Numerical Methods for Partial Eigenvalue and Eigenstructure Assignments in Vibrating Structures

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***Abstract*— Novel computational methods for feedback control in distributed parameter and matrix second-order systems, modeling a wide range of vibrating structures, are described.**

Unlike in standard engineering practice, the methods allow the computations to be carried out in their own mathematical formulations. Furthermore, the methods can be numerically implemented using only finite-dimensional control and numerically viable computational techniques. Thus these are practical methods for control, and stabilization of large vibrating structures.

I. INTRODUCTION

The general model for the vibration of distributed parameter systems, arising in a wide range of applications, especially in the design and analysis of vibrating structures, such as bridges, highways, buildings, airplanes, etc., can be written in the form

$$\mathbf{M}(x)\frac{\partial^2\nu(t,x)}{\partial t^2}+\mathbf{C}(x)\frac{\partial\nu(t,x)}{\partial t}+\mathbf{K}(x)\nu(t,x)=0, \quad (1.1)$$

where $\mathbf{M}, \mathbf{C} = \mathbf{D} + \mathbf{G}$ and $\mathbf{K}$ are differential operators in the x-domain (spatial domain) of the displacement function $\nu(t,x)$, where for all the t the $\nu(t,x)$ belong to some Hilbert space $\mathbb{H}$, that accounts for the boundary conditions of (1.1). The operators $\mathbf{M}, \mathbf{K}, \mathbf{D}$ and $\mathbf{G}$ are, respectively, called *mass, stiffness, damping and gyroscopic* operators. In many practical applications, $\mathbf{M}$ is self-adjoint and positive definite, $\mathbf{D}$ is self-adjoint and $\mathbf{G}$ is skew-symmetric.

Though it is desirable to solve a vibration problem in its own natural distributed parameter setting, very often in practice, due to lack of effective numerical methods to handle the system (1.1) directly, it is discretized to a finite-dimensional matrix second-order system of the form:

$$M\ddot{x}(t)+C\dot{x}(t)+Kx(t)=0, \quad (1.2)$$

where $M, C = D + G, K \in \mathbb{R}^{n\times n}$ and $\dot{x}(t)$, respectively, denote the first and second derivatives of the time dependent vector $x(t)$.

A vibration control problem is solved and a control law is implemented on the discretized system (1.2) and then applied to a real-life vibrating structure modeled by (1.1).

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A solution obtained this way naturally suffers from discretization error and after all, such a solution is just a finite-dimensional approximation of an infinite dimensional problem. Unfortunately, computational methods for solving vibration control problems using even this second-best alternative are not very well established. There are two standard approaches: solutions via a first-order realization and Independent Modal Space Control (IMSC) approach. Both of these approaches have severe engineering and computational disadvantages (See [9] and [10]).

If the standard first order transformation of (1.2)

$$\dot{z}(t)=\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C\end{pmatrix}z(t)+\begin{pmatrix} 0 \\ M^{-1}B\end{pmatrix}u(t),$$

$$\text{where } z(t)=\begin{pmatrix} x(t) \\ \dot{x}(t)\end{pmatrix}$$

is used, then the matrix M has to be inverted, and, if it is ill-conditioned, then the state matrix will not be computed accurately. Furthermore, all the exploitable properties, such as the definiteness, sparsity, bandness, etc. of the coefficient matrices M, D, and K, usually offered by a practical problem, will be completely destroyed. The use of a nonstandard first-order transformation, such as

$$\begin{pmatrix} M & 0 \\ 0 & M\end{pmatrix}\dot{z}(t)=\begin{pmatrix} 0 & M \\ -K & -C\end{pmatrix}z(t)+\begin{pmatrix} 0 \\ B\end{pmatrix}u(t)$$

will give rise to a *descriptor system* of the form $E\dot{z}(t) = Az(t)+\hat{B}u(t)$, and the eigenvalue and eigenstructure assignment methods for the descriptor systems, especially, when the matrix E is ill-conditioned, are not well developed.

Furthermore, with this formulation, though symmetry is preserved, other exploitable properties, such as positive definiteness, etc., are destroyed.

The *independent modal space control* (IMSC) *approach* also suffers from some serious computational difficulties and is *almost impossible to implement in practice*. The basic idea here is to decouple the problem into a set of n independent problems, solve each of these independent problems separately, and then piece the individual solutions together to obtain a solution of the given problem. The implementation of this idea requires knowledge of the **complete spectrum** and the associated eigenvectors of the pencil $P(\lambda) = \lambda^2 M + \lambda(D + G) + K$. Unfortunately, numerical methods for the quadratic eigenvalue problem are not well developed, especially for large and sparse problems. The state-of-the-art computational techniques are capable of computing only a few selected extremal eigenvalues and eigenvectors [12]). Furthermore, for decoupling of the right-hand sides of the associated modal equations, some stringent commutativity conditions need to be imposed (see [9]), which are almost impossible to satisfy in practice.

In the last few years, the author and his collaborators have developed **"Direct and Partial Modal"** algorithms for solving important feedback control problems in matrix second-order and distributed parameter systems [see ([3, 4, 6])]. The algorithms are direct, because they solve the problem directly in its given mathematical formulation; that is, in case the model is a discretized second-order model, no transformation to a first-order realization is invoked, and if the model is a DPS, then no discretization to a second-order system is necessary. They are partial-modal, because only a part of the spectrum and the corresponding eigenvectors (eigenfunctions) of the associated quadratic eigenvalue problem are necessary in implementing them.

The direct and partial-modal nature of these feedback algorithms make them suitable for practical use in stabilization and control of large vibrating structures, such as Large Space Structures (LSS), power systems, computer networks, aircrafts and others.

In this paper, we briefly review some of these algorithms. Specifically, we present, without proof, our new algorithms for partial eigenvalue assignment in DPS and partial eigenvalue and eigenstructure assignments in matrix section-order systems. The results on numerical experiment with real-life examples are given.

II. PARTIAL EIGENVALUE ASSIGNMENT IN DPS

The Partial Eigenvalue Assignment (PEVA) problem in the DPS (1.1) is the problem of re-assigning a small part of the spectrum, responsible for undesirable dangerous responses, such as resonance or instability, of the associated open-loop operator pencil

$$P(\lambda) = \lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}, \tag{2.1}$$

by a suitable feedback, in such a way that the remaining large part is not affected.

The problem is certainly practical in the sense that in most practical situations dealing with large problems, only a small part of the spectrum is troublesome, and thus, it makes sense to change that part only by feedback without solving a large full-order eigenvalue assignment problem. Furthermore, solving a large eigenvalue assignment is unpractical and the existing numerical methods are not suitable for large and sparse problems.

Mathematically, the PEVA in DPS is defined as follows:

Consider the controlling forces of the form

$$\sum_{k=1}^{m} \mathbf{f}_{1k}(x), \frac{\partial \nu(t,x)}{\partial t} + (\mathbf{f}_{2k}(x), \nu(t,x))\mathbf{b}_k(x),$$

where the functions $b_1(x), \ldots, b_m(x)$ are the *control functions*, and $\mathbf{f}_{1k}, \mathbf{f}_{2k} \in \mathcal{H}, k = 1, \ldots, m$ are the *velocity* and *position feedback functions*, respectively. Like in the finite-dimensional case, the spectrum and invariant subspace of the infinite-dimensional operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ govern the dynamics of (1.1) with these applied control forces. In infinite-dimensional setup, eigenvalue-eigenfunction pairs (λ, v) satisfy $\mathbf{P}(\lambda)v = 0$.

Assume that: (i) M is nonsingular, (ii) the open-loop operator pencil $\mathbf{P}(\lambda)$ has discrete spectrum without finite accumulation points and every eigenvalue of $\mathbf{P}(\lambda)$ is semi-simple, and (iii) the system of eigenfunctions of $\mathbf{P}(\lambda)$ is two-fold complete (see [7, 8]).

Let $\{\lambda_1, \ldots, \lambda_m\}$ be a finite small set of unwanted (bad) eigenvalues (*assumed to be available from measurements*) of $\mathbf{P}(\lambda)$ that are to be replaced by a user-chosen set $\{\mu_1, \mu_2, \ldots, \mu_m\}$.

The partial eigenvalue assignment problem in Distributed Parameter Systems is defined as follows:

Find **real** feedback functions $\mathbf{f}_{11}, \ldots, \mathbf{f}_{1m}$ and $\mathbf{f}_{21}, \ldots, \mathbf{f}_{2m}$ such that the spectrum of the closed-loop pencil

$$\begin{aligned} \mathbf{P}_c(\lambda)\phi = \lambda^2\mathbf{M}\phi + \lambda\left(\mathbf{C}\phi - \textstyle\sum_{k=1}^{m}(\mathbf{f}_{1k}, \phi)\mathbf{b}_k\right) + \\ \left(\mathbf{K}\phi - \textstyle\sum_{k=1}^{m}(\mathbf{f}_{2k}, \phi)\mathbf{b}_k\right) \end{aligned} \tag{2.2}$$

is the set $\mathcal{S} = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots\}$.

We now present our solution of the above problem in algorithmic form. The proof is based on an orthogonality relation between the eigenfunctions of the pencil (2.1), proved in [5, 11].

Algorithm 2.1 (Parametric Solution to the Partial Eigenvalue Assignment Problem in Distributed Parameter System)

Inputs:

a) The differential operators $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ of the open-loop pencil (2.1).
b) The m control functions $\mathbf{b}_1, \ldots, \mathbf{b}_m$.
c) The set of scalars $\{\mu_1, \ldots, \mu_p\}$, closed under complex conjugation.
d) The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \lambda_2, \ldots\}$ and the associated eigenfunction set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Outputs:

The feedback functions $\mathbf{f}_1, \ldots, \mathbf{f}_m$ and $\mathbf{f}_1, \ldots, \mathbf{f}_{2m}$ such that the spectrum of the closed-loop operator pencil (2.2) is the set $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \ldots\}$.

Assumptions:

a) The control functions $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are linearly independent.
b) The open-loop quadratic operator pencil $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ with control functions $\mathbf{b}_1, \ldots, \mathbf{b}_m$

is partially controllable with respect to the eigenvalues λ_1, …, λ_p.

c) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \lambda_{p+1}, \ldots\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint.

Step 1. Form $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$, $\mathbf{V}_1 = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and $\Lambda_{c1} = \text{diag}(\mu_1, \ldots, \mu_p)$.
Step 2. Choose arbitrary $m \times 1$ vectors $\gamma_1, \ldots, \gamma_p$ in such a way that $\overline{\mu_j} = \mu_k$ implies $\overline{\gamma_j} = \gamma_k$ and form $\Gamma = (\gamma_1, \ldots, \gamma_p)$.
Step 3. Solve the following Sylvester equation for Z_1:

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (\mathbf{v}_1, \mathbf{b}_1) & \cdots & (\mathbf{v}_1, \mathbf{b}_m) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_p, \mathbf{b}_1) & \cdots & (\mathbf{v}_p, \mathbf{b}_m) \end{pmatrix} \Gamma .$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\gamma_1, \ldots, \gamma_p$.
Step 4. Solve $\Phi Z_1 = \Gamma$ for $\Phi = (\Phi_{ij})$.
Step 5. If none of the $\lambda_1, \ldots, \lambda_p$ is zero, form for all $k = 1, \ldots, m$

$$\begin{aligned} \mathbf{f}_{1k} &= \sum_{j=1}^{p} \overline{\phi}_{kj} \mathbf{M}^* \mathbf{v}_j, \text{ and} \\ \mathbf{f}_{2k} &= -\sum_{j=1}^{p} (\overline{\phi}_{kj}/\overline{\lambda}_j) \mathbf{K}^* \mathbf{v}_j , \end{aligned}$$

otherwise form for all $k = 1, \ldots, m$,

$$\begin{aligned} \mathbf{f}_{1k} &= \sum_{j=1}^{p} \overline{\phi}_{kj} \mathbf{M}^* \mathbf{v}_j, \quad \text{and} \\ \mathbf{f}_{2k} &= \sum_{j=1}^{p} \overline{\phi}_{kj} (\overline{\lambda}_j \mathbf{M}^* \mathbf{v}_j + \mathbf{C}^* \mathbf{v}_j) . \end{aligned}$$

III. Partial Eigenvalue and Eigenstructure Assignment in Matrix Second-order Systems

If the mathematical model is the finite-element generated discretized second-order system (1.2), then the PEVA problem is defined as follows:
Given the matrices $M, C,$ and K of the model (1.2), the control matrix B of order $n \times m$ and the self-conjugate set $\{\mu_1, \cdots, \mu_p\}$, find real feedback matrices F_1 and F_2 such that the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ has the spectrum $\mu_1, \cdots, \mu_p,\ \lambda_{p+1}, \cdots, \lambda_{2n}\}$.

In this case, Algorithm 2.1 reduces to the following:

Algorithm 3.1 (Parametric Solution to the Partial Eigenvalue Assignment Problem in Matrix Second-order Systems).

Inputs:

a) The $n \times n$ matrices M, C, and K.

b) The $n \times m$ control matrix B.

c) The set $\{\mu_1, \ldots, \mu_p\}$, closed under complex conjugation.

d) The self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_{2n}\}$ and the associated right eigenvector set $\{y_1, \ldots, y_p\}$.

Outputs:
The feedback matrices F_1 and F_2 such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ is $\{\mu_1, \ldots, \mu_p;\ \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

Assumptions:

a) M is nonsingular and B has full rank.

b) The quadratic open-loop pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ with control matrix B is partially controllable with respect to the eigenvalues λ_1, …, λ_p.

c) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint.

Step 1. Form $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$, $Y_1 = (y_1, \ldots, y_p)$, and $\Lambda_{c1} = \text{diag}(\mu_1, \ldots, \mu_p)$.
Step 2. Choose arbitrary $m \times 1$ vectors $\gamma_1, \ldots, \gamma_p$ in such a way that $\overline{\mu_j} = \mu_k$ implies $\overline{\gamma_j} = \gamma_k$ and form $\Gamma = (\gamma_1, \ldots, \gamma_p)$.
Step 3. Find the unique solution Z_1 of the Sylvester equation

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B \Gamma .$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\gamma_1, \ldots, \gamma_p$.
Step 4. Solve $\Phi Z_1 = \Gamma$ for Φ.
Step 5. If none of the $\lambda_1, \ldots, \lambda_p$ is zero, form

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = -\Phi \Lambda_1^{-1} Y_1^H K ,$$

otherwise form

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C) .$$

Partial Eigenstructure Assignment Problem for Matrix Second-order Systems
Given

1) *Real $n \times n$ matrices $M = M^T > 0,\ C,\ K$.*
2) *The self-conjugate subset $\{\lambda_1, ..., \lambda_p\},\ p < 2n$ of the set of the open-loop eigenvalues $\{\lambda_1, ..., \lambda_{2n}\}$ of the pencil $P(\lambda)$ and the corresponding left eigenvector set $\{y_1, ..., y_p\}$.*
3) *The self-conjugate sets of scalars $\{\mu_1, ..., \mu_p\}$ and the set of vectors $\{x_{c1}, ..., x_{cp}\}$, such that $\mu_j = \overline{\mu k}$ implies $x_{cj} = \overline{x_{ck}}$.*

Find

Real control matrix B of order $n \times m (m < n)$ and real feedback matrices F_1 and F_2 of order $m \times n$ such that the spectrum of the closed-loop pencil (3) is the set $S = \{\mu_1, ..., \mu_p; \lambda_{p+1}, ..., \lambda_{2n}\}$ with $\{x_{c1}, ..., x_{cp}; x_{p+1}, ...x_{2n}\}$ as the associated eigenvector set, where $x_{p+1}, ..., x_{2n}$ are the eigenvectors of $P(\lambda)$ corresponding to $\lambda_{p+1}, ..., \lambda_{2n}$.

Algorithm 3.2 (An Algorithm for Partial Eigenstructure Assignment in Matrix Second-order Systems).
Inputs:

(a) The $n \times n$ matrices M, C, and K.
(b) The set of scalars $\{\mu_1, ..., \mu_p\}$ and the set of vectors $\{x_{c1}, ..., x_{cp}\}$, both closed under complex conjugation.
(c) The self-conjugate subset $\{\lambda_1, ..., \lambda_p\}$ of the open-loop spectrum $\{\lambda_1, ..., \lambda 2n\}$ and the associated right eigenvector set $\{y_1, ..., y_p\}$.

Outputs: The $n \times m$ control matrix B and the feedback matrices F_1 and F_2 such that the spectrum of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ is $\{\mu_1, ..., \mu_p; \quad \lambda_{p+1}, ..., \lambda_{2n}\}$ with the eigenvector matrix $X_c = (x_{c1}, ..., x_{cp}; x_{p+1}, ..., x_{2n})$.

Assumptions:
(a) M is nonsingular.
(b) The sets $\{\lambda_1, ..., \lambda_p\}, \{\lambda_{p+1}, ..., \lambda_{2n}\}$, and $\{\mu_1, ..., \mu_p\}$ are disjoint.

Step 1. Form $\Lambda_1 = \operatorname{diag}(\lambda_1, ..., \lambda_p), Y_1 = (y_1, ..., y_p), \Lambda_{c1} = \operatorname{diag}(\mu_1, ..., \mu_p)$, and $X_{c1} = (x_{c1}, ..., x_{cp})$.

Step 2. Form the matrix

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1}.$$

Stop if Z_1 is singular and conclude that the eigenstructure assignment with the given sets of eigenvalues and eigenvectors is not possible.

Step 3. Form the matrix T_c such that $T_c \Lambda_{c1} T_c^H$ is a real matrix.

Step 4. Form

$$B = (M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}) T_c^H,$$
$$F_1 = T_c Z_1^{-1} Y_1^H M, \text{ and}$$
$$F_2 = T_c Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C)$$

by solving the appropriate linear systems.

Computational and Engineering Features of the Algorithms:

As seen from Algorithms 2.1, 3.1 and 3.2, our new feedback scheme enjoys the following distinguished computational and engineering features:

(i) The computational requirements are minimal and the required tasks, namely, solutions of small Sylvester equations and linear algebraic systems, can be carried out in a numerically effective manner using the excellent state-of-the-art algorithms for small and dense problems [2].

(ii) The knowledge of only partial spectrum and associated eigenvectors (eigenfunctions) of the matrix (operator) pencil is sufficient to implement the scheme. These small number of eigenvalues and eigenvectors can be computed using the state-of-the-art computational techniques or can be measured in vibration laboratories.

(iii) Advantages can be taken of the exploitable structures, such as the sparsity, symmetry, bandness, etc., in a computational setting.

(iv) No spill-over phenomenon is guaranteed with mathematical proofs.

(v) The eigenvalue assignment algorithms are parametric in nature, which can be exploited to design numerically robust feedback schemes.

(vi) The algorithms are suitable for high-performance computing, since they are rich in BLAS-3 (Basic Linear Algebra Subroutines Level 3) computations.

IV. RESULTS OF NUMERICAL EXPERIMENTS

Some results of our numerical experiments on Algorithms 2.1, 3.1, and 3.2 are stated in this Section. The data for Algorithm 3.1 and 3.2 comes from a power plant obtained from the Benchmark Collections [1], and that for Algorithm 2.1 corresponds to a traveling string.

A. Vibrations of a Rotating Turbine Axle

Here we consider a large and sparse symmetric definite quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ of order $n = 211$ modeling a rotating axle in a power plant, where masses are assumed to be symmetric with respect to the axle. This is a damped *non-gyroscopic* model; that is $C = D$, $G = 0$.

The matrix

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n)$$

is positive definite and the damping and stiffness matrices given by

$$D = (d_{ij}), \text{ where } d_{ij} = \begin{cases} -\gamma_i & , \quad i+1=j \\ \gamma_{i-1} + \delta_i + \gamma_i & , \quad i=j \\ -\gamma_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$K = (k_{ij}), \text{ where } k_{ij} = \begin{cases} -\kappa_i & , \quad i+1=j \\ \kappa_{i-1} + \kappa_i & , \quad i=j \\ -\kappa_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

with $\gamma_0 = \gamma_n = \kappa_0 = \kappa_n = 0$ are both symmetric tridiagonal.

Using the data provided in the Benchmark Collection, the eigenvalues of the uncontrolled system are plotted, and it is seen that the decay rate of the vibrations of the axle is governed by its most unstable eigenvalue: $\lambda_1 = -1.3734 \cdot 10^{-6}$, whereas the other eigenvalues have much better stability properties, namely: $\operatorname{Re} \lambda_j \leq -0.016267\,,\ j = 2, 3, \ldots, 422$.

Since the largest contribution to shape of the transient response of the vibrating system is generated by the eigenvector that corresponds to the most unstable eigenvalue of the system, we use Algorithm 3.1 to assign λ_1 to $\mu_1 = -0.016$ and then Algorithm 3.2 to assign the eigenvector corresponding to λ_1 to the vector

$$x_{c1} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T\,,\ n = 211\,.$$

Eigenvalue assignment:
Algorithm 3.1 was applied with the control matrix $B = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^T$ and choosing the matrix $\Gamma = (-0.51454, -0.85747)^T)$ randomly.

The computed feedback matrices F_1 and F_2 are such that μ_1 was assigned accurately, and the two norm of the difference between the other eigenvalues of the corresponding open-loop and closed-loop pencils is about 1.7×10^{-6}. The 2×422 matrices F_1 and F_2 are such that $\|F_1\|_2 \approx 116$ and $\|F_2\| \approx 22$. Furthermore,

$$\frac{\|F_1\|_2}{\|C\|_2} \approx 0.57 \text{ and } \frac{\|F_2\|}{\|K\|_2|} \approx 1.5 \times 10^{-11}.$$

Eigenstructure Assignment: Algorithm 3.2 produces the 211×1 control matrix B with $||B||_2 \approx 2$ and the 1×211 feedback matrices F_1 and F_2 with $||F_1||_2 \approx 7.2$ and $||F_2||_2 \approx 1.4$, respectively, such that the prescribed eigenvalue and eigenvector are assigned correctly. Moreover, the relative changes in damping and stiffness matrices are given by:

$$\frac{||BF_1||_2}{||C||_2} \approx 0.07 \text{ and } \frac{||BF_2||_2}{||K||_2} \approx 1.8 \cdot 10^{-12}.$$

This shows that control forces required to suppress vibrations assigning the same eigenvalue are much less than those required by eigenvalue assignment with a priori given control matrix B. *To achieve this, however, we need more sophisticated actuators than those needed to implement the simple control force used in eigenvalue assignment.*

The 2-norms of the differences between the remaining eigenvalues of the open-loop pencil and the corresponding ones of the closed-loop pencil this time is about $2.2 \cdot 10^{-6}$ (MATLAB was used to compute the eigenvalues).

B. Small Oscillation of a Traveling String

Consider a gyroscopic distributed parameter system modeling the small oscillations of a uniform string traveling with constant velocity $\gamma < c$ over two fixed supports at $x = 0$ and $x = L$.

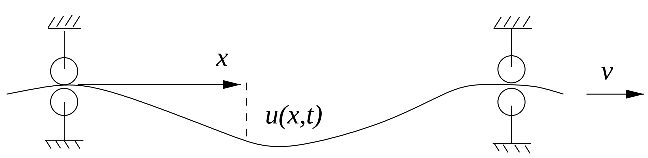

Figure 1. Small Oscillations of Traveling String

The motion of the moving string is governed by the partial differential equation

$$\nu_{tt} + 2\gamma\nu_{xt} + (\gamma^2 - c^2)\nu_{xx} = 0,$$

where $0 < x < L,\ t > 0,\ \gamma^2 < c^2$, with boundary conditions given by $\nu(o,t) = \nu(L,t) = 0$. With $L = 1,\ \gamma = 1/2$, and $c = 1$, the operators $\mathbf{M}, \mathbf{G}$, and $\mathbf{K}$, can be defined as

$$\mathbf{M}v = v, \quad \mathbf{G}v = \frac{\partial v}{\partial x}, \quad \mathbf{K}v = \frac{3}{4}\frac{\partial^2 v}{\partial x^2},$$

where $v(0) = v(1) = 0$. With respect to the scalar product $(v, w) = \int_0^1 \overline{v(x)}w(x)dx$, it can be easily shown that the operators $\mathbf{M}, \mathbf{G}$, and $\mathbf{K}$ have the following properties:

$$\mathbf{M}^* = \mathbf{M}, \mathbf{G}^* = -\mathbf{G} \text{ and } \mathbf{K}^* = \mathbf{K}.$$

The eigenvalues of $P(\lambda)$ are: $\lambda_k = \frac{3}{4}\pi k i, \quad k = \pm 1,\ \pm 2, \ldots$ and their corresponding left eigenfunctions are: $\mathbf{v}_{ck}(x) = e^{\frac{3}{2}\pi kix} - e^{-\frac{1}{2}\pi kix}$ where $0 \le x \le 1$. Algorithm 2.1 is used to assign λ_1 to $\mu_1 = -1 + i$ and $\overline{\lambda}_1$ to $\overline{\mu_1}$, choosing the two control functions $\mathbf{b}_1(x) = 1$ and $\mathbf{b}_2(x) = \sin(\pi x)$, where $0 \le x \le 1$. The step-wise results of our implementations are given in the following:

Step 1. $\Lambda_1 = \text{diag}(\frac{3}{4}\pi i, -\frac{3}{4}\pi i), \quad V_1 = (e^{\frac{3}{2}\pi ix} - e^{-\frac{1}{2}piix}, e^{-\frac{3}{2}\pi ix} - e^{\frac{1}{2}\pi ix})$ and

$$\Lambda_{c1} = \text{diag}(-1 + i,\ -1 - i).$$

Step 2. Choose

$$\Gamma = \begin{pmatrix} -.20575 + .8342i & -.20575 - .8342i \\ .22626 + .45888i & .22626 + .45888i \end{pmatrix}.$$

Step 3. Solving the Sylvester equation for Z_1,

$$\Lambda_1 Z_1 - Z_1\Lambda_{c1} = \begin{pmatrix} (v_1, b_1) & (v_1, b_2) \\ (v_2, b_1) & (v_2, b_2) \end{pmatrix}\Gamma,$$

we obtain

$$Z_1 = \begin{pmatrix} .18844 + .36623i & -.14535 - .84357i \\ -.14535 + .84357i & .188844 - .36623i \end{pmatrix}.$$

Step 4. Solving $\Phi Z_1 = \Gamma$ for Φ_1, we obtain

$$\Phi = \begin{pmatrix} .82911 - .50588i & .82911 + .50588i \\ .25486 + .45099i & .25486 - .45099i \end{pmatrix}.$$

Step 5. The velocity feedback functions $\mathbf{f}_{11}$ and $\mathbf{f}_{12}$ are plotted in Fig. 2, and the position feedback functions $\mathbf{f}_{21}$ and $\mathbf{f}_{22}$ are plotted in Fig. 3.

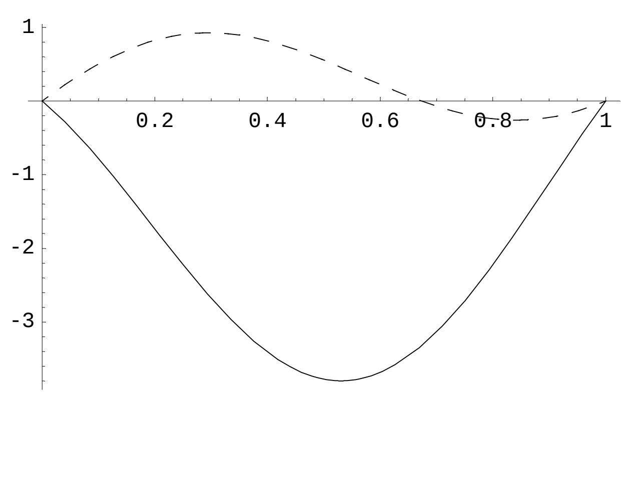

Figure 2. The Velocity Feedback Functions $\mathbf{f}_{11}$ and $\mathbf{f}_{12}$.

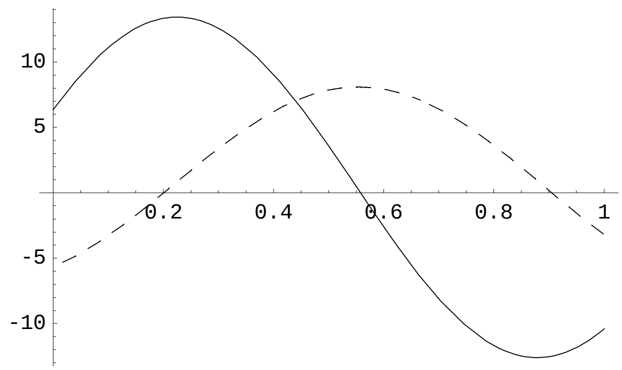

Figure 3. The Position Feedback Functions f_{21} and f_{22}.

The closed-loop operator pencil $P_c(\lambda)(\lambda)$ has the eigenvalues μ_1 and $\overline{\mu}_1$, with eigenfunctions given by

$$\begin{aligned} w_{c1} = \; & (0.4171 + 0.10287i) \\ & -(0.23671 + 0.20962i)e^{-2(1+i)x} \\ & -(0.19416 - 0.07776i)e^{\frac{2}{3}(1+i)x} \\ & -(0.0088786 - 0.037267i)e^{-\pi i x} \\ & +(0.022656 - 0.0082765i)e^{\pi i x} \end{aligned}$$

and $\overline{w_{c1}(x)}$, respectively. Furthermore, the eigenvalues $\lambda_k, k = \pm 2, \pm 3, \ldots$, of the open-loop pencil $P(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}$ remain unchanged.

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