the derived observer, applied in the CL system with nonzero excitations, generates the transients which are optimal, starting from the next step after appearance of excitations.

II. LQ regulator with state feedback

Let the DT state space model of a multi-variable plant takes the form

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{1}$$

where x, u and y are the vectors of state, input and output, n, r and p-dimensional, respectively; A, B, C are constant matrices of appropriate dimension and $t = 0, 1, 2, ...$ is discrete time.

Assume that A has a full rank i.e. rank $A = n$. This case occurs when the DT equations (1) result from discretization of a continuous-time model of the plant. This is very frequent case in applications.

Let the quadratic performance index takes the form

$$J = \sum_{t=0}^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)] \tag{2}$$

where the symmetric matrices $Q = D'D$ and R are semi-positive and positive definite, respectively. Assume, that the pair (A, B) is controllable and the pair (A, D) is detectable.

The solution of the DT LQR problem (1), (2) in the form of the state feedback is

$$u = -Kx \tag{3}$$

where the gain matrix K is determined by

$$K = (R + B'SB)^{-1}B'SA \tag{4}$$

where S is the solution of an appropriate algebraic Riccati equation [3].

The minimal value of the performance index (2) for given initial condition $x(0)$ of the plant is

$$J_{min} = x'(0)Sx(0) \tag{5}$$

The closed-loop (CL) system is described by

$$x(t+1) = (A - BK)x(t) \tag{6}$$

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the poles of the CL system (6).

III. Reduced order observer

The control law (3) may be implemented if all the state components are available (measured). Further on, the case when only the output variable y is available will be considered. It is known that in this case the control law (3) may be implemented if an appropriate observer estimating the state is applied.

The equations of the reduced order (Luenberger) observer for the plant (1) have the form [5]

$$\begin{aligned} \hat{v}(t+1) &= E\hat{v}(t) + Fy(t) + Gu(t) \\ \hat{x}(t) &= V\hat{v}(t) + Wy(t) \end{aligned} \tag{7}$$

where $\hat{x}$ and $\hat{v}$ are the vectors of the plant state and observer state estimates, n and m-dimensional respectively, $m = n - p$; E, F, G, V, W are the constant matrices of appropriate dimension; the choice of E is such that its eigenvalues $\bar{\lambda}_i$, $i = 1, 2, ..., m$ fulfill the inequality $|\bar{\lambda}_i| < 1$. Additionally, there exists a $m \times n$ matrix P fulfilling the equations

$$\begin{aligned} &PA - EP = FC \\ &G = PB \\ &WC + VP = I_n \end{aligned} \tag{8}$$

where I_n is $n \times n$ unit matrix.

To explain denote

$$v = Px \tag{9}$$

Multiplying both sides of (1) by P from left hand side and accounting (8), (1) and (9) we obtain.

$$\begin{aligned} v(t+1) &= Ev(t) + Fy(t) + Gu(t) \\ x(t) &= Vv(t) + Wy(t) \end{aligned} \tag{10}$$

Note that it is a freedom in choosing matrices E, F (and eigenvalues $\bar{\lambda}_i$) which guarantees the fast observer convergence.

The CL system with dynamic output feedback LQ regulator (DOFR) which may be implemented when only the output y is available is described by (1), (7) with accounting

$$u = -K\hat{x} \tag{11}$$

After transformations we obtain the description of the CL system in the form

$$\begin{aligned} x(t+1) &= (A - BK)x(t) + BKV\tilde{v}(t) \\ \tilde{v}(t+1) &= E\tilde{v}(t) \end{aligned} \tag{12}$$

$$\tilde{v}(t) = v(t) - \hat{v}(t) \tag{13}$$

From the form of (12) it results the known fact that the CL system with DOFR is of $(n+m)$-th order and has the poles $\lambda_1, \lambda_2, ..., \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_m$ being the union of the poles of the CL system (6) with LQ regulator and state feedback and of the observer (7).

Neither v nor x appearing in (10) are known, but for any initial conditions $\hat{v}(0)$ and $x(0)$ of the equations (7) and (1), respectively, we have $\hat{v}(t) \to v(t)$ and $\hat{x}(t) \to x(t)$ when $t \to \infty$. Thus $\hat{x}(t)$ determined from (7) is the estimate of $x(t)$. Similarly $\hat{v}(t) \to v(t)$ when $t \to \infty$.

In the case for which $\bar{\lambda}_i = 0$. $i = 1, 2, ...m$ the observer has the finite time of decaying of $\tilde{v}(t)$ to zero (smaller than or equal to m). This kind of observer is called the dead-beat observer.

Note that from the third equation of (7) it results that the matrices W, C, V, P, $n \times p$, $p \times n$, $n \times m$, $m \times n$, respectively, must have full ranks, that is rank $W = p$, rank $C = p$, rank $V = m$, rank $P = m$. Taking this into account we may formulate the following Lemma.

Lemma 1. Assume that $m = n - p \leq p$. Then for $E = 0$ there exist the matrices P, F, G, W, V which fulfill the equations (13).

Proof. Since maximal rank $(FC) = p$ and $m \leq p$ then we may choose the matrix F so that rank $(FC) = m$. The maximal rank $(PA) = m$. Then for the chosen F there exist the solution P of the matrix equation $PA = FC$. Matrix G results from the second equation of (8). Matrices V, W we obtain from

$$[V \; W] = \begin{bmatrix} P \\ C \end{bmatrix}^{-1} \tag{14}$$

which results from the third equation of (8).

$\diamond$

Corollary 1 The observer resulting from Lemma 1 with $E = 0$ detects the true value of the state $x(t)$ in one step. That is, for any unknown initial state $x(0)$ of the plant (1) and any assumed initial condition $\hat{v}(0)$ of the observer (7) we have $\hat{x}(t) = x(t)$ for $t \geq 1$.

Proof. Note that for the observer (7) with $E = 0$, from the second equation of (15) it results $\tilde{v}(t) = 0$ for $t \geq 1$ i.e. $\hat{v}(t) = v(t)$ for $t \geq 1$. Therefore from the second equations of (7) we obtain $\hat{x}(t) = x(t)$ for $t \geq 1$.

$\diamond$

The state space equations of the regulator based on the reduced order observer, result from (7) and (11) and take the form

$$\begin{aligned} \hat{v}(t+1) &= (E - GKV)\hat{v}(t) + (F - GKW)y(t) \\ u(t) &= -KV\hat{v}(t) - KWy(t) \end{aligned} \tag{15}$$

IV. Whether and in what sense the researched regulator is optimal?

One can suppose that the reduced order observer (7) determines the accurate estimate of the state (i.e. $\hat{x} = x$) if the initial condition $\hat{v}(0)$ of the observer is adequate to the initial condition $x(0)$ of the plant state. Note, that the second formula of (10) determines the transformation of the state $[v^T \; y^T]^T$ to the state x, which may be written in the form

$$x = [V \; W] \begin{bmatrix} v \\ y \end{bmatrix} \tag{16}$$

Accounting the third formula of (8) we obtain the inverse transformation in the form

$$\left[\begin{array}{c} v \\ y \end{array}\right] = \left[\begin{array}{c} P \\ C \end{array}\right] x \tag{17}$$

Lemma 2. Optimality of the regulator. For the initial conditions $x(0),\ \hat{v}(0)$ fulfilling the relation

$$\hat{v}(0) = Px(0) \tag{18}$$

the regulator (15) based on the reduced order observer, applied to the plant (1) is optimal in the sense that in the resulting CL system the performance index takes the optimal value equal to that obtained in the CL system with LQR and state feedback (for the same initial condition of the plant $x(0)$).

Proof. From (17) and (18) it results that $\hat{v}(0) = v(0)$, then $\tilde{v}(0) = 0$ and from (12) we obtain $\tilde{v}(t) = 0$ which gives $\hat{v}(t) = v(t)$ for any $t \geq 0$. Accounting (7) and (10) we obtain $\hat{x}(t) = x(t)$ for any $t \geq 0$. Therefore the formula (11) determines then the optimal control of the CL system with LQR and state feedback.

◇

The initial condition $\hat{v}(0)$ determined by (18) is called the adequate initial condition of the observer.

In the case when the initial condition of the observer is non-adequate (i.e. it does not fulfill the dependence (18), the properties of the CL system are somewhat different.

Lemma 3 about optimality of the regulator. In the case, when the initial condition $\hat{v}(0)$ of the observer does not fulfill (18), the regulator based on the reduced order observer with $E = 0$, working in the CL system with plant (1) is optimal, starting from time $t = 1$, i.e. we have

$$\sum_{t=1}^{\infty}[x'(t)Qx(t) + u'(t)Ru(t)] = x'(1)Sx(1) \tag{19}$$

where $x(1)$ is the state of the plant resulting from applying a non-optimal control $u(0)$ at time $t = 0$, while S is the matrix appearing in (5).

Proof results from the fact that for the observer with the matrix $E = 0$, a nonzero initial condition $\tilde{v}(0)$ decays to zero in one step. This means that $\hat{x}(t) = x(t)$ for $t \geq 1$.

◇

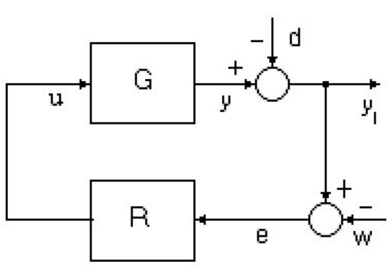

Fig. 1. Closed loop system.

V. Closed loop system with nonzero excitations

Consider the CL system shown in Fig. 1 with the plant G (1) and regulator-observer R described by (7), (11). The CL system for $t \geq 0$ is described by the equations

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{20}$$

$$\begin{array}{l} v^1(t+1) = Ev^1(t) + Fe(t) + Gu(t) \\ x^1(t) = Vv^1(t) + We(t) \end{array} \tag{21}$$

$$u(t) = -Kx^1(t) \tag{22}$$

$$e(t) = y(t) - d^*(t) - w^*(t) \tag{23}$$

and $d^*(t), w^*(t)$ denote some given vector functions. Assume that the initial condition of the plant $x(0) = 0$ and of the observer $v^1(0) = 0$, while the excitations take the form

$$d = d^*(t)\mathbf{1}(t), \qquad w = w^*(t)\mathbf{1}(t) \tag{24}$$

where $\mathbf{1}(t)$ denotes the unit step function ($\mathbf{1}(t) = 1$ for $t \geq 0$ and $\mathbf{1}(t) = 0$ for $t < 0$) and $d^*(t),\ w^*(t)$ are some p–dimensional vector functions, components of which describe the disturbances and set points for particular p outputs of the plant.

Assume that the vector functions $d^*(t),\ w^*(t)$ determined in the interval $-\infty < t < \infty$ would generate in the considered CL system a ”steady state” in this interval (if the excitations would be described by these functions in the interval $-\infty < t < \infty$). The components of these vector functions may for instance take the form of any constant and/or any periodic functions. One can also consider a generalized ”steady state” determined by the solutions of the equations (20)-(23) for any given functions $d^*(t)$, $w^*(t)$ determined in the interval $-\infty < t < \infty$.

Denote by the functions $x^* = x^*(t)$, $u^* = u^*(t)$, $y^* = y^*(t)$, $e^* = e^*(t)$, $v^{1*} = v^{1*}(t)$, $x^{1*} = x^{1*}(t)$ the solutions of the equations (20)-(23) in the interval $-\infty < t < \infty$. Further on the signals x^*, u^*, y^*, e^*, v^{1*}, x^{1*} will play the role of the reference signals and will determine the ”steady state” in the system.

Consider the performance index in the form

$$\bar{J} = \sum_{t=0}^{\infty}[\bar{x}'(t)Q\bar{x}(t) + \bar{u}'(t)R\bar{u}(t)] \tag{25}$$

where $\bar{x} = x - x^*$ and $\bar{u} = u - u^*$ denote the deviations of x and u from the reference signals x^* and u^* appearing in ”steady state”.

Theorem 1 about optimality of the regulator . Regulator (15) with the observer matrix $E = 0$, applied in the CL system shown in Fig. 1, with plant (1) and excitations (24) generates the control which minimizes the performance index (25) in the interval $t \geq 1$, starting from the state $\bar{x}(1)$ resulting from the non optimal control $u(0)$. That is we have

$$\sum_{t=1}^{\infty}[\bar{x}'(t)Q\bar{x}(t) + \bar{u}'(t)R\bar{u}(t) = \bar{x}'(1)S\bar{x}(1)] \tag{26}$$

where S is the matrix appearing in (5).

Proof. Accounting (23) in (21) and applying the superposition principle we obtain

$$v^1 = v - v^d, \qquad x^1 = x - x^d \tag{27}$$

where the functions $v = v(t)$ and $x = x(t)$ result from solving the equations

$$\begin{aligned} v(t+1) &= Ev(t) + Fy(t) + Gu(t) \\ x(t) &= Vv(t) + Wy(t) \end{aligned} \tag{28}$$

and (20) with the control u determined by (22) the output y determined by the second equation of (20) and initial condition $v(0) = 0$; the functions $v^d = v^d(t)$ and $x^d = x^d(t)$ result from solving the equations

$$\begin{aligned} v^d(t+1) &= Ev^d(t) + F[d^*(t) + w^*(t] \\ x^d(t) &= Vv^d(t) + W[d^*(t) + w^*(t)] \end{aligned} \tag{29}$$

with $v^d(0) = 0$. Denote also by $v^* = v^*(t)$ and $v^{d*} = v^{d*}(t)$ the solutions of the first equations of (28) and (29), respectively, with initial conditions $v^*(-\infty) = 0$ and $v^{d*}(-\infty) = 0$. From the second equations of (28) and (29) we obtain then appropriately the variables $x^* = x^*(t)$ and $x^{d*} = x^{d*}(t)$. We have also similarly as (27)

$$v^{1*} = v^* - v^{d*}, \qquad x^{1*} = x^* - x^{d*} \tag{30}$$

where as previously the variables with super-star denote the appropriate functions in "steady state" corresponding to excitations $d^*(t)$ i $w^*(t)$. Let us note that for $t \geq 1$ $\; v^d = v^d(t) = v^{d*}(t) = v^{d*}$ and $x^d = x^d(t) = x^{d*}(t) = x^{d*}$. From here and from (27) and (30) it results that for $t \geq 1$

$$\begin{aligned} \bar{x}^1 &= x^1 - x^{1*} = x - x^* = \bar{x} \\ \bar{v}^1 &= v^1 - v^{1*} = v - v^* = \bar{v} \end{aligned} \tag{31}$$

We also have

$$\begin{aligned} &e^* = y^* - d^* - w^* \\ &\bar{e} = e - e^* = y - d^* - w^* - (y^* - d^* - w^*) = \\ &= y - y^* = \bar{y} \end{aligned} \tag{32}$$

The reference signals x^*, u^*, y^*, e^*, v^{1*}, x^{1*} fulfill the equations (20)-(23) in the interval $-\infty < t < \infty$. Subtracting from the equations (20)-(23) (valid in the interval $t \geq 0$), the same equations with substituted in them the reference signals (valid in $-\infty < t < \infty$) and using (31) we obtain the following equations valid in the interval $t \geq 0$

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t), \qquad \bar{y}(t) = C\bar{x}(t) \tag{33}$$

$$\begin{aligned} v^1(t+1) &= E\bar{v}^1(t) + F\bar{e}(t) + G\bar{u}(t) \\ \bar{x}^1(t) &= V\bar{v}^1(t) + W\bar{e}(t) \end{aligned} \tag{34}$$

$$\bar{u}(t) = -K\bar{x}^1(t) \tag{35}$$

Accounting (31) and (32) in (33)-(35) we obtain the equations

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t), \qquad \bar{y}(t) = C\bar{x}(t) \tag{36}$$

$$\begin{aligned} \bar{v}(t+1) &= E\bar{v}(t) + F\bar{y}(t) + G\bar{u}(t) \\ \bar{x}(t) &= V\bar{v}(t) + W\bar{y}(t) \end{aligned} \tag{37}$$

$$\bar{u}(t) = -K\bar{x}(t) \tag{38}$$

which are valid for $t \geq 1$. From the second equation of (37) and from (17) it results that the initial conditions $\bar{x}(1)$ and $\bar{v}(1)$ fulfill the dependence $\bar{v}(1) = P\bar{x}(1)$. Thus from Lemma 1 it results that the control $\bar{u}(t)$ determined by (38) minimizes performance index (25), which means that the equality (26) is fulfilled.

$\diamond$

Let us notice that the performance index (25) accounts only the transients and does not take into account the accuracy in the "steady state" determined by the reference signals.

Corollary 2. Regulator (15) with the observer matrix $E = 0$, applied in the CL system with the plant (1) and excitations (24) generates for $t \geq 1$ the optimal transients. The accuracy of the "steady state" must be analyzed, separately.

VI. Example

Consider the plant with two-inputs two-outputs described by

$$A = \begin{bmatrix} -a & 0 & 0 \\ 0 & -a_1 & 0 \\ 0 & 0 & a_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & k_{12} \\ 0 & k_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} k_{11} & 1 & 0 \\ k_{21} & 0 & 1 \end{bmatrix} \tag{39}$$

where $a = -0.8187$, $k_{11} = 0.3625$, $k_{21} = 0.1813$, $a_1 = -0.7515$, $k_{12} = 0.3728$, $a_2 = -0.6703$, $k_{22} = 0.7253$.

Assuming the quadratic performance index (2) with

$$Q = C'C, \quad R = r\,diag\,[1, 1], \quad r = 0.001$$

we obtain the feedback control law (3) (using *dlqr* of MATLAB), where

$$K = \begin{bmatrix} 0.8049 & 2.7368 & -1.2485 \\ 0.0055 & -0.6751 & 1.2298 \end{bmatrix}$$

The CL system (6) has the poles $\lambda_1 = 0.0009, \lambda_2 = 0.0148, \lambda_3 = 0.7796$.

The considered plant is of third order and has two inputs and two outputs, then $m = 1 < p = 2$. To design the observer (7) with $E = 0$ we assume $F = [1 \;\; 1]$ and from equations (8) we obtain

$$P = [0.6642 \;\; 1.3307 \;\; 1.4918],$$

$$G = [0.6642 \;\; 1.5781]$$

The matrices W and V of dimension 3×2 and 3×1 we obtain from (14)

$$[W \;\; V] = \begin{bmatrix} 15.0112 & 16.8286 & -11.2806 \\ -4.4421 & -6.1010 & 4.0896 \\ -2.7211 & -2.0505 & 2.0448 \end{bmatrix}$$

The equations (15) of the regulator–observer take the form

$$\begin{aligned} \hat{v}(t+1) &= 0.7792\hat{v}(t+1) + [-0.7894 \;\; -1.2739]y(t) \\ u(t) &= \begin{bmatrix} 0.4401 \\ 0.3085 \end{bmatrix}\hat{v}(t) + \begin{bmatrix} -3.3225 & 0.5920 \\ 0.2645 & -1.6901 \end{bmatrix}y(t) \end{aligned} \tag{40}$$

The state equation of CL system with zero excitations (for analyzing stability) is

$$\begin{bmatrix} x(t+1) \\ \hat{v}(t+1) \end{bmatrix} = \\ = \begin{bmatrix} -0.2785 & -3.3225 & 0.5920 & 0.4401 \\ -0.0785 & 0.8501 & -0.6300 & 0.1150 \\ -0.1526 & 0,1919 & -0.5555 & 0.2238 \\ -0.5171 & -0.7894 & -1.2739 & 0.7792 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{v}(t) \end{bmatrix} \tag{41}$$

The CL system has the following poles $\lambda_1 = 0.0009, \lambda_2 = 0.0148, \lambda_3 = 0.7796, \lambda_4 = 0$.

From Theorem 1 it results that the CL system shown in Fig. 1 and composed of the plant (1), (39) and the regulator (40) (in the latter equations y is replaced by e) under excitations (24) has optimal transients for $t \geq 1$. It may be e.g. $d_i^*(t) = [A_i \sin(\omega_i t + \psi_i) + d_i]\mathbf{1}(t),\ w_i^*(t) = w_i \mathbf{1}(t),\ i = 1, 2,$ where $A_i,\ \omega_i,\ \psi_i,\ d_i,\ w_i,\ i = 1, 2$ are given. For the transients appearing in the system the performance index (25) takes the optimal value for $t \geq 1$.

VII. Conclusions

It is known, that the CL multi-variable DT system with the observer based LQ regulator has the poles being the union of those of the CL system with LQ regulator and state feedback and those of the observer [1].

In the present paper the additional property of the mentioned system has been noted. Namely, the regulator LQ based on the reduced order Luenberger observer is optimal when the observer starts from the adequate initial conditions.

The considerations concerning properties for the discussed regulators, when they start from non-adequate initial conditions of the observer, have the basic meaning utilized in further considerations. A special case of the dead-bear observer is considered, which in one step detects the true value of the state. It is shown that this kind of the observer may be designed if the number of the plant outputs is equal to at least half of the number of the plant states. It is also shown that in the CL system the discussed regulator with the considered observer is optimal, starting from the next step following the initial instance of time.

This property of the regulator is partially retained if the regulator is applied in the CL system with prescribed nonzero excitations appearing at time $t = 0$. In this case the property concern the transients generated in the system after appearance of the excitations. In this case the considered regulator generates the transients which are optimal, starting from the next step after appearance of the excitation.

In the case of the continuous-time systems the observer based regulators has partially similar but somewhat different properties [4].

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