A Pencil Equivalent of a General 2-D Polynomial Matrix

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Abstract— **In this paper, it is shown that any arbitrary 2-D polynomial matrix is equivalent to a *pencil* form. The exact form of both the matrix *pencil* and the transformation linking it to the original matrix are established.**

Keywords— **Matrix Pencils, 2-D Singular Systems, Zero-Coprime-Equivalence, Invariant Polynomials.**

I. INTRODUCTION

Matrix *pencils* play an important role in the theory of 1-D linear systems, see for example Rosenbrock [1], Verghese [2], Hayton et al. [3] and Karampetakis et al. [4]. In the 2-D case, matrix pencils arise in the description of 2-D singular state space systems such as those studied by for example Kaczorek [5]. The reduction of an arbitrary 2-D polynomial matrix to pencil form was first studied by Pugh et al.[6]. Their procedure consists of the application of a two stage algorithm which involves the removal of factors from certain matrices to ensure that the transformations linking the original matrix with the final matrix pencil are polynomial. The method does not give a priori the form of neither the resulting *2-D matrix pencil* nor the transformation linking it to the original polynomial matrix. In the present work, it will be shown that every 2-D polynomial matrix is equivalent to a pencil form and the exact form of both the matrix pencil and the transformation linking it to the polynomial matrix will be given in terms of coefficient matrices of the original polynomial matrix. The transformation linking the original matrix with its associated pencil is shown to be *zero coprime equivalence.* This type of equivalence has been studied by Levy [7] , Johnson [8] and Pugh et al. [9] and has been shown by Pugh et al. [10] to provide the connection between all least order polynomial realizations of a given 2-D transfer function matrix.

II. 2-D MATRIX PENCIL

Consider the following 2-D singular state space system as studied by Kaczorek [5]:

$$\begin{aligned} Ex(i+1,j+1) =& A_1x(i+1,j) + A_2x(i,j+1) \\ &+ A_0x(i,j) + B_1u(i+1,j) \\ &+ B_2u(i,j+1) + B_0u(i,j) \qquad (1) \\ y(i,j) =& Cx(i,j) + Du(i,j) \qquad (2) \end{aligned}$$

where $x(i,j)$ is the state vector, $u(i,j)$ is the input vector, $y(i,j)$ is the output vector, E, A_0, A_1, A_2, B_0, B_1, B_2, C, D are real constant matrices of appropriate dimensions and E may be singular. Then, taking the $2-D$ *z-transform* of (1) and (2) and assuming zero boundary conditions yields

$$\begin{bmatrix} szE - sA_1 - zA_2 - A_0 & -sB_1 - zB_2 - B_0 \\ C & D \end{bmatrix} \begin{bmatrix} \overline{x}(s,z) \\ \overline{u}(s,z) \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{y}(s,z) \end{bmatrix} \qquad (3)$$

The polynomial matrix over $\mathbb{R}[s,z]$,

$$szE - sA_1 - zA_2 - A_0 \qquad (4)$$

in (3), is called a *2-D matrix pencil.*

Definition 1: Two matrices $Q(s,z)$ and $M(s,z)$ of appropriate dimensions, are said to be *zero left coprime* if the compound matrix $\begin{bmatrix} Q(s,z) & M(s,z) \end{bmatrix}$ has full rank for all complex values of the indeterminate pair (s,z). Similarly, $P(s,z)$ and $N(s,z)$, of appropriate dimensions, are said to be *zero right coprime* if the compound matrix $\begin{bmatrix} P^T(s,z) & N^T(s,z) \end{bmatrix}^T$ has full rank for all complex values of the indeterminate pair (s,z).

Definition 2: Two matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m,n)$, where $\mathbb{P}(m,n)$ denotes the class of $(r+m)\times(r+n)$ polynomial matrices where m,n are fixed positive integers and r is variable and ranges over all integers greater than $max(-m,-n)$, are said to be *zero coprime equivalent* (*z.c.e.*) if they are related by the following

$$M(s,z)P(s,z) = Q(s,z)N(s,z) \qquad (5)$$

where $Q(s,z)$, $M(s,z)$ are *zero left coprime* and $P(s,z)$, $N(s,z)$ are *zero right coprime.*

Zero coprime equivalence is an extension of Fuhrmann's [11] *strict system equivalence* from the 1-D to the 2-D setting and has been found to preserve important polynomial matrix properties.

Lemma 1: (Pugh et al. 1996) : Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m,n)$, are related by *z.c.e.* and let $\varepsilon_1^{[P]}, \varepsilon_2^{[P]}, \ldots, \varepsilon_h^{[P]}$, where $h = \min(r^{[P]} + m, r^{[P]} + n)$, denote the invariant polynomials of P and $\varepsilon_1^{[Q]}, \varepsilon_2^{[Q]}, \ldots, \varepsilon_k^{[Q]}$, where $k = \min(r^{[Q]} + m, r^{[Q]} + n)$, denote the invariant polynomials of Q, then

$$\varepsilon_{h-i}^{[P]} = c_i \varepsilon_{k-i}^{[Q]} \quad \text{for } i = 0, 1, \ldots, \max(k-1, h-1) \qquad (6)$$

where $\varepsilon_j^{[P]} = 1, \varepsilon_j^{[Q]} = 1$ for any $j < 1$, $c_i \in \mathbb{R}\backslash\{0\}$.

III. EQUIVALENCE TO PENCIL FORM

Let $P(s,z)$ be an $m \times n$ polynomial matrix over $\mathbb{R}[s,z]$, then $P(s,z)$ can be written as :

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$$P(s,z)=\sum_{i=0}^{p}\sum_{j=0}^{q}P_{i,j}s^iz^j \tag{7}$$

where $P_{i,j}, i=0,1,...,p, j=0,1,...,q$ are $m\times n$ real constant matrices.

Now consider the following constant matrices

$$E=\begin{bmatrix} & 0_{n(pq-1),npq} & & \\ E_q & E_{q-1} & \cdots & E_1\end{bmatrix} \tag{8}$$

where

$$E_j=\begin{bmatrix} P_{p,j} & P_{p-1,j} & \cdots & P_{1,j}\end{bmatrix}, j=1,2,\dots,q. \tag{9}$$

$$A_0=Diag(-I_{n(pq-1)},P_{0,0}), \tag{10}$$

$$A_1=\begin{bmatrix} & 0_{np(q-1),npq} & & & \\ & 0_{n(p-1),n(pq-p+1)} & I_{n(p-1)} & & \\ 0_{m,np(q-1)} & -P_{p,0} & -P_{p-1,0} & \cdots & -P_{1,0}\end{bmatrix}, \tag{11}$$

and

$$A_2=\begin{bmatrix} 0_{np(q-1),np} & I_{np(q-1)} & & \\ & 0_{n(p-1),npq} & & \\ A_{2,q} & A_{2,q-1} & \cdots & A_{2,1}\end{bmatrix} \tag{12}$$

where

$$A_{2,j}=\begin{bmatrix} 0_{m,n(p-1)} & -P_{0,j}\end{bmatrix}, j=1,2,\dots,q. \tag{13}$$

Then, the $[n(pq-1)+m]\times npq$ polynomial matrix

$$Q_P(s,z)=szE-sA_1-zA_2-A_0 \tag{14}$$

is the *2-D matrix pencil* corresponding to the polynomial matrix $P(s,z)$.

IV. THEOREM 1

If $P(s,z)$ is an arbitrary $m\times n$ polynomial matrix over $\mathbb{R}[s,z]$ given by (7) and $Q_P(s,z)$ is the corresponding $[n(pq-1)+m]\times npq$ *2-D matrix pencil*, then $P(s,z)$ and $Q_P(s,z)$ are related by the following *z.c.e.* transformation

$$M(s,z)P(s,z)=Q_P(s,z)N(s,z) \tag{15}$$

where

$$M(s,z)=\begin{bmatrix} 0_{n(pq-1),m} \\ I_m\end{bmatrix}, N(s,z)=\begin{bmatrix} N_1\\ N_2\\ \vdots \\ N_q\end{bmatrix}\otimes I_n, \tag{16}$$

$\otimes$ denotes the Kronecker matrix product and

$$N_j=\begin{bmatrix} s^{p-1}z^{q-j} & s^{p-2}z^{q-j} & \cdots & z^{q-j}\end{bmatrix}^T, j=1,2,\dots,q. \tag{17}$$

Proof: From the construction of $Q_P(s,z)$, it can be easily verified that

$$M(s,z)P(s,z)=Q_P(s,z)N(s,z)=\begin{bmatrix} 0_{n(pq-1),n} \\ P(s,z)\end{bmatrix} \tag{18}$$

Now it remains to prove that the matrices $Q_P(s,z)$, $M(s,z)$ are *zero left coprime* and the matrices $P(s,z), N(s,z)$ are *zero right coprime*. This follows from the fact that the minor obtained by deleting the columns $n(pq-1)+1, n(pq-1)+2\dots,npq$ of the matrix

$$\begin{bmatrix} Q_P(s,z) & M(s,z)\end{bmatrix} \tag{19}$$

is equal to 1 and the minor obtained by deleting the rows $1,2,\dots,npq$ of the matrix

$$\begin{bmatrix} P(s,z) \\ N(s,z)\end{bmatrix} \tag{20}$$

is equal to 1. ■

V. EXAMPLE

Consider the 3×3 matrix $P(s,z)$ over $\mathbb{R}[s,z]$ given by

$$P(s,z)\equiv\begin{bmatrix} P_1(s,z) & P_2(s,z) & P_3(s,z)\end{bmatrix} \tag{21}$$

where

$$P_1(s,z)=\begin{bmatrix} -2(z+1)s^2+(3z+2)s-z+2 \\ 3s^2-1 \\ (z-1)s^2-(z-2)s\end{bmatrix} \tag{22}$$

$$P_2(s,z)=\begin{bmatrix} (z-4)s-z+4 \\ -zs-2z \\ (z+2)s^2-4z-1\end{bmatrix} \tag{23}$$

$$P_3(s,z)=\begin{bmatrix} s^2-2 \\ (z+1)s^2-(z-3)s-3z+1 \\ -2zs^2-(5z+2)s-2z+\end{bmatrix} \tag{24}$$

Here $m=n=3$, $p=2$ and $q=1$.
Using a Maple procedure, the invariant polynomials of $P(s,z)$ are computed as :

$$\begin{aligned}\varepsilon_1^{[P]} &= \varepsilon_2^{[P]} = 1\\ \varepsilon_3^{[P]} &= -(2z^3-5z^2-12z-10)s^6\\ &-\left(8z^3+20z^2+25z-12\right)s^5\\ &-\left(12z^3-6z^2-101z+55\right)s^4\\ &+\left(21z^3-z^2+42z+16\right)s^3\\ &+\left(26z^3+63z^2+160z-3\right)s^2\\ &-\left(33z^3+8z^2+25z+4\right)s\\ &+8z^3-9z^2-30z+12\end{aligned} \tag{25}$$

Writing $P(s,z)$ in the form (7), the coefficient matrices $P_{i,j}$ are given by

$$P_{0,0}=\begin{bmatrix} 2 & 4 & -2\\ -1 & 0 & 1\\ 0 & -1 & 3\end{bmatrix}, P_{0,1}=\begin{bmatrix} -1 & -1 & 0\\ 0 & -2 & -3\\ 0 & -4 & -2\end{bmatrix},$$

$$P_{1,0}=\begin{bmatrix} 2 & -4 & 0\\ 0 & 0 & 3\\ 2 & 0 & -2\end{bmatrix}, P_{1,1}=\begin{bmatrix} 3 & 1 & 0\\ 0 & 0 & -1\\ -1 & 0 & -5\end{bmatrix}, \tag{27}$$

$$P_{2,0}=\begin{bmatrix} -2 & 0 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}, P_{2,1}=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Then, constructing the 6×6 *2-D matrix pencil* $Q_P(s,z)$ corresponding to (14) gives

$$Q_P(s,z)=\begin{bmatrix} I_3 & -sI_3 \\ Q_1(s,z) & Q_2(s,z) \end{bmatrix} \quad (29)$$

where

$$Q_1(s,z)=\begin{bmatrix} -2s-2sz & 0 & s \\ 3s & -sz & s+sz \\ -s+sz & 2s+sz & -2sz \end{bmatrix}, \quad (30)$$

$$Q_2(s,z)=$$
$$\begin{bmatrix} 2-z+2s+3sz & 4-z-4s+sz & -2 \\ -1 & -2z & 1-3z+3s-sz \\ 2s-sz & -4z-1 & 3-2z-2s-5sz \end{bmatrix} \quad (31)$$

and the matrices E, A_0, A_1 and A_2 corresponding to $(8,10,11$ and $12)$ are given by

$$E=\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ -2&0&0&3&1&0 \\ 0&-1&1&0&0&-1 \\ 1&1&-2&-1&0&-5 \end{bmatrix},$$

$$A_0=\begin{bmatrix} -1&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&-2&-4&2 \\ 0&0&0&1&0&-1 \\ 0&0&0&0&1&-3 \end{bmatrix},$$

$$A_1=\begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 2&0&-1&-2&4&0 \\ -3&0&-1&0&0&-3 \\ 1&-2&0&-2&0&2 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&2&3 \\ 0&0&0&0&4&2 \end{bmatrix} \quad (32)$$

By virtue of Theorem 1, the polynomial matrix $P(s,z)$ in (21) and the *2-D matrix pencil* $Q_P(s,z)$ in (29) are related by the *zero coprime equivalence* transformation

$$M(s,z)P(s,z)=Q_P(s,z)N(s,z),$$

where

$$M(s,z)=\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}, N(s,z)=\begin{bmatrix} s&0&0 \\ 0&s&0 \\ 0&0&s \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \quad (33)$$

In fact it can be easily verified that

$$M(s,z)P(s,z)=Q_P(s,z)N(s,z)=\begin{bmatrix} 0_{3,3} \\ P(s,z) \end{bmatrix} \quad (34)$$

The matrices $Q_P(s,z), M(s,z)$ are *zero left coprime* and the matrices $P(s,z), N(s,z)$ are *zero right coprime* since the matrices

$$\begin{bmatrix} Q_P(s,z) & M(s,z) \end{bmatrix}, \begin{bmatrix} P(s,z) \\ N(s,z) \end{bmatrix} \quad (35)$$

have respectively a 6×6 and a 2×2 minor which is equal to 1.

The invariant polynomials of $Q_P(s,z)$ are :

$$\begin{aligned} \varepsilon_1^{[Q_P]} &= \varepsilon_2^{[Q_P]} = \varepsilon_3^{[Q_P]} = \varepsilon_4^{[Q_P]} = \varepsilon_5^{[Q_P]} = 1 \\ &= \varepsilon_1^{[P]} = \varepsilon_2^{[P]}, \\ \varepsilon_6^{[Q_P]} &= -(2z^3-5z^2-12z-10)s^6 \\ &\quad -\left(8z^3+20z^2+25z-12\right)s^5 \\ &\quad -\left(12z^3-6z^2-101z+55\right)s^4 \\ &\quad +\left(21z^3-z^2+42z+16\right)s^3 \\ &\quad +\left(26z^3+63z^2+160z-3\right)s^2 \\ &\quad -\left(33z^3+8z^2+25z+4\right)s \\ &\quad +8z^3-9z^2-30z+12 \\ &= \varepsilon_3^{[P]} \end{aligned} \quad (36)$$

which is in accord with Lemma 1.

VI. CONCLUSIONS

In this paper, a *2-D matrix pencil* equivalent of a given arbitrary polynomial matrix has been developed. The matrix obtained is one which arises in the context of the theory of generalized state space 2-D systems. The type and exact form of the equivalence linking the original matrix with its associated *pencil* has been set out and shown to be that of *zero coprime equivalence*. The resulting *2-D matrix pencil* may have a larger size than the one obtained by the algorithm given by Pugh et al. [6]. However the method presented in this paper has the advantage of providing a priori both the form of the final *2-D matrix pencil* and the transformations relating it to the original polynomial matrix.

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