ALGORITHMS FOR CONTROL OF A ROTATING PENDULUM

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Abstract: **This work deals with a physical model of the Rotating Pendulum which serves for research and education at the Faculty of Electrical Engineering, Czech Technical University in Prague. The paper describes design and implementation of a nonlinear control law and related tasks relevant for control theory courses. Issues of theory and practice are demonstrated on the physical system.**

Keywords: **Rotating Pendulum, Nonlinear modeling, Nonlinear control, Linear Observer**

I. INTRODUCTION

The paper describes the use of the physical model of a Rotating Pendulum in control related courses including Modeling and Simulation, Linear and Nonlinear Control Theory. Special attention is devoted to stabilization of the open loop unstable system with the energy based control algorithm. The pendulum in the Fig.1, was developed at the Department of Control Engineering at the Faculty of Electrical Engineering, Czech Technical University in Prague.

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The mathematical model of the physical pendulum presented in Chapter II is derived using Lagrange Equations. Simulation of the model dynamic is provided in Chapter III.

Chapter IV focuses on the regulation problem, stabilization of the system while eliminating small external disturbances. Chapter V describes the strategy for swinging up the pendulum to the upright position using energy of the pendulum. Chapter VI summarizes the work.

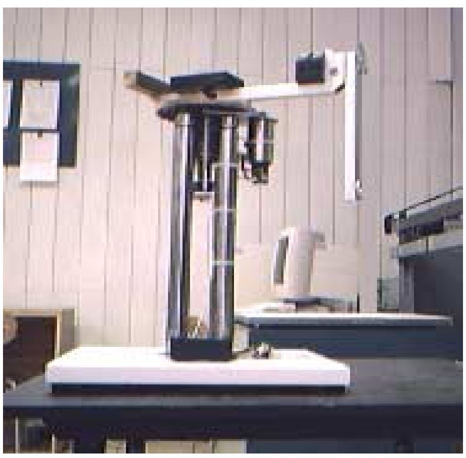

Fig. 1. The Rotating Pendulum

Control tasks for the Rotating Pendulum are well known problems. The mathematical model of the Rotating Pendulum was introduced e.g. in [2] and [3]. In [3] the parameter estimation of the mathematical model and simulations are described. [4] describes balancing the pendulum at the unstable position and swinging it up using the energy viewpoint. Swinging up the pendulum by periodic input and energy principles describes [4].

The pendulum is hinged to the arm, which is fixed on a cog-wheel driven by the DC motor. The arm angle φ_a and the pendulum angle φ_p are measured by optic incremental sensors. These signals are connected to a PC Lab Card, which interfaces the laboratory model to the computer. Detail description of the Rotary Pendulum (see Fig. 1.) and the PC-Pendulum interface can be found in [1]. To summarize, the system in Fig.2 has a single input u and two outputs, angles φ_p and φ_a.

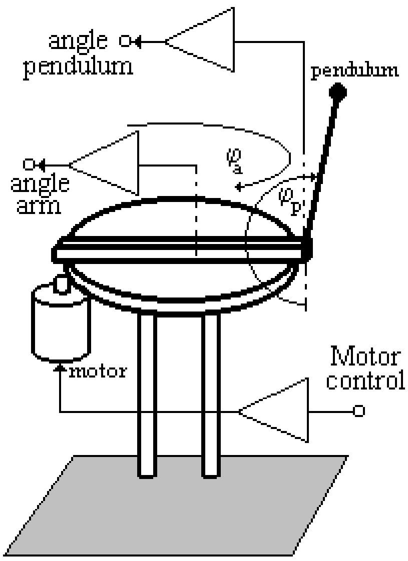

Fig. 2. Principal drawing of the Rotating Pendulum

II. MODELING

Deriving a mathematical model of the Rotating Pendulum is the first task for a student. The use of Lagrange Equations is recommended . The first objective is to define restrictions and limitations to be considered when deriving the model and then design the mathematical model in the form of a state-space description (1).

$$\dot{x} = f(x,u), \qquad y = g(x,u). \tag{1}$$

A. Theory for modeling

The simplifying assumptions for modeling lie in considering the viscose friction only and neglecting backlash of the gear of the DC motor. The dynamics of the motor is neglected in the modeling phase and the motor is considered as a source of torque.
There are many approaches to be used when deriving the mathematical model. However, Lagrange equations offer a systematic and error free way to do it. The general form of the Lagrange Function L of the system is given as

$$L = E_k - E_p, \tag{2}$$

where E_k is kinetic energy and E_p potential energy of the system. If the Lagrange function of the system is known, the mathematical model of the system is found in the form of

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \omega_a} - \frac{\partial L}{\partial \varphi_a} + \frac{\partial R}{\partial \omega_a} &= M_{m,} \\ \frac{d}{dt}\frac{\partial L}{\partial \omega_p} - \frac{\partial L}{\partial \varphi_p} + \frac{\partial R}{\partial \omega_p} &= 0, \end{aligned} \tag{3}$$

where φ_a and φ_p are the arm and pendulum angles, ω_a is the arm speed of rotation and ω_p is the pendulum speed of rotation. M_m is the torque of the motor and R is the Rayleigh function. The Rayleigh function expresses the dissipation energy of the system such as viscose friction. R can get the form of

$$R = \frac{1}{2}b_a\omega_a^2 + \frac{1}{2}b_p\omega_p^2, \tag{4}$$

where b_a is the constant of the viscose friction of the arm and b_a is the same for the pendulum.

B. Modeling of the Rotating Pendulum

The task for students is to derive the equations of motion for the Rotating Pendulum. Often, the most efficient way for mechanical systems is to apply the Lagrange equations. The first step involves computing the Lagrange function (2). The differential equations of the Rotating Pendulum are found by setting several partial derivatives of the Lagrange function in the equations (3). Using the sketch in the Fig. 2 we have for the kinetic energy of the system

$$\begin{aligned} E_k = \frac{1}{2}(J_1 + mr^2)\omega_a^2 + \frac{1}{2}(J_2 + \frac{1}{4}ml^2)\omega_p^2 + \\ \frac{1}{2}mrl\omega_a\omega_p\cos\varphi_p, \end{aligned} \tag{5}$$

where J_1 is moment of inertia of the arm and the DC motor, J_2 is moment of inertia of the pendulum,

m is weight of the pendulum, r is radius of the arm and l is length of the pendulum.
For the potential energy of the Rotating Pendulum we obtain

$$E_p = mgh_0 - \frac{1}{2} mgl \cos\varphi_p, \tag{6}$$

where h_0 is the relative height the mass point of the pendulum and g is gravity acceleration. Substituting (5) and (6) in (2) we obtain Lagrange function L. Putting equal

$$\begin{aligned} &J_a = J_1 + mr^2,\ J_p = J_2 + \frac{1}{4} ml^2,\ J = mrl, \\ &K_p = mgl,\ M_m = K_a u, \end{aligned} \tag{7}$$

substituting several partial derivatives of the Lagrange function, Rayleigh function and equations (7) in equations (3), we obtain the Nonlinear Model of the Rotating Pendulum as follows

$$\begin{aligned} &J_a \dot{\omega}_a + \frac{1}{2} J \cos\varphi_p \dot{\omega}_p - \frac{1}{2} J \sin\varphi_p \omega_p^2 \\ &\quad + b_a \omega_a \ = \ K_a u_a, \\ &J_p \dot{\omega}_p + \frac{1}{2} J \cos\varphi_p \dot{\omega}_a + \frac{1}{2} K_p \sin\varphi_p \\ &\quad + b_p \omega_p = \ 0, \\ &\dot{\varphi}_a = \omega_a, \\ &\dot{\varphi}_p = \omega_p, \end{aligned} \tag{8}$$

where u is the system input (Motor control). The model (8) may be transformed into the form of (1)

$$\begin{aligned} &\dot{\omega}_a = \frac{1}{4J_a J_p - J^2 \cos^2\varphi_p} \\ &\cdot (-2J_p J \sin\varphi_p \omega_p^2 - 4J_p b_a \omega_a \\ &+ 2Jb_p \cos\varphi_p \omega_p + 4J_p K_a u_a \\ &+ JK_p \sin\varphi_p \cos\varphi_p), \\ &\dot{\omega}_p = \frac{1}{4J_a J_p - J^2 \cos^2\varphi_p} \\ &\cdot (-2J^2 \sin\varphi_p \cos\varphi_p \omega_p^2 - 4J_a b_p \omega_p \\ &+ 2Jb_a \cos\varphi_p \omega_a + 2JK_a \cos\varphi_p u_a \\ &- 2J_a K_p \sin\varphi_p), \\ &\dot{\varphi}_a = \omega_a, \\ &\dot{\varphi}_p = \omega_a, \end{aligned} \tag{9}$$

where

$$x = (\omega_a, \omega_p, \varphi_a, \varphi_p)^T, y = (\varphi_a, \varphi_p)^T \ u = (u_a).$$

The state space model (9) of the Rotating Pendulum can be simplified to get a form useful for design of a control algorithm. In the case when

$$4J_a J_p >> J^2$$

and the pendulum angle φ_p is kept close to π by regulation, then the equations (9) can be modifed to

$$\begin{aligned} &\dot{\omega}_a = \alpha_1 \omega_a + \alpha_2 (u + u_0) + \alpha_3 \sin\varphi_p \\ &\quad + \alpha_4 \sin\varphi_p \omega_p^2, \\ &\dot{\omega}_p = \beta_1 \omega_p + \beta_2 \sin\varphi_p + \beta_3 (u + u_0) \cos\varphi_p \\ &\quad + \beta_4 \cos\varphi_p \omega_a, \\ &\dot{\varphi}_a = \omega_a, \\ &\dot{\varphi}_p = \omega_p. \end{aligned} \tag{10}$$

The model (10) will be used for design of pendulum controllers. Similar model of the Rotating Pendulum can be found in [3].

III. SIMULATION

Simulation block diagram and selecting an appropriate method of a numerical solver is another task for students to solve. The model (9) and (10) serves as basis for the task. The simulation and validation of the model structure and parameters are followed by the system analysis phase.

A. *Theory*

Students design the simulation diagram following the mathematical model derived previously and set identified parameters of the model. For simulations, students choose a suitable software including a solver. MATLAB and *Simulink* is used. Students choose the numerical method and solver options from the following list:

1) *Methods*

- *Bogacki-Shampine* (ode23): non-stiff differential equations, low order method
- *Dormand-Prince* (ode45): non-stiff differential equations, medium order method
- *Adams* (ode113): non-stiff differential equations, variable order method
- *stiff/NDF* (ode15s): stiff differential equations, variable order method
- *stiff/Mod. Rosenbrock* (ode23s): stiff differential equations, low order method

2) *Type*

- Fixed-step
- Variable-step

3) Solver options and parameters

- Maximal step size
- Relative tolerance
- Absolute tolerance

There are two ways how to choose suitable numerical methods for a solver. The first way is to compare quality of each method with respect to stability of the solution. The second way is to compare performance of the selected method with real measured data. Bad choice of the solver type and parameters can cause unstable behaviour.

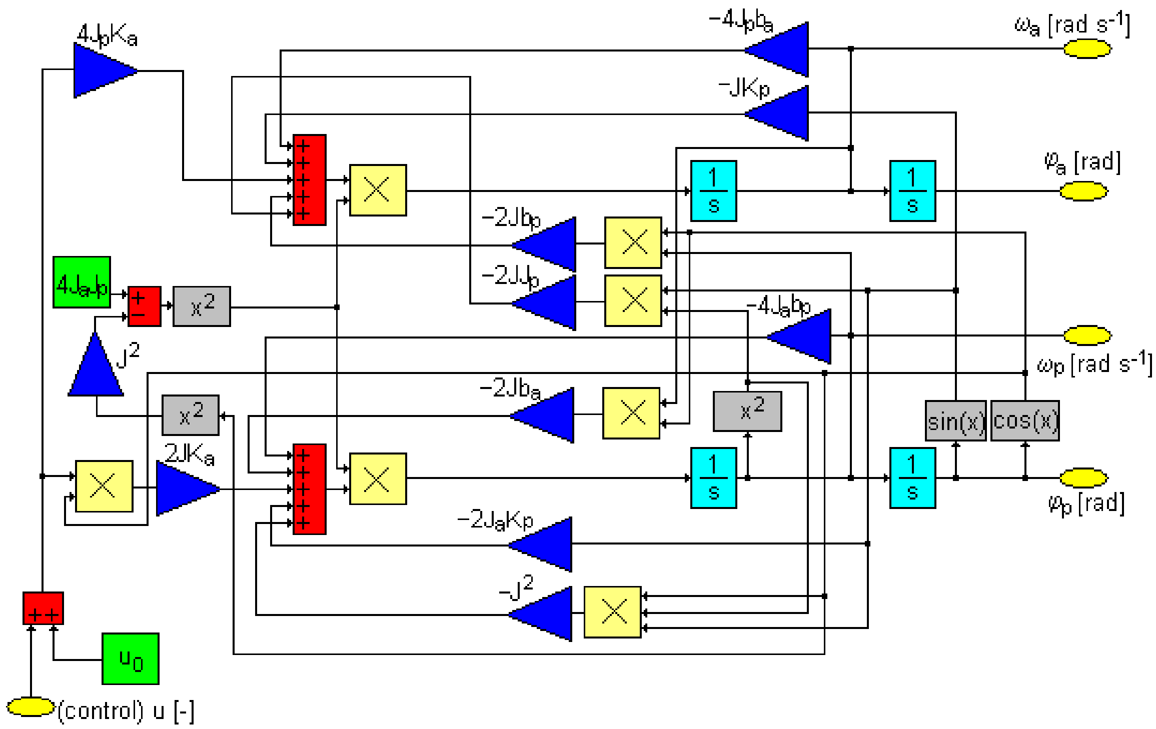

Fig. 3. Simulation diagram of the mathematical model (9)

After the model is validated, we analyze the model characteristics and linearize the nonlinear model for Linear Control Design to get the model in the form

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du. \tag{11}$$

B. Simulations

The parameters of the block diagram in Fig. 3 are initialized by values specified in the Appendix 1. All simulations use *ode45* solver. Fig.4 shows model validation where simulation run is compared to the real recording. Fig. 4 shows reaction of the system to the pulse input $u = 0.1$ applied during the interval [0,2.5] seconds. Experiment starts from zero initial conditions.

The parameters of the block diagram in Fig. 5 are initialized by values specified in the Appendix 2. All simulations use *ode45* solver. Fig.5 shows model validation where simulation run is compared to the real recording. Fig. 5 shows reaction of the system to the pulse input $u = 0.1$ applied during the interval [0,2.5] seconds. Experiment starts from zero initial conditions.

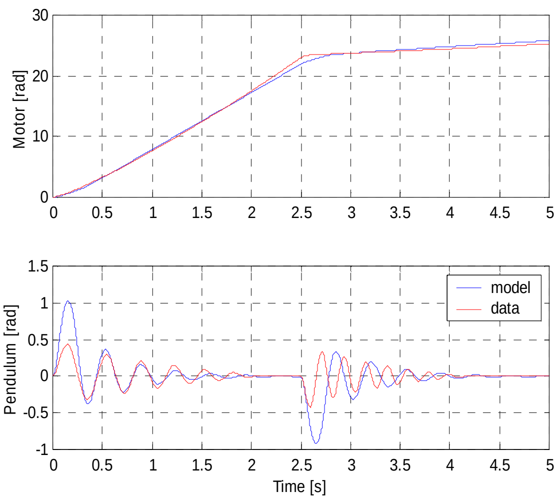

Fig. 4. Model (9) validation

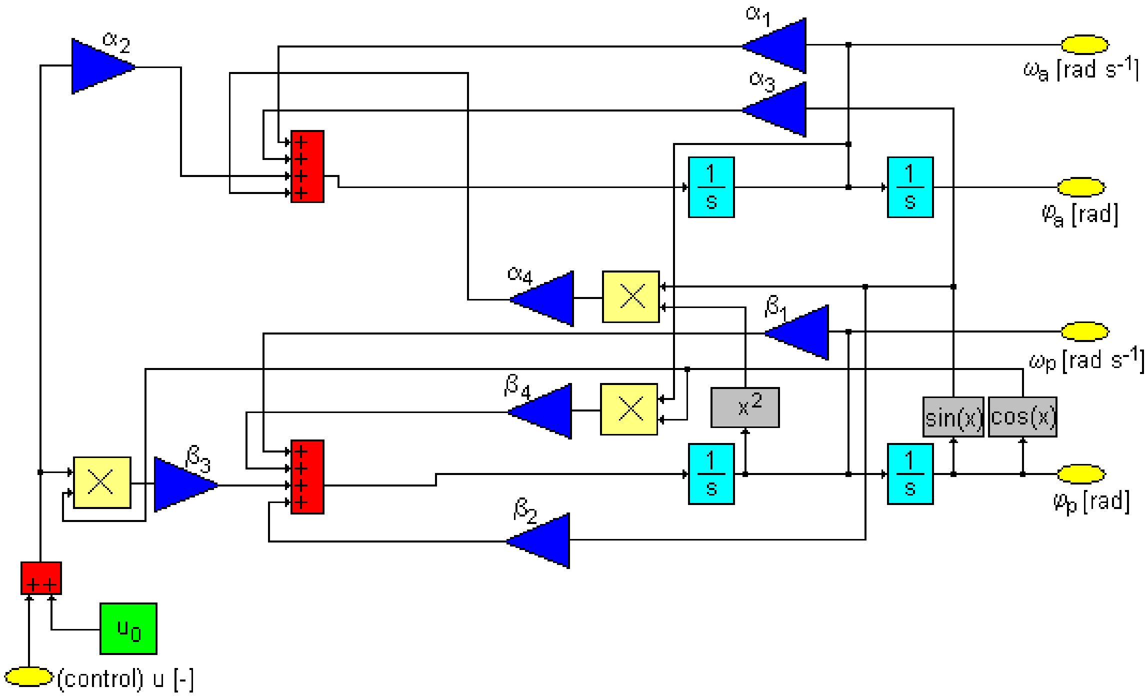

Fig. 5. Simulation diagram of the mathematical model (10)

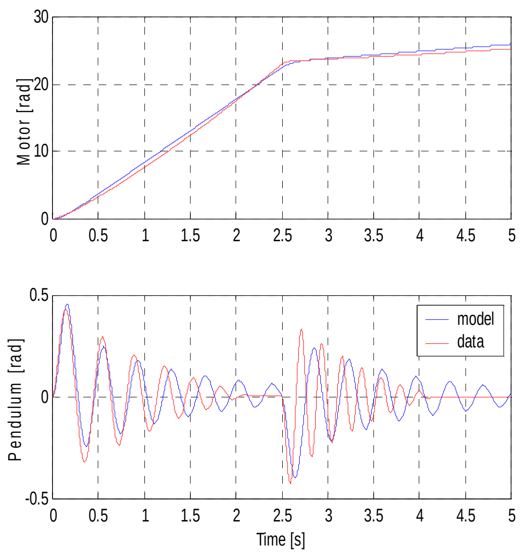

Fig. 6. Model (10) validation

C. *Model Analysis*

Design of control algorithms is based on the mathematical model (10). After linearization of the nonlinear model (10) around the state

$$x_0 = (0,0,0,\pi)^T.$$

the LTI system (11) is obtained. The LTI system matrices are as follows

$$A = \begin{bmatrix} -9 & 0 & 0 & 0 \\ 12 & -1.5 & 0 & 600 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 800 \\ -1450 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

with the matrix A eigenvalues

$$\lambda_1 = 0, \; \lambda_2 = -9, \; \lambda_3 = -25.25, \; \lambda_4 = 23.75.$$

Computing the observability matrix P, we have

$$P = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, rank(P) = 4,$$

therefore it is feasible to build the observer.

Computing the controllability matrix R, we have

$$R = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}, \; rank(R) = 4,$$

therefore it is feasible to use the state feedback.

IV. BALANCING THE PENDULUM

Balancing the Rotating Pendulum at the upright unstable position is a well known control task. In this Chapter students design the controllers using Linear and Nonlinear Control Theory (Linear State Feedback, Nonlinear State Feedback) for stabilization the pendulum at the upright position while eliminating small external disturbances. Finally students compare the designed control algorithms and explain differences against the theory.

A. Observer Design

If we want to control the Pendulum, we need to know state variables ω_a, ω_p, φ_a, and φ_p from equations (10) or their estimations. In this work we use the linear observer to reconstruct unmeasured states. For design of the observer the linearized model of the system should be known. The output vector of the system (10) is

$$y = (\varphi_a, \varphi_p)^T$$

Because only the angles of the arm and the pendulum are measured, the arm and the pendulum speed of rotation are to be estimated. We will design the observer based on the pole placement method. In this case we need to know the matrix L of the observer gains. We specify the desired poles λ_{o1}, λ_{o2}, λ_{o3}, and λ_{o4} of the observer. To calculate the matrix of gains we have to solve the following equation

$$\det(\lambda I - A + LC) = \\ = (\lambda - \lambda_{o1})(\lambda - \lambda_{o2})(\lambda - \lambda_{o3})(\lambda - \lambda_{o4}).$$

The desired poles of the control system with linear state feedback LSF are chosen to be the stable poles of the LTI system (11), i.e. λ_2 and λ_3. The remaining two poles are chosen to be stable and near to the limit of stability. After simulations and real experiments the poles of the State Feedback were set as follows

$$\lambda_2 = -9,\ \lambda_3 = -25,\ \lambda_1 = \lambda_4 = -3.$$

Validation of the linear state feedback is shown in Fig.7, where the real experiment is compared with the simulaton.

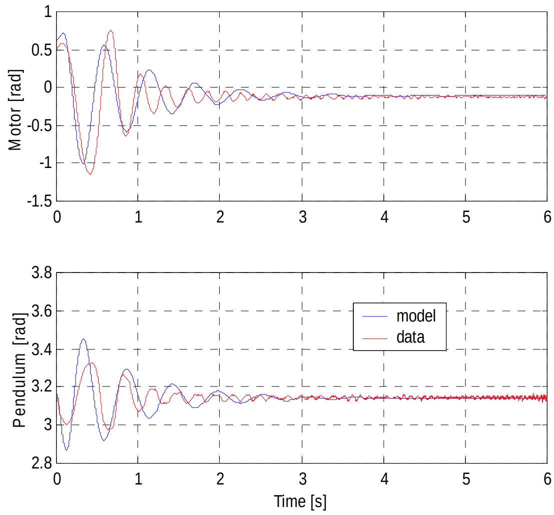

Fig. 7. Linear State Feedback validation

B. Nonlinear State Feedback

The nonlinear control law is based on transformation of the nonlinear state model (10) to a linear equation [2]

$$\ddot{\varphi}_p + A\dot{\varphi}_p + B\varphi_p = C. \tag{12}$$

Substituting the second equation (10) into (12), we obtain

$$\begin{aligned} &\ddot{\varphi}_p + A\dot{\varphi}_p + B\varphi_p - C \\ &= \ddot{\varphi}_p - \beta_1 \dot{\varphi}_p - \beta_2 \sin\varphi_p \\ &-\beta_3 (u + u_0)\cos\varphi_p - \beta_4 \cos\varphi_p \omega_a. \end{aligned} \tag{13}$$

The equation (13) gives the required control law

$$\begin{aligned} u = (&b_1 \dot{\varphi}_p + b_2 \sin\varphi_p + b_3 \cos\varphi_p \\ &+ b_4 \cos\varphi_p \omega_a + b_5 \varphi_p + b_6) / \beta_3 \cos\varphi_p, \end{aligned} \tag{14}$$

where

$$b_1 = -A - \beta_1,\ b_2 = -\beta_2,\ b_3 = -\beta_3 u_0, \\ b_4 = -\beta_4,\ b_5 = -B,\ b_6 = C.$$

Because the pendulum is balancing at the upright position, it is necessary to set $\varphi_p(\infty) = \pi$. Therefore it is possible to compute the parameter C from (12)

$$C = \pi B.$$

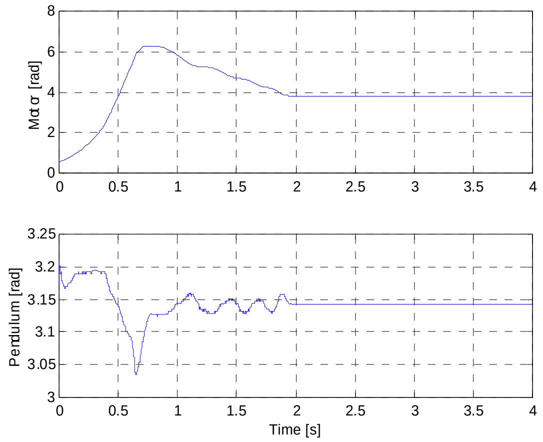

Fig. 8. The real Nonlinear State Feedback

The parameters A and B of the linear pendulum equation (12) may be obtained from characteristic equation for designed poles. The poles must be stable and chosen with respect to the poles of the LTI system (11). Real-time experiments and simulations help to choose the poles as follows

$$\lambda_1 = -10 + 60i, \ \lambda_2 = -10 - 60i.$$

Then $A = 20$ and $B = 1000$ are obtained from (12) which finally determines the control law.

V. SWINGING UP THE PENDULUM

In this section students design a control law for swinging up the pendulum. The control law is based on minimum time strategy, which minimizes total energy of the pendulum. The task for students is to design parameters of this control law and to balance the pendulum after swinging.

The control law [3] for input of the system is

$$u = K_u \,|\, E_{ref} - E \,|\, \omega_p \cos\varphi_p , \qquad (15)$$

where E_{ref} is total energy in stable position of the pendulum

$$E_{ref} = K_p ,$$

and E is the total energy of motion of the pendulum

$$E = \frac{1}{2} J_p \omega_p^2 + \frac{1}{2} K_p (1 - \cos\varphi_p). \qquad (16)$$

The control law (15) may be written as

$$u = K_u \,|\, K_1 (1 + \cos\varphi_p) - K_2 \omega_p^2 \,|\, \omega_p \cos\varphi_p . \qquad (17)$$

where $K_1 = \frac{1}{2} K_p$ and $K_2 = \frac{1}{2} J_p$.

Setting of parameters of the control law for swinging up the pendulum, having the form of the equation (17), is based on constants K_p and J_p which can be found in the Appendix 1. The remaining parameter K_u of the control law (17) is to be determined in simulations and real experiments. Finally, the parameters K_1 and K_2 from (17) get the following values

$$K_1 = 0.05 \ \mathrm{s}^{-2}, \ K_2 = 1.15 \ 10^{-3} \ \mathrm{kg} \ \mathrm{m}^2.$$

After finishing experiments it is required to determine the parameter K_u and to modify setting of parameter K_2 which results in

$$K_u = 0.05, \ K_2 = 1.05 \ 10^{-5} \ \mathrm{kg} \ \mathrm{m}^2.$$

In real control experiments with the Rotating Pendulum introduced in Chapter IV and Chapter V *Real-Time Toolbox* for *MATLAB* was used. Selection of the sampling period T_S for real-timer control experiments is limited by characteristics of the *RT Toolbox* and frequency range of the LTI system (11) stabilized by linear state feedback. Nyquist frequency read from Bode plot of the controled system is imporatnt for estimating the lower limit of the sampling rate. For $\omega_N = 13.3$ *rad* s^{-1}, the upper limit of the sampling period, respecting Shannon Theorem is

$$T_{SH} = 0.2362 \ s$$

The upper limit of the sampling rate is determined by physical constraints due to the PC hardware, the operating system Matlab and *RT Toolbox* characteristics. For the lower limit of the samling period we get

$$T_{SL} = 1 \ ms.$$

This lower limit of the sampling period was used in real-time experiments.

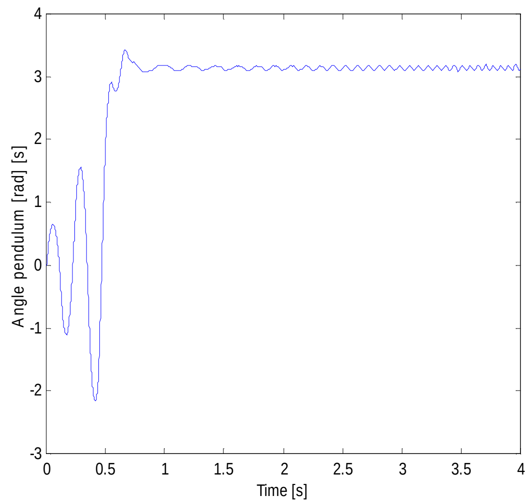

Fig. 9. Real swinging up and balancing the pendulum by Linear State Feedback

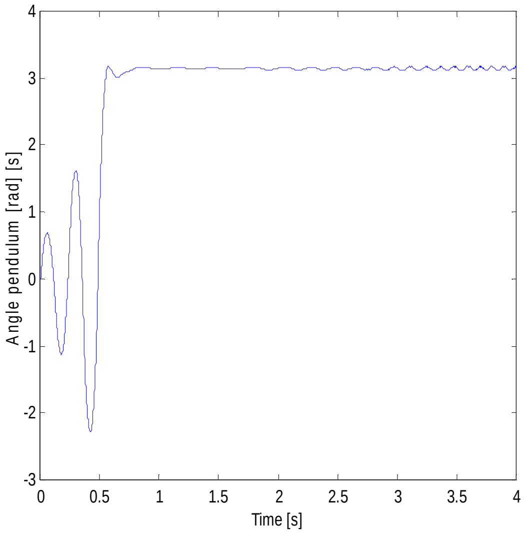

Fig. 10. Real swinging up and balancing the pendulum by Nonlinear State Feedback

In the Fig. 9 and Fig.10 the real swing ups and control of the pendulum using Linear and Nonlinear State Feedback are presented.

VI. CONCLUSION

The important stimulation for this work was to provide students with the material to learn modeling, simulations and control of the Nonlinear Systems efficiently and in the attractive way.

Chapter II introduces the efficient method for modeling the nonlinear mechanical systems. Modeling of the Rotating Pendulum shows straightforward application of Lagrange Equations for this system. Chapter III introduces a task of modeling and simulation of the Nonlinear System including discussion on selection of suitable ODE solver. The experiments which follow show the Model Validation. Chapters IV and V describe control tasks of the Rotating Pendulum.

REFERENCES

[1] Ernest, P. (2002). *Control of a Double Rotating Pendulum*. Diploma Thesis, Czech Technical University in Prague, Czech Republic.

[2] Winklund, M., Kristenson, A. and Astrom, K., J. (1994). *A new strategy for swinging up An Inverted Pendulum*. In: INSPEC, Vol. 9, 151-154, Lund, Sweden.

[3] Ling, K.,V., Lai, Y.,K. and Chew, K.,B. (2000). *An Online Internet Laboratory for Control Experiments*. IEEE Trans. On Fuzzy Syst., Vol. 8, No. 6, 1-4, Nanyang, Singapore.

[4] Furuta, T., T., K., Suzuki, S., H., Sugiki, A. (2002). *Swing-up Control of Inverted Pendulum by Periodic Input*. IFAC, 15th Triennial World Congress, Barcelona, Spain.

APPENDIX 1

Parameter	*Value*	
J_a	$6.5\ 10^{-3}$	kg m^2
J_p	$2.3\ 10^{-3}$	kg m^2
J	$1.5\ 10^{-3}$	kg m^2
b_a	$4.5\ 10^{-2}$	kg m^2s^{-1}
b_p	$8.8\ 10^{-5}$	kg m^2s^{-1}
K_a	3.75	kg m s^{-2}
K_p	0.1	s^{-2}

APPENDIX 2

Parameter	*Value*
α_1	- 9
α_2	800
α_3	0
α_4	0
α_5	0
β_1	- 1.5
β_2	- 600
β_3	1450
β_4	- 12
β_5	0