STATIC H_2 AND H_∞ OUTPUT-FEEDBACK OF STATE MULTIPLICATIVE LINEAR SYSTEMS

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Keywords: Stochastic static H_∞ output-feedback, state-multiplicative systems, robust output-feedback.

Abstract

A linear parameter dependent approach for designing a static output- feedback controller for linear time-invariant systems with multiplicative noise which achieves a minimum bound on either the stochastic H_2 or the H_∞ performance level is introduced. A solution is obtained also for the case where, in addition to the stochastic parameters, the system matrices reside in a given polytope. The theory developed is demonstrated by a simple example.

1 Introduction

The analysis and design of controllers for systems with stochastic uncertainties received much attention in the past (see [1] and the references therein) where mainly robust stability has been considered. Recently, a renewed interest in this problem has been encountered and solutions to the stochastic control problem have been derived that ensure a worst case performance bound in the H_∞ sense [2]-[9] .

Systems with parameter uncertainties that are modeled as white noise processes in a linear setting have been treated in [4]-[6], for the continuous-time case and in [3],[7]-[9] for the discrete-time case. Such models of uncertainties are encountered in many areas of applications (see [3] and the references therein) such as: nuclear fission and heat transfer, population models and immunology.

The deterministic static output-feedback problem has attracted the attention of many in the past [10], [11]. The main advantage of the static output-feedback is the simplicity of its implementation and the ability it provides for designing controllers of prescribed structure such as PI and PID. An algorithm has been presented recently by [12] which under some assumptions is found to converge in stationary infinite horizon examples without uncertainty. A sufficient condition for the existence of a solution to a special case of the static output-feedback problem has been obtained in [13]. This condition is in some cases quite conservative.

In the present paper we solve the stochastic state-multiplicative H_2 and H_∞ static control problems for discrete-time linear systems that contain state-multiplicative white-noise parameter uncertainties in the matrices of the state-space model that describes the system. We apply the simple design method of [14] for deriving the static output-feedback gain that satisfies prescribed stochastic H_2 and H_∞ performance criteria. Since a constant gain cannot be achieved in practice and all amplifiers have some finite bandwidth, we add, in series to the measured output of the system, a simple low-pass component with a transference close to identity. A parameter dependent Lyapunov function (LPD) is then described for the augmented system which is obtained by incorporating the states of the additional component into the state space description.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-positive definite). We denote by $L^2(\Omega, \mathcal{R}^n)$ the space of square-summable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the space of nonanticipative stochastic processes $\{f_k\}=\{f_k\}_{k\in[0,\infty)}$ in $\mathcal{R}^n$ with respect to $(\mathcal{F}_k)_{k\in[0,\infty)}$ satisfying

$$||f_k||_{\tilde{l}_2}^2=E\{\sum_0^\infty ||f_k||^2\}=\sum_0^\infty E\{||f_k||^2\}<\infty,\ \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n)$$

where $||\cdot||$ is the standard Euclidean norm. We denote by Tr$\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function.

2 Problem Formulation

We consider the following linear system:

$$\begin{array}{l} x_{k+1}=(A+D\nu_k)x_k+B_1w_k+(B_2+G\zeta_k)u_k \\ y_k=C_2x_k+D_{21}n_k, \quad x_0=0 \end{array} \tag{1}$$

with the objective vector

$$z_k=C_1x_k+D_{12}u_k, \tag{2}$$

where $x\in\mathcal{R}^n$ is the system state vector, $w\in\mathcal{R}^q$ is the exogenous disturbance signal, $n\in\mathcal{R}^p$ is the the measurement noise signal, $u\in\mathcal{R}^\ell$ is the control input, $y\in\mathcal{R}^m$ is the measured output and $z\in\mathcal{R}^r\subset\mathcal{R}^n$ is the state combination (objective function signal) to be regulated and where the variables $\{\zeta_k\}$ and $\{\nu_k\}$ are zero-mean real scalar white-noise sequences that satisfy:

$$E\{\nu_k\nu_j\}=\delta_{kj},\ E\{\zeta_k\zeta_j\}=\delta_{kj},\ E\{\zeta_k\nu_j\}=0,$$

$\forall k,j \geq 0$. The matrices in (1), (2) are constant matrices of appropriate dimensions.

We seek a constant output-feedback controller

$$u_k=Ky_k, \tag{3}$$

that achieves a certain performance requirement. We treat the following two different performance criteria.

- **The stochastic H_2 control problem :** Assuming that $\{w_k\}$, $\{n_k\}$ are realizations of a unit variance, stationary, white noise sequences that are un correlated with $\{\nu_k\},\ \{\zeta_k\}$, the following performance index should be minimized:

$$J_2\triangleq \underset{w,n}{E}\{||z_k||_{\tilde{l}_2}^2\}. \tag{4}$$

- **The stochastic H_∞ control problem:** Assuming that the exogenous disturbance signal is energy bounded, a static control gain is sought which, for a prescribed scalar $\gamma>0$ and for all nonzero $w\in\mathcal{R}^q,\ n\in\mathcal{R}^p$, guarantees $J_\infty<0$ where

$$J_\infty\triangleq ||z_k||_{\tilde{l}_2}^2-\gamma^2[||w_k||_{\tilde{l}_2}^2+||n_{k+1}||_{\tilde{l}_2}^2]. \tag{5}$$

Instead of considering the purely constant output-feedback controller (3) we consider the following proper (but not strictly) controller

$$\eta_{k+1}=-\epsilon\eta_k+(1+\epsilon)y_{k+1},\quad u_k=K\eta_k,\quad \eta_0=0 \tag{6a-c}$$

where $\eta\in\mathcal{R}^m$ and $\epsilon<<1$ is a positive scalar such that the transference from y to η is very close to I_m. The latter controller is introduced in order to facilitate the convexity of the design method below. It represents, however, the actual situation where 'constant' gains are achieved in practice by amplifiers of finite bandwidth.

Augmenting system (1) and (2) to include the states of (6a,b) we define the augmented state vector $\xi = col\{x,\ \eta\}$ and obtain the following representation to the closed-loop system.

$$\begin{array}{lcl} \xi_{k+1} &=& \tilde{A}_k\xi_k+\tilde{B}_k\tilde{w}_k+\tilde{D}_x\xi_x\nu_k+\tilde{G}_k\xi_k\zeta_k \\ z_k &=& \tilde{C}_k\xi_k, \quad \xi_0=0 \end{array} \tag{7a,b}$$

where:

$$\tilde{w}_k\triangleq\begin{bmatrix} w_k \\ n_{k+1}\end{bmatrix},\ \tilde{A}_k\triangleq\begin{bmatrix} A & B_2K \\ \bar{\epsilon}C_2A & \bar{\epsilon}C_2B_2K-\epsilon I_m\end{bmatrix},$$

$$\tilde{B}_k \triangleq \begin{bmatrix} B_1 & 0 \\ \bar{\epsilon}C_2B_1 & \bar{\epsilon}D_{21} \end{bmatrix}, \tilde{C}_k \triangleq \begin{bmatrix} C_1 & D_{12}K \end{bmatrix},$$

$$\tilde{D}_k \triangleq \begin{bmatrix} D & 0 \\ \bar{\epsilon}C_2D & 0 \end{bmatrix}, \quad \tilde{G}_k \triangleq \begin{bmatrix} 0 & GK \\ 0 & \bar{\epsilon}C_2GK \end{bmatrix}, \qquad \text{(7c-h)}$$

where $\bar{\epsilon} \triangleq (1+\epsilon)$.

We consider the following Lyapunov function

$$V_L = \xi^T \tilde{P} \xi \text{ with } \quad \tilde{P} = \begin{bmatrix} P & -\alpha^{-1}PC_2^T \\ -\alpha^{-1}C_2P & \hat{P} \end{bmatrix}, \ \tilde{P} > 0, \qquad \text{(8a-c)}$$

where $P \in \mathcal{R}^{n\times n}$ and $\hat{P} \in \mathcal{R}^{m\times m}$. The parameter α is a positive scalar tuning parameter.

2.1 The stochastic H_2 control problem

Applying to (8) the derivation of the stochastic H_2 control results [1] we obtain that $J_2 < \delta^2$ for a prescribed δ if there exists a positive definite solution $\tilde{Q} = \tilde{P}^{-1}$, where the latter is of the structure (8b), and $H \in \mathcal{R}^{(q+p)\times(q+p)}$ to the following Linear Matrix Inequalities (LMIs) :

$$\begin{bmatrix} -\tilde{Q} & \tilde{A}\tilde{Q} & 0 & 0 & 0 \\ \tilde{Q}\tilde{A}^T & -\tilde{Q} & \tilde{Q}\tilde{C}^T & \tilde{Q}\tilde{D}^T & \tilde{Q}\tilde{G}^T \\ 0 & \tilde{C}\tilde{Q} & -I_r & 0 & 0 \\ 0 & \tilde{D}\tilde{Q} & 0 & -\tilde{Q} & 0 \\ 0 & \tilde{G}\tilde{Q} & 0 & 0 & -\tilde{Q} \end{bmatrix} < 0, \begin{bmatrix} H & \tilde{B}^T \\ \tilde{B} & \tilde{Q} \end{bmatrix} > 0, \qquad \text{(9a-c)}$$

and $trace\{H\} < \delta^2$. Applying [14] it is found that $\tilde{Q}$ possesses the following structure:

$$\tilde{Q} = \begin{bmatrix} Q & C_2^T\hat{Q} \\ \hat{Q}C_2 & \alpha\hat{Q} \end{bmatrix}, \qquad (10)$$

where $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$. Substituting for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, $\tilde{G}$ into the latter LMIs we obtain the following:

Theorem 1 *Consider the system of (1),(2). The output-feedback control law (6b) achieves a prescribed H_2-norm bound $0 < \delta$, for some positive scalar $\epsilon << 1$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$, $Y \in \mathcal{R}^{\ell\times m}$ and $H \in \mathcal{R}^{(q+p)\times(q+p)}$ that, for some tuning scalar $0 < \alpha$, satisfy the following LMIs:*

$$\begin{bmatrix} -Q & -C_2^T\hat{Q} & \tilde{\Gamma}(1,3) & \tilde{\Gamma}(1,4) & 0 & 0 & 0 & 0 & 0 \\ * & -\alpha\hat{Q} & \tilde{\Gamma}(2,3) & \tilde{\Gamma}(2,4) & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & -C_2^T\hat{Q} & \tilde{\Gamma}(3,5) & QD^T & \tilde{\Gamma}(3,7) & C_2^TY^TG^T & \tilde{\Gamma}(3,9) \\ * & * & * & -\alpha\hat{Q} & \tilde{\Gamma}(4,5) & \hat{Q}C_2D^T & \tilde{\Gamma}(4,7) & \alpha Y^TG^T & \tilde{\Gamma}(4,9) \\ * & * & * & * & -I_r & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & -\alpha\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & * & -Q & -C_2^T\hat{Q} \\ * & * & * & * & * & * & * & * & -\alpha\hat{Q} \end{bmatrix} < 0,$$

$$\begin{bmatrix} H_{11} & H_{12} & B_1^T & \bar{\epsilon}B_1^TC_2^T \\ * & H_{22} & 0 & \bar{\epsilon}D_{21}^T \\ * & * & Q & C_2^T\hat{Q} \\ * & * & * & \alpha\hat{Q} \end{bmatrix} > 0, \quad trace\{H\} < \delta^2, \qquad \text{(11a-d)}$$

where $H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$, $\tilde{\Gamma}(1,3) = AQ + B_2YC_2$,

$$\tilde{\Gamma}(1,4) = \alpha B_2Y + AC_2^T\hat{Q}, \quad \tilde{\Gamma}(2,3) = \bar{\epsilon}[C_2AQ + C_2B_2YC_2 - \hat{Q}C_2] + \hat{Q}C_2, \ \tilde{\Gamma}(2,4) = \bar{\epsilon}[C_2AC_2^T\hat{Q} + \alpha C_2B_2Y - \alpha\hat{Q}] + \alpha\hat{Q},$$
$$\tilde{\Gamma}(3,5) = QC_1^T + C_2^TY^TD_{12}^T, \quad \tilde{\Gamma}(3,7) = \bar{\epsilon}QD^TC_2^T,$$
$$\tilde{\Gamma}(3,9) = \bar{\epsilon}C_2^TY^TG^TC_2^T, \quad \tilde{\Gamma}(4,5) = \alpha Y^TD_{12}^T + \hat{Q}C_2C_1^T,$$

$$\tilde{\Gamma}(4,7) = \bar{\epsilon}\hat{Q}C_2D^TC_2^T, \ \tilde{\Gamma}(4,9) = \alpha\bar{\epsilon}Y^TG^TC_2^T, \qquad (12)$$

and $\bar{\epsilon} = 1 + \epsilon$.

If a solution to the latter LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y\hat{Q}^{-1}. \qquad (13)$$

2.2 The Stochastic H_∞ problem

The LMIs of Theorem 1 provide a sufficient condition for the existence of a static output-feedback gain that achieves a prescribed H_2-norm for the system (7). A similar result can be obtained if the H_∞ -norm of the latter system is considered. Given a prescribed desired bound $0 < \gamma$ on the H_∞ -norm of the system, the inequalities in (9) are replaced by the following Bounded Real Lemma (BRL) condition [7].

$$\begin{bmatrix} -\tilde{Q} & \tilde{A}\tilde{Q} & \tilde{B} & 0 & 0 & 0 \\ \tilde{Q}\tilde{A}^T & -\tilde{Q} & 0 & \tilde{Q}\tilde{C}^T & \tilde{Q}\tilde{D}^T & \tilde{Q}\tilde{G}^T \\ \tilde{B}^T & 0 & -\gamma^2 I_{q+p} & 0 & 0 & 0 \\ 0 & \tilde{C}^T\tilde{Q} & 0 & -I_r & 0 & 0 \\ 0 & \tilde{D}\tilde{Q} & 0 & 0 & -\tilde{Q} & 0 \\ 0 & \tilde{G}\tilde{Q} & 0 & 0 & 0 & -\tilde{Q} \end{bmatrix} < 0. \qquad (14)$$

Using the definition of (10), multiplying (14), from both sides, by $diag\{\tilde{Q},\ \tilde{Q},\ I_{q+p},\ I_r,\ \tilde{Q},\ \tilde{Q}\}$, where $\tilde{Q}$ is defined in (10), and substituting for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, $\tilde{G}$ in the latter LMI we obtain the following.

Theorem 2 *Consider the system of (1), (2) . The control law (6b) achieves a prescribed H_∞ -norm* *bound $0 < \gamma$, for some positive scalar $\epsilon << 1$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalar $0 < \alpha$, satisfy $\hat{\Gamma} < 0$ where*

$$\hat{\Gamma} = \left[\begin{array}{ccccccc} -Q & -C_2^T\hat{Q} & \tilde{\Gamma}(1,3) & \tilde{\Gamma}(1,4) & B_1 & 0 & 0 \\ * & -\alpha\hat{Q} & \tilde{\Gamma}(2,3) & \tilde{\Gamma}(2,4) & \bar{\epsilon}C_2B_1 & \bar{\epsilon}D_{21} & 0 \\ * & * & -Q & -C_2^T\hat{Q} & 0 & 0 & \tilde{\Gamma}(3,5) \\ * & * & * & -\alpha\hat{Q} & 0 & 0 & \tilde{\Gamma}(4,5) \\ * & * & * & * & -\gamma^2 I_q & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_p & 0 \\ * & * & * & * & * & * & -I_r \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array} \right.$$

$$\left. \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ QD^T & \tilde{\Gamma}(3,7) & C_2^TY^TG^T & \tilde{\Gamma}(3,9) \\ \hat{Q}C_2D^T & \tilde{\Gamma}(4,7) & \alpha Y^TG^T & \tilde{\Gamma}(4,9) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -Q & -C_2^T\hat{Q} & 0 & 0 \\ * & -\alpha\hat{Q} & 0 & 0 \\ * & * & -Q & -C_2^T\hat{Q} \\ * & * & * & -\alpha\hat{Q} \end{array} \right]. \quad (15)$$

and where $\tilde{\Gamma}(1,3), \tilde{\Gamma}(1,4), \tilde{\Gamma}(2,3), \tilde{\Gamma}(2,4), \tilde{\Gamma}(3,5)-\tilde{\Gamma}(3,9), \tilde{\Gamma}(4,5)-\tilde{\Gamma}(4,9)$ are defined in (12).
If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by (13).

3 The robust stochastic H_2 static output-feedback controller

The system considered in Section 2.1 assumes that all the parameters of the system are known, including the matrices D and G. In the present section we consider the system (1),(2) whose matrices are not exactly known. Denoting

$$\Omega = \left[\begin{array}{cccccccc} A & B_1 & B_2 & C_1 & D_{12} & D_{21} & D & G \end{array} \right],$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j=1,...N\}$, namely,

$$\Omega = \sum_{j=1}^N f_j\Omega_j \quad \text{for some} \quad 0 \le f_j \le 1,\ \sum_{j=1}^N f_j = 1 \qquad (16)$$

where the vertices of the polytope are described by

$$\Omega_j = \left[\begin{array}{cccccccc} A^{(j)} & B_1^{(j)} & B_2^{(j)} & C_1^{(j)} & D_{12}^{(j)} & D_{21}^{(j)} & D^{(j)} & G^{(j)} \end{array} \right],$$

for $j = 1,2...,N$. The solution of the robust problem in this section is based on the derivation of a specially devised BRL for polytopic-type uncertainties [15] with a simple straightforward adaptation to the stochastic case. Considering (9), multiplying it by $diag\{I_{n+m},\ \tilde{Q}^{-1},\ I_r,\ I_{n+m},\ I_{n+m}\}$, from the left and the right and using the method of [15] we obtain that a sufficient condition for achieving the H_2-norm bound of δ for the system at the i-th vertex of Ω is that there exists a solution $\tilde{Q}_i,\ Z,\ H$ to the following LMIs:

$$\left[\begin{array}{ccccc} -\tilde{Q}_i & \tilde{A}_iZ & 0 & 0 & 0 \\ Z^T\tilde{A}_i^T & \tilde{Q}_i - Z - Z^T & Z^T\tilde{C}_i^T & Z^T\tilde{D}_i^T & Z^T\tilde{G}_i^T \\ 0 & \tilde{C}_iZ & -I_r & 0 & 0 \\ 0 & \tilde{D}_iZ & 0 & -\tilde{Q}_i & 0 \\ 0 & \tilde{G}_iZ & 0 & 0 & -\tilde{Q}_i \end{array} \right] < 0, \qquad (17)$$

with $\left[\begin{array}{cc} H & \tilde{B}_i^T \\ \tilde{B}_i & \tilde{Q}_i \end{array} \right] > 0$ and $trace\{H\} < \delta^2$, where $H \in \mathcal{R}^{(q+p)\times(q+p)}$.

Denoting

$$Z \triangleq \left[\begin{array}{c} Z_1 \\ Z_2[C_2 \ \ \beta I_m] \end{array} \right] \qquad (18)$$

where β is a tuning scalar, we arrive at the following result:

Theorem 3 *Consider the uncertain system of (1), (2). The control law (6b) guarantees, for some positive scalar $\epsilon << 1$, a prescribed H_2-norm bound $0 < \delta$ over the entire uncertainty polytope Ω if there exist $Z_1 \in \mathcal{R}^{n\times(n+m)}$, $Z_2 \in \mathcal{R}^{m\times m}$, $\tilde{Q}_i \in \mathcal{R}^{(n+m)\times(n+m)}$, $i = 1,2...,N$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalar β, satisfy the following LMIs:*

$$\left[\begin{array}{cccccc} -\tilde{Q}_{i11} & -\tilde{Q}_{i12} & \Upsilon(1,3) & \Upsilon(1,4) & 0 & 0 \\ * & -\tilde{Q}_{i22} & \Upsilon(2,3) & \Upsilon(2,4) & 0 & 0 \\ * & * & \Upsilon(3,3) & \Upsilon(3,4) & \Upsilon(3,5) & Z_{11}^TD_i^T \\ * & * & * & \Upsilon(4,4) & \Upsilon(4,5) & Z_{12}^TD_i^T \\ * & * & * & * & -I_r & 0 \\ * & * & * & * & * & -\tilde{Q}_{i11} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array} \right.$$

$$\left. \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{\epsilon}Z_{11}^TD_i^TC_2^T & C_2^TY^TG_i^T & \Upsilon(3,9) \\ \bar{\epsilon}Z_{12}^TD_i^TC_2^T & \beta Y^TG_i^T & \Upsilon(4,9) \\ 0 & 0 & 0 \\ -\tilde{Q}_{i12} & 0 & 0 \\ -\tilde{Q}_{i22} & 0 & 0 \\ * & -\tilde{Q}_{i11} & -\tilde{Q}_{i12} \\ * & * & -\tilde{Q}_{i22} \end{array} \right] < 0,$$

$$\left[\begin{array}{cccc} H_{11} & H_{12} & B_1^T & \bar{\epsilon}B_1^TC_2^T \\ * & H_{22} & 0 & \bar{\epsilon}D_{21}^T \\ * & * & \tilde{Q}_{i11} & \tilde{Q}_{i12} \\ * & * & * & \tilde{Q}_{i22} \end{array} \right] > 0,\ trace\{H\} < \delta^2.$$

(19a-c)

where

$$\tilde{Q}_{i11} = \bar{\Upsilon}_1 \tilde{Q}_i \bar{\Upsilon}_1^T,\ \tilde{Q}_{i12} = \bar{\Upsilon}_1 \tilde{Q}_i \bar{\Upsilon}_2^T,\ \tilde{Q}_{i22} = \bar{\Upsilon}_2 \tilde{Q}_i \bar{\Upsilon}_2^T,$$

$$Z_{11} \triangleq Z_1 \bar{\Upsilon}_1^T,\ Z_{12} \triangleq Z_1 \bar{\Upsilon}_2^T, \bar{\Upsilon}_1 \triangleq [I_n\ \ 0],\ \bar{\Upsilon}_2 \triangleq [0\ \ I_m], \tag{20a-g}$$

$$\begin{aligned}
&\Upsilon(1,3) = A_i Z_{11} + B_{2i} Y C_2, \quad \Upsilon(1,4) = A_i Z_{12} + \beta B_{2i} Y,\\
&\Upsilon(2,3) = \bar{\epsilon}[C_2 A_i Z_{11} + C_2 B_{2i} Y C_2 - Z_2 C_2] + Z_2 C_2\\
&\Upsilon(2,4) = \bar{\epsilon}[\beta C_2 B_{2i} Y + C_2 A_i Z_{12} - \beta Z_2] + \beta Z_2,\\
&\Upsilon(3,3) = \tilde{Q}_{i11} - Z_{11} - Z_{11}^T, \quad \Upsilon(3,4) = \tilde{Q}_{i12} - Z_{12} - C_2^T Z_2^T,\\
&\Upsilon(3,5) = Z_1^T C_{1i}^T + C_2^T Y^T D_{12i}^T + \beta Y^T D_{12i}^T,\\
&\Upsilon(3,9) = \bar{\epsilon} C_2^T Y^T G_i^T C_2^T, \quad \Upsilon(4,4) = Q_{i22} - \beta[Z_2 + Z_2^T],
\end{aligned}$$

$$\Upsilon(4,9) = \beta\bar{\epsilon} Y^T G_i^T C_2^T, \tag{21}$$

where $\bar{\epsilon} = 1 + \epsilon$
If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y Z_2^{-T}. \tag{22}$$

Remark 1 : We note that the existence of Z_2^{-T} is guaranteed if the condition of (17) is fulfilled.

4 The robust H_∞ control

Similarly to the previous section, at each point in Ω, say the one that is obtained by $\sum_{j=1}^{N} \alpha_j \Omega_j$ for some $0 \le \alpha_j \le 1,\ \sum_{j=1}^{N} \alpha_j = 1$ we assign a special matrix solution $\tilde{Q}$. For each vertex of Ω, say the i-th, the inequality of (14) can be written, following a modified result of [15] as the following LMI:

$$\begin{bmatrix}
-\tilde{Q}_i & \tilde{A}_i Z & \tilde{B}_i & 0 & 0 & 0\\
Z^T \tilde{A}_i^T & \tilde{Q}_{i,22} & 0 & Z^T \tilde{C}_i^T & Z^T \tilde{D}_i^T & Z^T \tilde{G}_i^T\\
\tilde{B}_i^T & 0 & -\gamma^2 I_{q+p} & 0 & 0 & 0\\
0 & \tilde{C}_i Z & 0 & -I_r & 0 & 0\\
0 & \tilde{D}_i Z & 0 & 0 & -\tilde{Q}_i & 0\\
0 & \tilde{G}_i Z & 0 & 0 & 0 & -\tilde{Q}_i
\end{bmatrix} < 0 \tag{23}$$

where $\tilde{Q}_{i,22} = \tilde{Q}_i - Z - Z^T$ and where Z is defined in (18). We arrive at the following result:

Theorem 4 *Consider the uncertain system of (1), (2). The control law (6b) guarantees, for some positive scalar $\epsilon << 1$, a prescribed H_∞-norm $0 < \gamma$ over the entire uncertainty polytope Ω if there exist $Z_1 \in \mathcal{R}^{n\times(n+m)}$, $Z_2 \in \mathcal{R}^{m\times m}$, $\tilde{Q}_i \in \mathcal{R}^{(n+m)\times(n+m)}$, $i = 1,2...,N$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalar β, satisfy $\tilde{\Upsilon} < 0$ where $\tilde{\Upsilon}$ is the following LMI:*

$$\begin{bmatrix}
-\tilde{Q}_{i11} & -\tilde{Q}_{i12} & \Upsilon(1,3) & \Upsilon(1,4) & B_{1i} & 0 & 0 & 0 & 0 & 0 & 0\\
* & -\tilde{Q}_{i22} & \Upsilon(2,3) & \Upsilon(2,4) & \bar{\epsilon} C_2 B_{1i} & \bar{\epsilon} D_{21i} & 0 & 0 & 0 & 0 & 0\\
* & * & \Upsilon(3,3) & \Upsilon(3,4) & 0 & 0 & \Upsilon(3,5) & Z_{11}^T D_i^T & \bar{\epsilon} Z_{11}^T D_i^T C_2^T & C_2^T Y^T G_i^T & \Upsilon(3,9)\\
* & * & * & \Upsilon(4,4) & 0 & 0 & \Upsilon(4,5) & Z_{12}^T D_i^T & \bar{\epsilon} Z_{12}^T D_i^T C_2^T & \beta Y^T G_i^T & \Upsilon(4,9)\\
* & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -I_r & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -\tilde{Q}_{i22} & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\tilde{Q}_{i11} & -\tilde{Q}_{i12}\\
* & * & * & * & * & * & * & * & * & * & -\tilde{Q}_{i22}\\
* & * & * & * & * & * & * & * & * & * &
\end{bmatrix},$$

where $\tilde{Q}_{i11},\ \tilde{Q}_{i12},\ \tilde{Q}_{i22},$ and $Z_{11},\ Z_{12}$ are given in (20a-e), $\Upsilon(1,3), \Upsilon(1,4), \Upsilon(2,3), \Upsilon(2,4), \Upsilon(3,3) - \Upsilon(3,9), \Upsilon(4,4) - \Upsilon(4,9)$ are given in (21) and $\bar{\epsilon} = 1 + \epsilon$.

If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by (22).

5 Example

To demonstrate the solvability of the various LMIs in this paper we bring a 3th-order two output one input example. We consider the following discrete-time linear system:

$$A = \begin{bmatrix} 0.9813 & 0.3420 & 1.3986\\ 0.0052 & 0.9840 & -0.1656\\ 0 & 0 & 0.5488 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0198 & 0.0034 & 0.0156\\ 0.0001 & 0.0198 & -0.0018\\ 0 & 0 & 0.0150 \end{bmatrix},\ B_2 = \begin{bmatrix} -1.47\\ -0.0604\\ 0.4512 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix},\ D = \begin{bmatrix} 0.2 & 0 & 0\\ 0.005 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},$$

with $D_{21} = 0,\ G = 0$. Applying the result of Theorem 2 and solving (15) a minimum value of $\gamma = 0.868$ for $\alpha = 2.4$ and $\epsilon = 10^{-5}$, is obtained.

The corresponding static output-feedback controller of (22) is:

$$K = \left[\begin{array}{cc} 0.4888 & 0.7185 \end{array} \right],$$

where the closed-loop poles are $-0.1474,\ 0.9498 + 0.1152i,\ \ 0.9498 - 0.1152i$. For the nominal case where $D = 0$ (i.e with no state-multiplicative noise), we obtain for the above values of α and ϵ an attenuation level of $\gamma = 0.6572$.

6 Conclusions

A convex programing method is presented which provides an efficient design of robust static output-feedback controllers for linear systems with state multiplicative noise. Linear systems with polytopic type uncertainties are considered and a sufficient condition is derived, based on a linear parameter dependent Lyapunov function, for the existence of a constant output-feedback gain that stabilizes the system and achieves a prescribed bound on its performance over the entire uncertainty polytope.

Both stochastic H_2 and H_∞ performance criteria have been considered. For both, conditions for quadratic stabilizing solution have been obtained. The conservatism entailed in these conditions can be reduced either by using a recent method that enables the use of parameter dependent Lyapunov based optimization, or by treating the vertices of the uncertainty polytope as distinct plants. The latter solution cannot guarantee the stability and the performance within the polytope whereas the former optimization method achieves the required bound over the entire polytope.

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