A dominance-based approach to robustness analysis

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Abstract— ***In this paper simple symmetric interval bounds on the singular values of a matrix based on its Gershgorin disks are proposed. This allows the Gershgorin theorem to be used not only to provide information about the location of the eigenvalues of a matrix but also its singular values. This is utilised for the proposition of the first design technique for singular value loop shaping based on the diagonal dominance methodology for design of linear multivariable plants. A design example is given demonstrating the effectiveness of this approach.***

Keywords— ***Diagonal dominance, SVD analysis, loop-shaping, linear multivariable controller design***

I. Introduction

FOR a matrix $A = [a_{ij}] \in \mathbf{C}^{m\times m}$, the radius of its column Gershgorin disks $C_j(A)$ also referred to as the *deleted absolute column sum* and row gershgorin disks $R_i(A)$ also referred to as the *deleted absolute row sum* are defined respectively as

$$C_j(A) = \sum_{\substack{i=1\\ i\neq j}}^{m} \mid a_{ij} \mid \tag{1}$$

$$R_i(A) = \sum_{\substack{j=1\\ i\neq j}}^{m} \mid a_{ij} \mid . \tag{2}$$

Gershgorin's theorem [1] states that the eigenvalues of A lie inside the region defined by these disks centered on the diagonal entries of A

$$G_R(A) \equiv \bigcup_{i=1}^{m} \{s \in \mathbf{C} : \mid s - a_{ii} \mid \leq R_i(A)\} \tag{3}$$

$$G_C(A) \equiv \bigcup_{j=1}^{m} \{s \in \mathbf{C} : \mid s - a_{jj} \mid \leq C_j(A)\} \tag{4}$$

Note that both $G_R(A)$ and $G_C(A)$ must include the eigenvalues, hence their intersection $G_\mu(A) = G_R(A) \cap G_C(A)$ is the only subset the eigenvalues can truly exist in. $G_\mu(A)$ is referred to as a *minimal Gershgorin set* [2] and other minimal sets may be obtained by considering the intersection of all the Gershgorin sets corresponding to *similar* operators to A (e.g. $\check{A} = S^{-1}AS$). In this work, however, we are concerned with the standard Gershgorin sets.

Rosenborck [3] used Gershgorin's theorem to propose the first frequency-based linear multivariable controller design technique based on the concept of Diagonal Dominance. This is a design technique that converts a linear multivariable design problem into several single-loop design problems which can then be solved using any number of available single-loop design techniques. In the case of column dominance, for a plant with transfer function matrix $G(s) = [g_{ij}(s)] \in \mathbf{C}^{m\times m}$, this involves finding a pre-compensator matrix $K = [k_{ij}(s)] \in \mathbf{C}^{m\times m}$, such that the resulting open-loop system with transfer function matrix $Q(s) = G(s)K$ satisfies the inequality

$$\mid q_{ii}(s) \mid \geq\geq \sum_{\substack{j=1\\ j\neq i}}^{m} \mid q_{ij}(s) \mid \quad (i = 1, \ldots, m) \tag{5}$$

where '$\geq\geq$' denotes 'at least equal to, but as much greater than as possible'. If such a K can be found, $Q(s)$ may be replaced by $\tilde{Q}(s) = diag\{Q(s)\} = [q_{ii}(s)] \in \mathbf{C}^{m\times m}$. Next, a diagonal controller matrix $D(s) = [d_{ii}(s)] \in \mathbf{C}^{m\times m}$ can be found such that $\tilde{q}_{ii}(s)d_{ii}(s)$ is as close as possible to $m_i(s)(1-m_i(s))$, where $M(s) = diag\{m_i(s)\}$ is the desired transfer function matrix of the closed-loop system, whose actual overall transfer function matrix is $T(s)$.

Note that, since $\tilde{q}_{ij}(s)d_{ij}(s) = 0 \quad (\forall\ i \neq j)$, the design of $D(s)$ can be broken down into m single-loop design problems; the transfer function matrix of the corresponding multivariable controller is $C(s) = KD(s)$. If the pre-compensator matrix K satisfies the inequality (5), then this inequality is also satisfied for $Q(s)D(s)$ since $D(s)$ post-multiplies each column of $Q(s)$ by the same gain at each frequency.

A major handicap with diagonal dominance is that the controller design only focuses on very few properties of the system [4] and issues such as robustness, disturbance rejection, etc... are implied from the process and not inherently addressed by it. This makes the motivations for this work very clear. By showing that the Gershgorin disks can be used to bound the singular values as well, it allows the designer to not only use the Gershgorin disks to asses the system's interaction, but also guaranteed bounds on the whereabouts of its singular values. In turn, the design process can be made to cater for cases where, for example, there are specifications on the H_∞ norm of the system or the behavior of its singular values.

II. A symmetric interval bound

Consider the regular splitting of a matrix $A = [a_{ij}] \in \mathbf{C}^{m\times m}$ into $\tilde{A} = [a_{ii}] \in \mathbf{C}^{m\times m}$ and $E = [e_{ij} = a_{ij,\ i\neq j}] \in \mathbf{C}^{m\times m}$, such that $A = \tilde{A} + E$, where $\tilde{A}$ contains the diagonal entries of A and the remainder is contained in E. By basic manipulation and use of the triangular inequity, the following can be arrived at

$$\mid \parallel \tilde{A} \parallel - \parallel A \parallel \mid \leq \parallel E \parallel, \tag{6}$$

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where $\| \bullet \|$ can be any matrix norm including the spectral norm which is defined as

$$\begin{aligned} \| \Gamma \|_2 &= \max_{x \neq 0} \frac{\| \Gamma x \|_2}{\| x \|_2} \\ &= \sqrt{\rho(\Gamma^H \Gamma)} \\ &= \max_i \sigma_i(\Gamma) = \sigma_1(\Gamma) \end{aligned} \tag{7}$$

where Γ represents a generic matrix in $\mathbf{C}^{m \times m}$ and $\rho(\Gamma)$ denotes the spectral radius of Γ defined as $\rho(\Gamma) = \max_i | \lambda_i(\Gamma) |$. Choosing the matrix norm in (6) to be the spectral norm, and using the fact that $\| \tilde{A} \|_2 = \max_i | a_{ii} |$ allows (6) to be rewritten as

$$| \max_i | a_{ii} | - \sigma_1(A) | \leq \| E \|_2. \tag{8}$$

Consider again the matrix Γ and its spectral radius $\rho(\Gamma) = | \lambda_{max}(\Gamma) |$, where $\lambda_{max}(\Gamma)$ is the largest eigenvalue of Γ. Let W be the matrix whose every column is the eigenvector corresponding to λ_{max} such that $\lambda_{max} W = \Gamma W$. Observe that for any matrix norm

$$\begin{aligned} | \lambda | \| W \| &= \| \Gamma W \| \\ &\leq \| \Gamma \| \| W \| \\ &\therefore \\ \rho(\Gamma) &= | \lambda_{max} | \leq \| \Gamma \|. \end{aligned} \tag{9}$$

Therefore, the spectral radius of a matrix is always bounded by any of its norms. Hence,

$$\begin{aligned} \| E \|_2 &= \sigma_1(E) \\ &= [\rho(E^H E)]^{1/2} \\ &\leq \| E^H E \|_\beta^{1/2} \\ &\leq (\| E^H \|_\beta \| E \|_\beta)^{1/2}, \end{aligned} \tag{10}$$

where

$$\| \Gamma \|_\beta = \begin{cases} \max_{1 \leq j \leq m} \sum_{i=1}^m | \gamma_{ij} | & for \quad \beta = 1 \\ \| \Gamma^T \|_1 & for \quad \beta = \infty \end{cases} \tag{11}$$

Note from the Cauchy-Schwarz inequality that $\| \Gamma \|_1 \leq Cm(\Gamma) + \max_j | \gamma_{jj} |$ and $\| \Gamma \|_\infty \leq Rm(\Gamma) + \max_i | \gamma_{ii} |$ where $Cm(\Gamma) = \max_j C_j(\Gamma)$ and $Rm(\Gamma) = \max_i R_i(\Gamma)$. Further, let $mD(\Gamma)$ =max$\{Cm(\Gamma), Rm(\Gamma)\}$ and observe that $mD(\Gamma)$ equals the radius of the largest Gershgorin disk of Γ. Consequently

$$\begin{aligned} \| E \|_2 &\leq \sqrt{\| E \|_\infty \| E \|_1} \\ &\leq \sqrt{(Cm(E) + \max_j | e_{jj} |)(Rm(E) + \max_i | e_{ii} |)} \\ &\leq \sqrt{mD(A)^2} = mD(A). \end{aligned} \tag{12}$$

Using (6) and (12), one can finally arrive at

$$| \max_i | a_{ii} | - \sigma_1(A) | \leq mD(A), \tag{13}$$

which shows that the largest singular value of A is within the radius of the largest Gershgorin disk of A relative to the absolute value of its largest element. To arrive at the final result, the interlacing theorem for singular values of a matrix is used. This states that if Γ is a given matrix and $\breve{\Gamma}$ is the matrix obtained by deleting any one column or row from Γ, then the following holds true

$$\sigma_1(\Gamma) \geq \sigma_1(\breve{\Gamma}) \geq \sigma_2(\Gamma) \geq \cdots \geq \sigma_m(\breve{\Gamma}) \geq 0 \tag{14}$$

Suppose one deletes the row and column from A which contains $\max_i | a_{ii} |$ and lets the resulting matrix be $\breve{A}$, then from (13) it is evident that

$$| \max_i | \breve{a}_{ii} | - \sigma_1(\breve{A}) | \leq mD(\breve{A}). \tag{15}$$

However, since $0 \leq \sigma_1(\breve{A}) \leq \sigma_1(A)$ and further because $Cm(\bullet)$ and $Rm(\bullet)$ are absolute norms, then $mD(\breve{A}) \leq mD(A)$, and therefore

$$| \max_i | \breve{a}_{ii} | - \sigma_2(A) | \leq mD(A) \tag{16}$$

Hence, the second singular value of A is also bounded around the *second* largest diagonal element of A by $mD(A)$. This procedure may be repeated to show that each singular value of A is bounded by $mD(A)$ around *each* of its diagonal elements. Henceforth, $mD(A)$ will be referred to as the *singular interval* and it can be concluded that all the singular values of A must lie in the region

$$\Upsilon(A) \equiv \bigcup_{i=1}^m \{x \in \mathbf{R} : | x - | a_{ii} | | \leq mD(A), \ x \geq 0\} \tag{17}$$

which will be termed the *Gershgorin singular region*. In the next section, a graphical interpreting of this result is shown.

III. A Graphical interpretation

On each diagonal element of A draw a circle with radius $mD(A)$. These disks will be referred to as the *singular disks* and it should be noted that they are concentric with the Gershgorin disks. Then rotate each disk with respect to the origin until it passes and leaves a trace on the real positive axis. *Within each trace left on the real positive axis, there is at least one singular value of A.* Note that the term *at least* is used, since if any given two interval are intersecting, the possibility exists that the singular value of each interval could lie in the intersection region. This in effect means that each interval contains two singular values. For example, consider the matrix

$$A = \begin{pmatrix} -10 + 10i & 0 & 2 \\ 1 & -20 & 0 \\ -1 + i & i & -3 - 12i \end{pmatrix} \tag{18}$$

Figure (1) shows the Gershgorin disks (solid disks) of A on its diagonal elements. For this matrix, $mD(A) = Cm(A) = C_1(A) = 2.41$ which is the Gershgorin disk of the first column of A. This is why the Gershgorin and singular disk (dashed) centered at $-10 + 10i$ have the same radius. The three intervals $\{9.95 \leq sv1 \leq 14.8\}$, $\{11.7 \leq sv2 \leq 16.6\}$ and $\{17.6 \leq sv3 \leq 22.4\}$ are traces left on the positive real axis after the rotation of each singular disk around the origin, and each will contain at least one singular value of A. The actual singular values of A in this case are $\{20.1,\ 14.7,\ 11.9\}$ (rounded). Note that, in this example, $sv1$ and $sv2$ both contain two singular values since the two singular values $\{14.7,\ 11.9\}$ are in the region $sv1 \cap sv2$. Observe that if the matrix is made diagonal dominant enough (via scaling or compensation), these intervals will shrink and become disjoint resulting in the bound becoming sharper. This ties in perfectly with the first stage of a diagonal dominance controller design process, which aims to pre-compensate the system such that the Gershgorin disks are minimised. In the second stage, the designer would then be able to shape each *singular band* (bands swept out by the singular disks) to meet some given constraints.

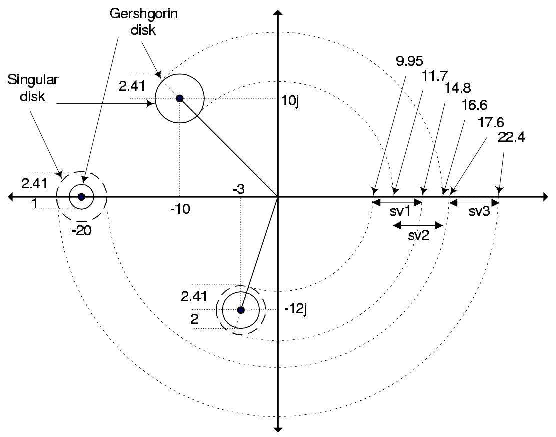

Fig. 1. The gershgorin singular value bound

IV. A design study: The Automotive Gas Turbine

Consider the automotive gas turbine (AGT), as described in [5]. The open loop transfer function matrix model of the AGT is as follows:-

$$G(S) = \tag{19}$$
$$\begin{pmatrix} \frac{0.806s+0.264}{s^2+1.15s+0.202} & \frac{-15s-1.42}{s^3+12.8s^2+13.6s+2.36} \\ \frac{1.95s^2+2.12s+0.49}{s^3+9.15s^2+9.39s+1.62} & \frac{7.14s^2+25.8s+9.35}{s^4+20.8s^3+116.4s^2+111.6s+18.8} \end{pmatrix}$$

The Nyquist array (NA) of the open loop system is shown in Figure (2), where it is evident that the system is interacting. Whilst the first loop is dominant up to a point, the second loop is not dominant at any frequency. More insight may be obtained from Figure (3), which contains the *composite Bode plot* (CBP). The composite Bode plot contains the response of the diagonal elements, the Gershgorin bands for each column, the plot of the singular values of the system, and also the line of upper bound on the singular values as obtained from inequality (13). In plotting the CBP the following convention is adhered to; diagonal elements (solid line), singular values if plotted (dashed line), Gershgorin bands (dotted line), and finally the singular value bounds (dashed-dotted line).

The analysis of the CBP is fairly straightforward. For a given column, dominance is lost if the Bode response of the diagonal element of that column falls below the magnitude Gershgorin band. This is the Bode domain equivalent of the zero exclusion theorem in the NA plot and follows directly from the definition of dominance. Note, however, one advantage here that is not present with the NA is the information regarding the exact frequency at which dominance is lost, which can easily be read from the CBP. In this case study, the singular values of the system are also plotted to confirm that indeed in each case they are within the bounds calculated. One of the important things that one should observe from Figure (3) is the direct link between the degree of dominance at each frequency and the values of the upper bound. For example, observe that in the frequency range $0.2 \leq \omega \leq 1$ the upper bound bulges even though both diagonals are decreasing in value. The reason for this is the increase in the interactions in that frequency range which is indicated directly from the Gershgorin bands.

Suppose at this point one wishes to design a dominance based controller that can satisfy constraints on the behavior of the singular values of the system. The next stage, would still be exactly similar to a normal diagonal dominance based design study. Namely, in the next step the aim would be to find a pre-compensator (preferably dynamic) that would achieve as high a degree of diagonal dominance as possible. In this case, a technique proposed in [6] was used to design the following pre-compensator

$$K(S) = \begin{pmatrix} 0.049813\frac{(s+12)}{s} & 0.86524\frac{(s+0.1)}{s(s+0.2)} \\ -0.404\frac{(s+0.9)}{s} & 0.038834\frac{s(s+0.45)(s+13)}{(s+0.25)} \end{pmatrix}. \tag{20}$$

The NA of $Q(s) = G(S)K(S)$ is shown in Figure (4). The very high degree of dominance achieved is apparent. It can further be confirmed from the CBP shown in Figure (5). Note the following very interesting phenomenon. The upper singular bound is now much sharper. In fact the biggest singular value is 'trapped' inside a very narrow region around the magnitude Bode plot of the largest diagonal element. Further, from the reasoning made earlier with regards to the existence of such a symmetric bound around all other diagonal elements, it is also known that within the same narrow region around the other Bode plot, lies the second singular value. At this point, one can treat the singular value loop shaping problem as two SISO loop shaping design exercises. Using the diagonal elements of $D(S)$, the diagonal Bode plots of $G(S)K(S)$ are shaped 'as if they were its singular values' such that together with the singular value bounds they satisfy the given constraints . Suppose, for example, that the closed-loop design specifications were as follows:-

- Zero steady-state error to a step input in all input directions
- The H_∞ norm to be less than 0.5dB
- 1 rads/second bandwidth

• Minimum 20dB/decade roll-off for the complementary sensitivity function past the bandwidth frequency

Then (having the upper bound in mind, as the indicator of how close the two are), one can shape the diagonal Bode responses such that together with the bounds, they meet the constraints on the *singular values*. For example, in this case the shaping TFM was found to be simply

$$D(S) \quad = \quad \begin{pmatrix} 0.9 & \\ & 2\frac{1+3.2s}{(1+1.5s)} \end{pmatrix} \tag{21}$$

Figure (6) shows the CBP of the closed loop system, where it is clear that all constraints are met. To illustrate this more clearly, the plot of the singular values of the complementary sensitivity function is shown separately in Figure (7). Finally, the step response of the initial open loop and the final closed loop systems are shown in Figures (8) and (9).

V. Final Remarks

In this work we proposed, and successfully developed, the idea of using the Gershgorin disks of a matrix to bound its singular values. This was illustrated by two examples.

Based on this, a simple approach to the problem of shaping the singular values of a multivariable system was proposed. This approach is unique in that unlike other proposed techniques, it constitutes a *design* rather than a *synthesis* technique. The advantages of a design technique versus a synthesis technique are well known and will not be rehearsed here. However, in this case, it provides a critical advantage to the designer; namely, that the complexity of the resulting controllers is purely a matter for the designer to choose and is not in any way dictated by the technique. Further, whereas a synthesis technique would fail if the problem formulation is deemed to be unfeasible, a design process would instead produce a result as close as possible to the unfeasible requirements.

Finally, it is acknowledged that this idea is very much in its infancy and much further research is needed to fully develop and further extend the technique.

Acknowledgments

The authors would like to thanks UMIST, EPSRC and the IEE for supporting this research.

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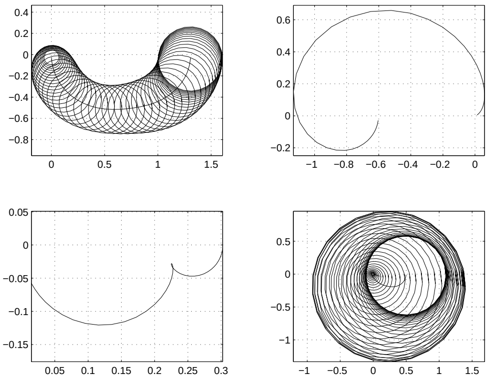

Fig. 2. Nyquist array of the open loop AGT

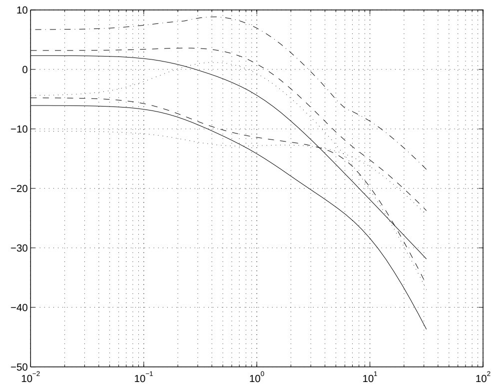

Fig. 3. CBP of the open loop AGT

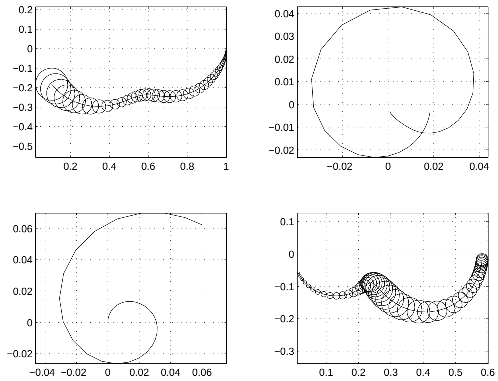

Fig. 4. Nyquist array of the open loop pre-compensated AGT

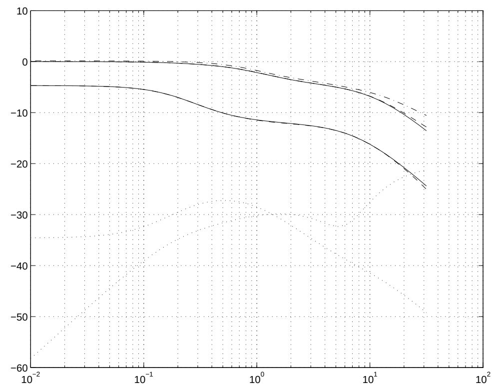

Fig. 5. CBP of the open loop pre-compensated AGT

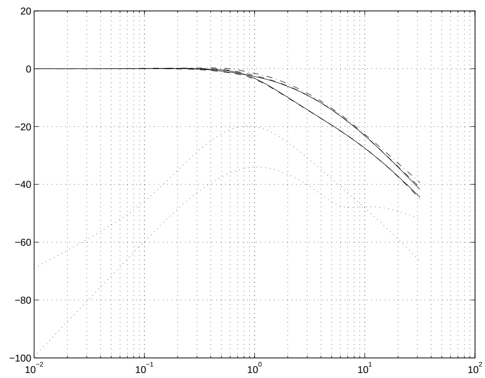

Fig. 6. CBP of the closed loop pre-compensated AGT with the controller

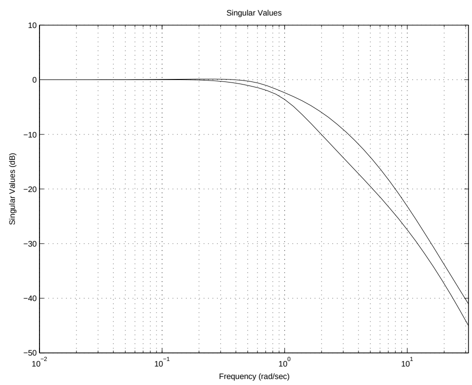

Fig. 7. Singular value plot of the closed loop pre-compensated AGT with the controller

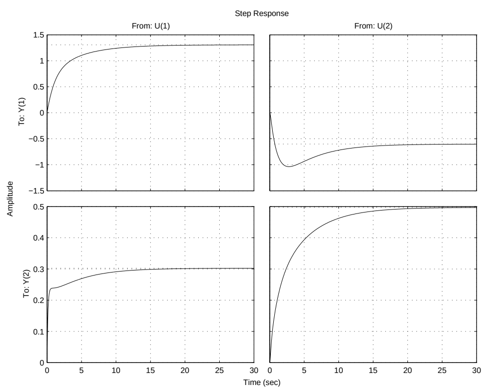

Fig. 8. Open loop step response of the AGT

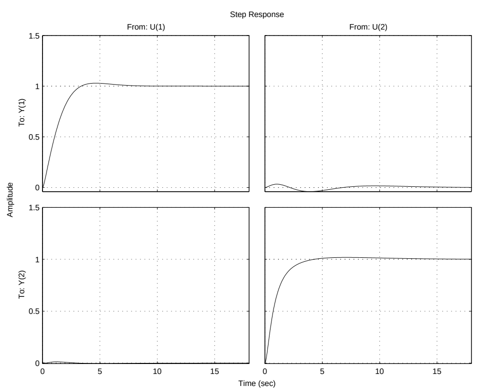

Fig. 9. Step response of the closed loop pre-compensated AGT with the controller