On the boundary control of passive infinite dimensional systems

Ömer Morgül

Abstract— **We will consider the feedback stabilization of a class of passive infinite dimensional systems by means of boundary control. Such systems usually possess an internal energy, and along their solutions a conservation of energy equation hold. This equation shows the balance between the internal and external powers. By utilizing this balance, we will prove various stability results. We will also give some examples on the application of the proposed technique to some well known passive systems.**

Keywords— **Boundary Control, Infinite Dimensional Systems, Passivity.**

I. Introduction

Many mechanical systems, such as spacecraft with flexible attachments, or robots with flexible links, and many practical systems such as power systems, and mass transport systems contain certain parts whose dynamic behaviour can be rigorously described only by partial differential equations (PDE). In such systems, to achieve high precision demands, the dynamic effect of the system parts whose behaviour are described by PDE's on the overall system has to be taken into account in designing the controllers.

In recent years, boundary control of systems represented by PDE's has become an important research area. This idea is first applied to the systems represented by the wave equation (e.g. elastic strings, cables), see e.g. [1], [5], and extended to beam equations, [2], and to the rotating flexible structures, see [9], [10]. In particular, it has been shown that for a string which is clamped at one end and is free at the other end, a single *non-dynamic* boundary control applied at the free end is sufficient to exponentially stabilize the system, see [1]. For an extension of these ideas to dynamic boundary controllers, see [9], [10]. For more references on the subject the reader is referred to [6], [7].

The stabilization of systems is an important research area in the control theory. While the stabilization is an important subject in its own right, it could also be viewed as a first step in designing controllers to achieve some additional tasks such as tracking, disturbance rejection, robustness, etc. In this sense, when a system to be controlled is given, it would be desirable to determine a relatively large class of stabilizing controllers, if possible all. Then within this class of controllers one may try to find suitable ones to solve additional problems like tracking, disturbance rejection, etc.

In this work we will consider the boundary control of a class of *passive* infinite dimensional systems. We follow the general framework introduced in [7] for such systems. We will develop some general results for the stabilization of this class of infinite dimensional systems by means of boundary control techniques. In this class of systems the inputs and outputs are assumed to act on the boundaries of the system. For this class of systems, we will first investigate the effect of a simple feedback law and prove certain stability results. We also show that some of the examples frequently encountered in the literature (e.g. the wave equation, the Euler-Bernoulli and the Timoshenko beam equations) can be viewed in this class and we present the stability results for such systems. We also consider certain examples and apply the proposed approach for the stabilization of these systems.

Ö. Morgül is with the Bilkent University, Dept. of Electrical Eng., 06800, Bilkent, Ankara, Turkey. E-mail: morgul@bilkent.edu.tr .

II. A General Framework

First we consider a general case to motivate the concept of passivity. Let $\mathcal{S}$ be a dynamical system, let $u, y \in \mathbf{R}^m$ be its input and output vectors, respectively, let X be a Hilbert space in which the solutions of $\mathcal{S}$ evolve and let $E : X \to \mathbf{R}$ be an appropriate "energy" function which depends on the solutions of $\mathcal{S}$. Assume that the following holds

$$\dot{E} = u^T y = \sum_{i=1}^{m} u_i y_i, \tag{1}$$

where the derivative is taken along the solutions of $\mathcal{S}$ and we set $u = (u_1 \ldots u_m)^T, y = (y_1 \ldots y_m)^T \in \mathbf{R}^m$, the superscript T denotes the transpose. Here we may view E as the internal "energy" of the system and (1) may be viewed as the conservation of energy, where the right hand side of (1) may be viewed as the "external power" supplied to the system, and the left hand side may be viewed as "internal power". Hence, we may also consider (1) as a "balance of power" equation. It follows from (1) that a natural choice for the control inputs u_i for the stabilization is the following

$$u_i = -\alpha_i y_i, \quad \alpha_i \geq 0. \tag{2}$$

If we use (2) in (1), the latter becomes

$$\dot{E} = -\sum_{i=1}^{m} \alpha_i (y_i)^2. \tag{3}$$

Therefore as a result of the feedback law given by (2), the energy of the system decreases along the solutions and under appropriate assumptions some stabilization results may be deduced.

To elaborate further, let H be a Hilbert space, let $< \cdot, \cdot >_H$ and $\|\cdot\|_H$ denote the inner-product and the associated norm for H, respectively. Consider the following second order systems:

$$w_{tt} + Aw = 0 \quad , \tag{4}$$

where a subscript denotes the partial derivative with respect to the corresponding variable, and A is a linear (not necessarily bounded) operator on H. Assume that A depends on the (one dimensional) spatial variable x and that $x \in [0,1]$. Assume that the system given by (4) has the following boundary conditions

$$(\ B_i^1 w\ )(0) = f_i^1, i = 1,..,k\ ,\ (\ B_i^2 w\ )(1) = f_i^2, i = 1,..,l, \tag{5}$$

$$(\ B_i^3 w\ )(0) = 0, i = 1,..,p\ ,\ (\ B_i^4 w\ )(1) = 0, i = 1,..,r, \tag{6}$$

where B_i^j are various linear (not necessarily bounded) operators on H, k, l, p, r are some appropriate integers, and f_i^j are control inputs of our systems. In the sequel we will not state the range of indices, which should be obvious from the context. We note that here $(\ B_i^j w\ )(\cdot) : [0,1] \to H$ and $(\ B_i^j w\ )(c)$ denotes the value of $B_i^j w$ at $x = c$.

We first define the following sets :

$$\mathcal{S}_1 = \{w \in H \mid (\ B_i^1 w\ )(0) = 0 \quad , \quad (\ B_i^2 w\ )(1) = 0\}, \tag{7}$$

$$\mathcal{S}_2 = \{w \in H \mid (\ B_i^3 w\ )(0) = 0 \quad , \quad (\ B_i^4 w\ )(1) = 0\}. \tag{8}$$

Let $D(A) \subset H$ be the domain of A, which may be given as

$$D(A) = \{w \in H \mid Aw \in H\ \}. \tag{9}$$

We also define the operator A_{uc} as A with the following domain

$$D(A_{uc}) = D(A) \cap \mathcal{S}_1 \cap \mathcal{S}_2. \tag{10}$$

We make the following assumptions

Assumption 1 : D(A) is dense in H. □

Assumption 2 : $D(A_{uc})$ is dense in H, A_{uc} is self-adjoint and coercive in H, i.e. the following holds for some $\alpha > 0$

$$< w, A_{uc} w >_H \geq \alpha \|w\|_H^2 \quad , \quad w \in D(A_{uc}).\ \square \tag{11}$$

It follows then that $A_{uc}^{1/2}$ exists, is self-adjoint and non-negative. Let V be defined as :

$$V = D(A_{uc}^{1/2}). \tag{12}$$

For technical reasons we assume the following.

Assumption 3 : The set $V \subset H$ satisfies the following

$$V \cap \mathcal{S}_1 \neq V \quad , \quad V \cap \mathcal{S}_2 = V.\quad \square \tag{13}$$

In most of the examples, the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ impose certain conditions at the boundaries, and one may easily modify V without changing the density assumptions.

Consider the system given by (4)-(6) with $f_i^1 = f_j^2 = 0$. Since the control inputs are set to zero we call the resulting system as *uncontrolled*. Now (4) can be rewritten as

$$\dot{z} = \mathcal{A} z \quad , \quad z(0) \in X, \tag{14}$$

where $X = V \times H$, $z = (w\ w_t)^T \in X$, and $\mathcal{A}$ is a linear operator defined on X as

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \tag{15}$$

with $D(\mathcal{A}) = D(A_{uc}) \times V$. Here, and in the sequel, the superscript T denotes the transpose. For $z_1 = (u_1\ v_1)^T, z_2 = (u_2\ v_2)^T \in X$, the inner-product on X is given as

$$< z_1, z_2 >_X = < A_{uc}^{1/2} u_1, A_{uc}^{1/2} u_2 >_H + < v_1, v_2 >_H, \tag{16}$$

and for $z = (u\ v)^T$ the norm in X is given as

$$\|z\|_X^2 = \|A_{uc}^{1/2} u\|_H^2 + \|v\|_H^2, \tag{17}$$

Consider the system given by (4)-(6). Our control problem is to determine appropriate forms for f_i^j such that the resulting closed loop system is well posed and stable in certain senses.

To define a feedback control law, we need an output function. On the other hand, the balance of power equation given by (1) also imposes a certain constraint on the choices of outputs. Hence the selection of appropriate outputs are quite important. The next assumption suggests an appropriate choice for the outputs.

Assumption 4 : Let $D_1 = D(A) \cap \mathcal{S}_2$ and $D = D_1 \times V$. D_1 is dense in $D(A_{uc})$ and the following holds

$$\begin{aligned} < z, \mathcal{A} z >_X &= \textstyle\sum_{i=1}^k (\ B_i^1 u\ )(0)(\ O_i^1 v\ )(0) \\ &\quad + \textstyle\sum_{i=1}^l (\ B_i^2 u\ )(1)(\ O_i^2 v\ )(1), \end{aligned} \tag{18}$$

where $z = (u\ v)^T \in D$ and O_i^j, $i = 1, \ldots k$ or l, $j = 1, 2$, whichever appropriate, are linear (not necessarily bounded) operators on H. We will call (18) as the *power form* for the system given by (14). (cf. (1)). □

Remark 1 : Let the operator $\mathcal{A}$ generate a C_0 semigroup of contractions $T(t)$, and for $z(0) \in D(\mathcal{A})$, let $z(t)$ be the solution of (14). Let us define the *energy* $E(t)$ of the solutions of (14) as

$$E(t) = \frac{1}{2} < z(t), z(t) >_X . \tag{19}$$

Since $z(t)$ is then differentiable, see [7], by differentiating (19), using (14), and (18), we obtain $\dot{E} = 0$. Therefore, in the uncontrolled system the energy E is conserved. By utilizing (18) we will propose appropriate control laws which results in asymptotically stable closed loop systems. □

Let $z = (w\ w_t)^T$ be the solution of (14). It follows from (18) that appropriate outputs y_i^j of the system (14) may be given as

$$y_i^1 = (\ O_i^1 w_t\ )(0)\ ,\ i = 1,..,k\ ,\ y_i^2 = (\ O_i^2 w_t\ )(1)\ ,\ i = 1,..,l. \tag{20}$$

Let us assume that the Assumptions 1-4 hold for the system given by (4)-(6). Let us choose the outputs as given by (20). We will denote the resulting system as $\mathcal{S}$. Then (18) may be rewritten as

$$< z, \mathcal{A} z >_X = \sum_{i=1}^k f_i^1 y_i^1 + \sum_{i=1}^l f_i^2 y_i^2. \tag{21}$$

For the system $\mathcal{S}$, the control problem we consider is to find appropriate control laws for f_i^j by using the outputs y_i^j such that the resulting closed-loop system is well-posed and asymptotically stable. The following simple feedback law is frequently used in the literature

$$f_i^j = -\alpha_i^j y_i^j, \tag{22}$$

where $\alpha_i^j \geq 0$, (cf. (2)). This choice is quite natural since then (21) becomes the following

$$< z, \mathcal{A}z >_X = -\sum_{i=1}^{k} \alpha_i^1 (y_i^1)^2 - \sum_{i=1}^{l} \alpha_i^2 (y_i^2)^2. \tag{23}$$

Hence $\mathcal{A}$ becomes dissipative with this control law, which is quite important in proving both the well-posedness and the asymptotical stability of the closed-loop system. For the asymptotic stability, we may use the energy E defined by (19) as a Lyapunov function. Note that $\dot{E}$ is then given by (23), cf. (3). From LaSalle's invariance Theorem, it may be concluded then that under certain conditions all solutions of system $\mathcal{S}$ asymptotically tend to the maximal invariant set contained in

$$\mathcal{O} = \{ z \in X \mid \ < z, \mathcal{A}z >_X = 0 \} \ , \tag{24}$$

see e.g. [7]. In this case, the inputs, as well as for any $\alpha_i^j > 0$ the corresponding outputs become zero. If we can prove that, under these conditions the only possible solution of the system $\mathcal{S}$ is the zero solution, then by LaSalle's invariance theorem, we may conclude that all solutions of the system $\mathcal{S}$ asymptotically decay to zero, [7]. We also note that in this case, the question of asymptotic stability is also related to the observability, see [3].

By using (20) and (22) in (5), (6), we obtain the following boundary conditions for the closed loop system

$$(\ B_i^1 w + \alpha_i^1 O_i^1 w_t \ )(0) = 0 \ , \ i = 1, \ldots, k, \tag{25}$$

$$(\ B_i^2 w + \alpha_i^2 O_i^2 w_t \ )(1) = 0 \ , \ i = 1, \ldots, l. \tag{26}$$

Let us consider the boundary conditions (25) and (26). To incorporate these in the closed-loop system, we define the following set

$$\begin{aligned} \mathcal{S}_{1c} \ = \{ (u \ v)^T \in H \times H \mid & (\ B_j^1 u + \alpha_j^1 O_j^1 v \ )(0) = 0 \ . \\ & (\ B_i^2 u + \alpha_i^2 O_i^2 v \ )(1) = 0 \\ & j = 1, \ldots, k, \quad i = 1, \ldots, l \} \end{aligned} \tag{27}$$

We also define the following set

$$D(A_c) = D(A) \cap \mathcal{S}_2, \tag{28}$$

where $\mathcal{S}_2$ is given by (8). The closed loop system can be rewritten as

$$\dot{z} = \mathcal{A}z \ , \ z(0) \in X, \tag{29}$$

where $X = V \times H$, the operator $\mathcal{A}$ is given by (15) and

$$D(\mathcal{A}) = (D(A_c) \times V) \cap \mathcal{S}_{1c}. \tag{30}$$

This system will be referred as the system $\mathcal{S}_c$. For this system we will make the following assumption.

Assumption 5 : The operator $\lambda I - \mathcal{A} : D(\mathcal{A}) \subset X \to X$ is onto for all $\lambda > 0$. □

A simple consequence of this assumption is given in the following theorem.

Theorem 1 : Consider the system $\mathcal{S}_c$ given by (29) and let the Assumptions 1-5 hold. Then the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $T(t)$ on X. If $z(0) \in D(\mathcal{A})$, then $z(t) = T(t)z(0)$ is the unique *classical* solution of (29) and $z(t) \in D(\mathcal{A})$ for $t \geq 0$. If $z(0) \in X$, then $z(t) = T(t)z(0)$ is the unique *weak* solution of (29).

Proof : The proof easily follows from the assumptions and the Lümer-Phillips Theorem, see [15], [7]. □

The following assumptions are required to establish some asymptotic stability results.

Assumption 6 : The operator $(\lambda I - \mathcal{A})^{-1} : X \to X$ is compact for $\lambda > 0$. □

Assumption 7 : The only invariant solution of (29) in the set $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$ is the zero solution, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are given by (7), (8) and $\mathcal{S}_3$ is given by

$$\begin{aligned} \mathcal{S}_3 \ = \{ (u \ v)^T \in H \times H \mid & \qquad \square \\ (\ O_i^1 v \ )(0) = 0 \ , \ (\ O_j^2 v \ )(1) = 0 & \\ for \ \alpha_i^1 > 0, i = 1, \ldots, k, \ for \ \alpha_j^2 > 0, j = 1, \ldots, l \}. & \end{aligned} \tag{31}$$

Remark 2 : We note that in most of the examples encountered in the literature the Assumptions 5 and 6 are satisfied, see e.g. [4, p. 187]. For the assumption 7, we need to solve (4) in $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$. In most of the cases $\mathcal{S}_3$ introduces extra boundary conditions, and due to these conditions in most of the examples the Assumption 7 is also satisfied. □

Theorem 2 : Let the assumptions 1-7 hold, consider the system $\mathcal{S}_c$ given by (29), and let $T(t)$ be the unique C_0-semigroup generated by $\mathcal{A}$. Then, the system $\mathcal{S}_c$ is globally asymptotically stable, that is for any $z(0) \in X$, the unique (clasical or weak) solution $z(t) = T(t)z(0)$ of (29) asymptotically approaches to zero, i.e. $\lim_{t\to\infty} \|z(t)\|_X = 0$.

Proof : Proof follows from the assumptions and the LaSalle's invariance theorem, see [7]. □

To establish the exponential stability, we may use the following well-known result.

Theorem 3 : Let the assumptions 1-5 hold, consider the system $\mathcal{S}_c$ given by (29), and let $T(t)$ be the unique C_0-semigroup generated by $\mathcal{A}$. Then $T(t)$ is exponentially stable, i.e. the following holds for some $M > 0$, $\delta > 0$

$$\|T(t)\|_X \leq M e^{-\delta t} \|z(0)\|_X \ \ , \tag{32}$$

if and only if the imaginary axis belongs to the resolvent set of $\mathcal{A}$ and the following holds

$$\sup_{\omega} \|(j\omega I - \mathcal{A})^{-1}\|_X < \infty \tag{33}$$

Proof : This result is known as Huang's Theorem, see e.g. [7] □

In the applications, the difficult part in using the Theorem 3 is to establish (33). Alternatively, we may use the so-called energy multiplier methods. One such result is given below.

Theorem 4 : Consider the system $\mathcal{S}_c$ given by (29) and let the assumptions 1-5 hold. Let $T(t)$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Let $z = (u\ v)^T \in H$ and let us define the projections $P_1 : X \to V$, $P_2 : X \to H$ as $P_1 z = u$, $P_2 z = v$. Let $z(0) \in D(\mathcal{A})$ and let $z(t)$ denote the solution of (29). Assume that for a linear map $O : H \to H$ the following holds

$$|< P_2 z(t), O P_1 z(t) >_H| \leq C E(t), \tag{34}$$

$$\begin{aligned}\frac{d}{dt} < P_2 z(t), O P_1 z(t) >_H \leq -E(t) \\ + \sum_{i=1}^{k} a_i^1 (f_i^1)^2 + \sum_{i=1}^{l} a_i^2 (f_i^2)^2,\end{aligned} \tag{35}$$

where $C > 0$ and a_i^j are arbitrary constants. Then the system $\mathcal{S}_c$ is exponentially stable, i.e. (32) holds.

Proof : See e.g. [7] □

The result given above can be used rather easily. However, note that this is only a sufficient condition, and that it may not be applicable to certain cases.

III. Examples

Consider the wave equation given below. For convenience the relevant coefficients are assumed to have unit value :

$$w_{tt} - w_{xx} = 0\ ,\ 0 < x < 1\ ,\ t \geq 0, \tag{36}$$

$$w(0,t) = \ ,\ w_x(1,t) = f(t), \tag{37}$$

where $f(t)$ is the boundary control input. We set $H = L_2(0,1)$, $Au = -u''$ where a prime denotes derivative, and $D(A) = \{u \in H \mid u, u', u'' \in H\}$. Note that $D(A)$ is dense in H, hence the Assumption 1 holds. By comparing (37) with (5)- (6) we see that B_i^1 and B_i^4 do not exist, (i.e. $k = r = 0$), $l = p = 1$, $B_1^2 w = w'$, $B_1^3 w = w$ and $f_1^2 = f$. Hence we have $\mathcal{S}_1 = \{w \in H \mid w'(1) = 0\}$, $\mathcal{S}_2 = \{w \in H \mid w(0) = 0\}$. It then follows from (10) that $D(A_{uc}) = \{w \in H \mid w \in D(A), w(0) = w'(1) = 0\}$. Note that $D(A_{uc})$ is dense in H, moreover we have :

$$< w, A_{uc} w > = -\int_0^1 w w'' dx = \int_0^1 (w')^2 dx \tag{38}$$

By using (37), we obtain :

$$\int_0^1 w^2 dx \leq \int_0^1 (w')^2 dx \ \ . \tag{39}$$

Hence, A_{uc} is coercive and the Assumption 2 holds. It can be shown that we can choose V as

$$V = \{w \in H \mid w, w' \in H, w(0) = 0\}, \tag{40}$$

see [7], hence the Assumption 3 also holds. Accordingly we have $X = V \times H$, with the following inner-product

$$< z_1, z_2 >_X = \int_0^1 u_1' u_2' dx + \int_0^1 v_1 v_2 dx, \tag{41}$$

where $z_1 = (u_1\ v_1)^T, z_2 = (u_2\ v_2)^T$, see (16). To check the Assumption 4, first note that $D_1 = \{w \in H \mid w \in D(A), w(0) = 0\}$. For $z = (u\ v)^T \in D_1 \times V$, by using (41) we obtain

$$\begin{aligned}< z, \mathcal{A} z >_X &= \textstyle\int_0^1 u'v' dx + \int_0^1 v u'' dx \\ &= u'(1)v(1) = (\ B_1^2 u\ )(1) v(1).\end{aligned} \tag{42}$$

Note that (42) has the same form as (18) with $O_1^2 v = v$. Hence, the Assumption 4 also holds. For the system (36)-(37), according to the power form, the appropriate output is

$$y_1^2(t) = w_t(1,t). \tag{43}$$

By using (22) and (43), we obtain :

$$w_x(1,t) = -\alpha w_t(1,t). \tag{44}$$

By using (27) and (28) we obtain

$$\mathcal{S}_{1c} = \{(u\ v)^T \in H \times H \mid u'(1) + \alpha v(1) = 0\},$$

$$D(A_c) = \{u \in D(A) \mid u(0) = 0\}\ \ . \tag{45}$$

Therefore, the system given above can be put into the form (29). Note that in this case $D(\mathcal{A})$ given by (30) becomes

$$\begin{aligned}D(\mathcal{A}) &= \{(u\ v)^T \in X \mid u \in D(A_c), v \in V, \\ &\qquad u'(1) + \alpha v(1) = 0\}, \\ &= \{(u\ v)^T \in X \mid u, u', u'' \in H, v, v' \in V, \\ &\qquad u(0) = v(0) = 0, u'(1) + \alpha v(1) = 0\}.\end{aligned} \tag{46}$$

It can be shown that $\lambda I - \mathcal{A} : D(\mathcal{A}) \subset X \to X$ is onto for $\lambda > 0$, see e.g. [12]. Hence, by Theorem 1, $\mathcal{A}$ generates a C_0-semigroup of contractions on X. It can be shown that the Assumptions 6-7 also hold in this case as well, hence $T(t)$ is asymptotically stable as well. To prove exponential stability, we may use both Theorem 3 and 4. For the latter, note that $E(t)$ given by (19) takes the following form

$$E(t) = \frac{1}{2}\int_0^1 w_t^2 dx + \frac{1}{2}\int_0^1 w_x^2 dx, \tag{47}$$

see (17). An appropriate function O could be given as

$$< P_2 z(t), O P_1 z(t) >_H = \int_0^1 x w_t w_x dx. \tag{48}$$

By comparing (34) with (48) we see that $(\ O w\ )(x) = x w'(x)$ for $w \in V$. After straightforward calculations we obtain the following

$$|\int_0^1 x w_t w_x dx\ \ | \leq \int_0^1 |\ w_t w_x \ | \, dx \leq 2E(t), \tag{49}$$

hence (34) is satisfied with $C = 2$. Next, by taking the derivative and noting that w is a solution of (36)-(37), we obtain

$$\frac{d}{dt}\int_0^1 x w_t w_x dx = \int_0^1 x w_{xx} w_x dx + \int_0^1 x w_t w_{xt} dx. \tag{50}$$

By using integration by parts, we obtain

$$\int_0^1 x w_{xx} w_x dx = \frac{1}{2} w_x^2(1,t) - \frac{1}{2}\int_0^1 w_x^2 dx, \tag{51}$$

$$\int_0^1 x w_t w_{xt} dx = \frac{1}{2} w_t^2(1,t) - \frac{1}{2}\int_0^1 w_t^2 dx, \quad (52)$$

where we used $w(0,t) = 0$. By using (51), (52) and (44) in (50) we see that (35) is satisfied with $a_1^2 = (\alpha^2+1)/2\alpha^2$. Hence by the Theorem 4, it follows that $T(t)$ is exponentially stable.

As a second example, let us consider the following coupled wave equation :

$$u_{tt} - u_{xx} = \alpha(v-u)\ ,\ 0 < x < 1, t \geq 0, \quad (53)$$

$$v_{tt} - v_{xx} = \alpha(u-v)\ ,\ 0 < x < 1, t \geq 0, \quad (54)$$

$$u(0,t) = \ ,\ u_x(1,t) = f(t), \quad (55)$$

$$v(0,t) = \ ,\ v_x(1,t) = g(t), \quad (56)$$

see e.g. [14]. Here, $\alpha > 0$ is the coupling constant, $f(t)$ and $g(t)$ are the boundary control forces. We set $H = L_2(0,1) \times L_2(0,1)$. The operator $A : H \to H$ is defined as

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u'' - \alpha(v-u) \\ -v'' - \alpha(u-v) \end{pmatrix} \quad . \quad (57)$$

Similar to previous example, we have

$$D(A) = \{(u\ v)^T \in H \mid u, u', u'', v, v', v'' \in H\}\ .$$

Since $D(A)$ is dense in H, the Assumption 1 holds. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ can be found as

$$\mathcal{S}_1 = \{(u\ v)^T \in H \mid u(0) = v(0) = 0\}\ ,$$

$$\mathcal{S}_2 = \{(u\ v)^T \in H \mid u'(1) = v'(1) = 0\}\ .$$

Consequently, $D(A_{uc})$ is found as

$$\begin{aligned} D(A_{uc}) &= \{(u\ v)^T \in H \mid (u\ v)^T \in D(A)\ , \\ &\quad u(0) = v(0) = 0\ ,\ u'(1) = v'(1) = 0\}\ . \end{aligned}$$

For $z = (u\ v)^T$, we obtain

$$\begin{aligned} < z, A_{uc} z >_H &= \textstyle\int_0^1 [u(-u'' - \alpha(v-u)) \\ &\quad + v(-v'' - \alpha(u-v))]dx \\ &= \textstyle\int_0^1 ((u')^2 + (v')^2)dx + \alpha\int_0^1 (u-v)^2 dx \end{aligned} \quad (58)$$

From (39) it follows that A_{uc} is coercive, hence the Assumption 2 holds. As in previous example, we may choose V as

$$V = \{(u\ v)^T \in H \mid (u', v')^T \in H, \quad u(0) = v(0) = 0\ ,\ \}$$

It then easily follows that the Assumption 3 is also satisfied. Accordingly we have $X = V \times H$ with the usual extension of the inner product in $L^2(0,1)$.

To show that the Assumption 4 is also satisfied, first note that $D_1 = D(A) \cap \mathcal{S}_2$ is dense in $D(A_{uc})$. Let us set $z = (u\ v\ u_1\ v_1)^T \in X$, and $\tilde{z}$ similarly. From (58) it follows that the appropriate inner product in X is the following :

$$\begin{aligned} < z, \tilde{z} >_X &= \tfrac{1}{2}(\textstyle\int_0^1 (u\tilde{u} + v\tilde{v} + u_1\tilde{u}_1 + v_1\tilde{v}_1 \\ &\quad + \alpha(u-v)(\tilde{u}-\tilde{v}))dx) \end{aligned} \quad (59)$$

By using the inner product given in (59), using integration by parts, after straightforward calculations we obtain the following

$$< z, \mathcal{A}z >_X = u'(1)u_1(1) + v'(1)v_1(1) \quad (60)$$

for any $z \in D_1 \times V$. It then follows easily that the Assumption 4 is also satisfied. Let $z = (u\ v\ u_t\ v_t)^T \in D(\mathcal{A})$ be the solution of (54)-(56). Note that the Energy expression given by (19) becomes

$$\begin{aligned} E(t) &= \tfrac{1}{2} < z(t), z(t) >_X = \tfrac{1}{2}(\textstyle\int_0^1 (u_t^2 + v_t^2 \\ &\quad + (u')^2 + (v')^2 + \alpha(u-v)^2)dx) \end{aligned} \quad (61)$$

Hence from (59)-(61) we obtain :

$$\frac{dE}{dt} = f(t)u_t(1,t) + g(t)v_t(1,t) \quad (62)$$

Therefore, the outputs y_1 and y_2 should be chosen as :

$$y_1 = u_t(1,t) \quad , \quad y_2 = v_t(1,t) \quad (63)$$

By using (22) and (43), we obtain :

$$f(t) = -\alpha_1 u_t(1,t) \ \ ,\ \ g(t) = -\alpha_2 v_t(1,t)\ . \quad (64)$$

By using (27) and (28) we obtain

$$\mathcal{S}_{1c} = \{(z \in X \mid u'(1) + \alpha_1 u_1(1) = 0, v'(1) + \alpha_2 v_1(1) = 0\},$$

$$D(A_c) = \{u \in D(A) \mid u(0) = v(0) = 0\}$$

Therefore, the system given above can be put into the form (29). Note that in this case $D(\mathcal{A})$ given by (30) becomes

$$\begin{aligned} D(\mathcal{A}) &= \{z \in X \mid (u\ v)^T \in D(A_c), (u_1\ v_1)^T \in V, \\ &\quad u'(1) + \alpha_1 u_1(1) = 0 \quad , \quad v'(1) + \alpha_2 v_1(1) = 0\}, \end{aligned} \quad (65)$$

Similar to the previous example, it can be shown that $\lambda I - \mathcal{A} : D(\mathcal{A}) \subset X \to X$ is onto for $\lambda > 0$, see e.g. [12]. Hence, by Theorem 1, $\mathcal{A}$ generates a C_0-semigroup of contractions on X. As in previous example, the Assumption 6 is also satisfied. To prove the assumption 7, let us assume that $\alpha_1 > 0$ and $\alpha_2 = 0$, i.e. only one boundary control force is active. In this case, the set $\mathcal{S}_3$ given by (31) is found as

$$\mathcal{S}_3 = \{z \in X \mid u_1 = 0\}\ .$$

Hence accordingly we should look at the nonzero solutions of the system given by (53)-(56) with

$$f(t) = 0 \ \ ,\ \ g(t) = 0 \ \ ,\ \ u_t(1,t) = 0\ .$$

By using separation of variables, see e.g., [13], we could find the possible solutions of this system. Note that, by using $w^+ = u + v$, $w^- = u - w$, this system of equations can be reduced to two decoupled system of equations of the form

$$w_{tt}^+ - w_{xx}^+ = 0 \ \ ,\ \ w^+(0) = 0 \ \ ,\ \ w^{+\prime}(1) = 0\ ,$$

$$w_{tt}^- - w_{xx}^- + 2\alpha w^- = 0 \ \ ,\ \ w^-(0) = 0 \ \ ,\ \ w^{-\prime}(1) = 0\ .$$

It can be shown that the natural frequencies of the first system are given by $\omega_i^+ = \frac{(2i+1)\pi}{2}$, $i = 0, 1, \ldots$ (i.e. the eigenvalues are $\lambda_i = \jmath\omega_i^+$). Similarly, the natural frequencies of the second system are given by $\omega_i^- = \sqrt{2\alpha + (\omega_i^+)^2}$. By using these and the eigenvalue expansion, and noting that $2u = w^+ + w^-$, it follows that to have a nontrivial solution satisfying $u_t(1,t) = 0$, for some i and j, we must have $\omega_i^+ = \omega_j^-$. Therefore, if this equation is not satisfied, then the only possible solution of this system is the trivial (i.e. zero) solution. Hence we conclude that if

$$\alpha \neq \frac{1}{2}((\omega_i^+)^2 - (\omega_j^+)^2)$$

for any i, j, then the system given above is asymptotically stable. It can also be shown that in this case exponential stability does not hold; and when $\alpha_2 > 0$ holds as well, this system is exponentially stable, see [14]. We simulated this system for $\alpha_1 = 0$, $\alpha_2 = 0.1$, $\alpha = 1$, and the simulation results are shown in the following figures. As can be seen, the asymptotic stability holds.

IV. Conclusions

In this work we considered the feedback stabilization of a class of *passive* infinite dimensional systems, by means of boundary control. We utilized the general framework introduced in [7] for such systems. We first gave some general results for the stabilization of this class of infinite dimensional systems by means of boundary control techniques. In this class of systems the inputs and outputs are assumed to act on the boundaries of the system. For this class of systems, we first investigated the effect of a simple feedback law and proved certain stability results. Some of the examples frequently encountered in the literature (e.g. the wave equation, the Euler-Bernoulli and the Timoshenko beam equations) can be viewed in this class. We then considered certain examples and applied the proposed approach for the stabilization of these systems. We also presented some simulation results. We note that this approach could be generalized to dynamic boundary controllers, see [7].

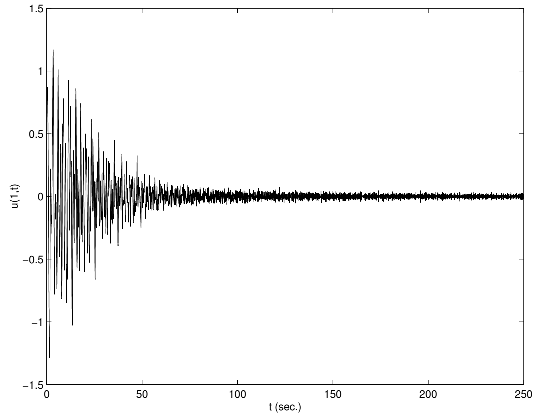

Fig. 1. u(1,t) vs. t

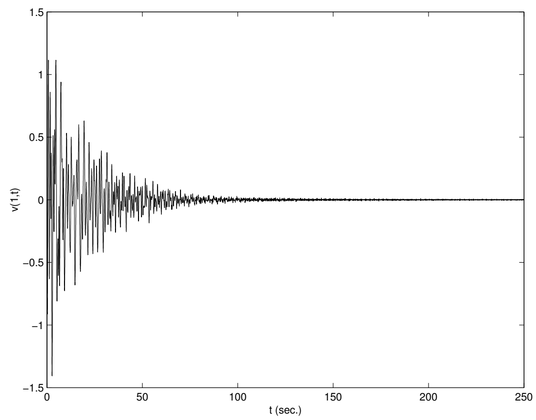

Fig. 2. v(1,t) vs. t

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