On Hurwitz stability of weighted diamond polynomials

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Abstract— **In this paper, the results on the stability of a weighted diamond of polynomials which was studied by Kharitonov and Tempo[4] are simplified for low-order polynomials. Specifically, for n =3 and 4, the number of polynomials required to check the stability is four and six, respectively, instead of eight. Further, diamond of polynomials which was studied by Barmish, Tempo, Hollot and Kang[2] are also investigated.**

Keywords— **Robust stability, Weighted diamond polynomials, Hurwitz stability.**

I. Introduction

SINCE Kharitonov's theorem was discovered[3], many papers have been studied for interval polynomials. In those papers, there is a simplifications' study for low order polynomials, that is, the paper of Anderson, July and Mansour[1] in which was shown that the number of polynomials required to check robust Hurwitz stability can be reduced. After that, Barmish, Tempo, Hollot and Kang[2] studied vertex polynomials for robust stability of a diamond of polynomials. Further, Kharitonov and Tempo[4] studied vertex polynomials for robust stability of a weighted diamond of polynomials. However, there are no studies about a simplifications for low order polynomials for a diamond and / or a weighted diamond of polynomials.

In this paper, it is shown that the results on the stability of a weighted diamond of polynomials which was studied by [4] are simplified for low-order polynomials. Further, diamond of polynomials which was studied by [2] are also investigated.

II. Preliminaries

For given the $(n+1)$-dimensional vector $q^* := [q_n^*, q_{n-1}^*, \cdots, q_1^*, q_0^*] \in \boldsymbol{R}^{n+1}$, a positive number $r > 0$ and weights $w_i > 0$ $(i = 0, 1, \cdots, n)$, the following set

$$Q_w := \{q = [q_n, q_{n-1}, \cdots, q_1, q_0] \in \boldsymbol{R}^{n+1} \ : \ w_n \mid q_n - q_n^* \mid + w_{n-1} \mid q_{n-1} - q_{n-1}^* \mid + \cdots + w_0 \mid q_0 - q_0^* \mid \leq r\}$$

is said to be a weighted diamond of $\boldsymbol{R}^{n+1}$.

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Consider the following n-th order polynomials given by

$$p(s, q) = q_n s^n + q_{n-1} s^{n-1} + \cdots + q_1 s + q_0,$$

whose coefficients vector $q := [q_n, q_{n-1}, \cdots, q_1, q_0]$ is in a weighted diamond Q_w. Then, the set of polynomials $D_w^n(s) := \{p(s, q) : q \in Q_w\}$ is said to be a weighted diamond of polynomials with the center q^* and the radius r. Especially, in the case of $w_i = 1$, $(i = 0, 1, \cdots, n)$, $D^n(s) := D_w^n(s)$ is said to be a diamond of polynomials. The weighted diamond of polynomials $D_w^n(s)$ is said to be Hurwitz stable if $p(s, q)$ has all its roots in the open left half plane for all $p(s, q) \in D_w^n(s)$. In order to investigate the stability of $D_w^n(s)$ we assume that coefficients of all polynomials in $D_w^n(s)$ are positive, that is, assume that $q_i^* > r/w_i$ $(i = 0, 1, \cdots, n)$.

Now, define the following eight vertex polynomials of weighted diamond of polynomials $D_w^n(s)$ given by

$$p_{w1}(s) := p(s, q^*) + r/w_0,$$

$$p_{w2}(s) := p(s, q^*) - r/w_0,$$

$$p_{w3}(s) := p(s, q^*) + rs/w_1,$$

$$p_{w4}(s) := p(s, q^*) - rs/w_1,$$

$$p_{w5}(s) := p(s, q^*) + rs^{n-1}/w_{n-1},$$

$$p_{w6}(s) := p(s, q^*) - rs^{n-1}/w_{n-1},$$

$$p_{w7}(s) := p(s, q^*) + rs^n/w_n,$$

$$p_{w8}(s) := p(s, q^*) - rs^n/w_n.$$

Then, the following theorem was given by Kharitonov and Tempo[4]

Theorem 1: [4] Assume that the following condition (i) or (ii) holds. However, in the case of $n = 1, 2$, the following conditions are not necessary.

(i) When n $(n \geq 3)$ is odd,

$$w_0 w_n = w_1 w_{n-1},$$

$$\sqrt[2k]{w_0}\ \sqrt[n-1]{w_{n-1}} \leq \sqrt[n-1]{w_0}\ \sqrt[2k]{w_{2k}} \quad (k = 1, 2, \cdots, (n-1)/2),$$

$$\sqrt[2k]{w_1}\,\sqrt[n-1]{w_n} \le \sqrt[n-1]{w_1}\,\sqrt[2k]{w_{2k+1}} \quad (k = 1, 2, \cdots, (n-1)/2).$$

(ii) When n $(n \ge 4)$ is even,

$$\sqrt[n]{w_0}\,\sqrt[n-2]{w_{n-1}} = \sqrt[n-2]{w_1}\,\sqrt[n]{w_n},$$

$$\sqrt[2k]{w_0}\,\sqrt[n]{w_n} \le \sqrt[n]{w_0}\,\sqrt[2k]{w_{2k}} \quad (k = 1, 2, \cdots, (n-2)/2),$$

$$\sqrt[2k]{w_1}\,\sqrt[n-2]{w_{n-1}} \le \sqrt[n-2]{w_1}\,\sqrt[2k]{w_{2k+1}} \quad (k = 1, 2, \cdots, (n-2)/2).$$

Then, $D_w^n(s)$ is Hurwitz stable if and only if $p_{wi}(s)$ $(i = 1, \cdots, 8)$ are Hurwitz stable. ■

Now, define the following eight vertex polynomials of a diamond of polynomials $D^n(s)$ given by

$$p_1(s) := p(s,q^*) + r, \quad p_2(s) := p(s,q^*) - r,$$

$$p_3(s) := p(s,q^*) + rs, \quad p_4(s) := p(s,q^*) - rs,$$

$$p_5(s) := p(s,q^*) + rs^{n-1}, \quad p_6(s) := p(s,q^*) - rs^{n-1},$$

$$p_7(s) := p(s,q^*) + rs^n, \quad p_8(s) := p(s,q^*) - rs^n.$$

Then, the following theorem was given by Barmish, Tempo, Hollot and Kang[2].

Theorem 2: [2] $D^n(s)$ is Hurwitz stable if and only if $p_i(s)$ $(i = 1, \cdots, 8)$ are Hurwitz stable. ■

The following lemma can be used in Section III.

Lemma 3: [5] (Liénard-Chipart) Suppose that n-th polynomial $p(s,q) = q_n s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + q_0$ has the positive coefficients q_i $(i = 0, 1, \cdots, n)$ which are necessary conditions for that to be Hurwitz stable. Define the following matrices and determinants given by

$$M_k^n := \begin{bmatrix} q_{n-1} & q_{n-3} & q_{n-5} & \cdots & 0 \\ q_n & q_{n-2} & q_{n-4} & \cdots & 0 \\ 0 & q_{n-1} & q_{n-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-k} \end{bmatrix} \quad (k = 1, \cdots, n),$$

$$H_1^n(p) := |M_1^n| = |\, q_{n-1} \,|, \; H_2^n(p) := |M_2^n| = \begin{vmatrix} q_{n-1} & q_{n-3} \\ q_n & q_{n-2} \end{vmatrix},$$

$$\cdots, \quad H_n^n(p) := |\, M_n^n \,|.$$

Then, $p(s,q)$ is Hurwitz stable if and only if the following condition (i) or (ii) holds.

(i) $H_i^n(p) > 0$ $(i = 1, 3, 5, \cdots)$.

(ii) $H_i^n(p) > 0$ $(i = 2, 4, 6, \cdots)$. ■

III. Simplifications for low order polynomials

The case of n=1.

Consider the following 1-st order weighted diamond of polynomials $D_w^1(s)$ given by

$$p(s,q) := q_1 s + q_0,$$

where $p(s,q^*) := q_1^* s + q_0^*$, $q^* := [q_1^*, q_0^*] \in \boldsymbol{R}^2$, $w_1|q_1 - q_1^*| + w_0|q_0 - q_0^*| \le r$, $q_i^* > r/w_i$ $(i = 0, 1)$.

Now, the following four vertex polynomials of $D_w^1(s)$ are as follows :

$$p_{w1}(s) = p_{w5}(s) = q_1^* s + (q_0^* + r/w_0),$$

$$p_{w2}(s) = p_{w6}(s) = q_1^* s + (q_0^* - r/w_0),$$

$$p_{w3}(s) = p_{w7}(s) = (q_1^* + r/w_1)s + q_0^*,$$

$$p_{w4}(s) = p_{w8}(s) = (q_1^* - r/w_1)s + q_0^*.$$

Then, since $H_1^1(p_{w1}) = H_1^1(p_{w5}) = q_0^* + r/w_0 > 0$, $H_1^1(p_{w2}) = H_1^1(p_{w6}) = q_0^* - r/w_0 > 0$, and $H_1^1(p_{w3}) = H_1^1(p_{w7}) = H_1^1(p_{w4}) = H_1^1(p_{w8}) = q_0^* > 0$, it follows from Theorem 1 and Lemma 3 that $D_w^1(s)$ is always stable.

The case of n =2.

Consider the following 2-nd order weighted diamond of polynomials $D_w^2(s)$ given by

$$p(s,q) := q_2 s^2 + q_1 s + q_0,$$

where $p(s,q^*) := q_2^* s^2 + q_1^* s + q_0^*$, $q^* := [q_2^*, q_1^*, q_0^*] \in \boldsymbol{R}^3$, $w_2|q_2 - q_2^*| + w_1|q_1 - q_1^*| + w_0|q_0 - q_0^*| \le r$, $q_i^* > r/w_i$ $(i = 0, 1, 2)$.

Now, the following six vertex polynomials of $D_w^2(s)$ are as follows :

$$p_{w1}(s) = q_2^* s^2 + q_1^* s + (q_0^* + r/w_0),$$

$$p_{w2}(s) = q_2^* s^2 + q_1^* s + (q_0^* - r/w_0),$$

$$p_{w3}(s) = p_{w5}(s) = q_2^* s^2 + (q_1^* + r/w_1)s + q_0^*,$$

$$p_{w4}(s) = p_{w6}(s) = q_2^* s^2 + (q_1^* - r/w_1)s + q_0^*,$$

$$p_{w7}(s) = (q_2^* + r/w_2)s^2 + q_1^* s + q_0^*,$$

$$p_{w8}(s) = (q_2^* - r/w_2)s^2 + q_1^* s + q_0^*.$$

Then, since $H_1^2(p_{w1}) = H_1^2(p_{w2}) = q_1^* > 0$, $H_1^2(p_{w3}) = H_1^2(p_{w5}) = q_1^* + r/w_1 > 0$, $H_1^2(p_{w4}) = H_1^2(p_{w6}) = q_1^* - r/w_1 > 0$, $H_1^2(p_{w7}) = H_1^2(p_{w8}) = q_1^* > 0$, it follows from

Theorem 1 and Lemma 3 that $D_w^2(s)$ is always stable.

The case of n =3.

Consider the following 3-rd order weighted diamond of polynomials $D_w^3(s)$ given by

$$p(s,q) := q_3 s^3 + q_2 s^2 + q_1 s + q_0,$$

where $p(s,q^*) := q_3^* s^3 + q_2^* s^2 + q_1^* s + q_0^*$, $q^* := [q_3^*, q_2^*, q_1^*, q_0^*] \in \boldsymbol{R}^4$, $w_3|q_3 - q_3^*| + w_2|q_2 - q_2^*| + w_1|q_1 - q_1^*| + w_0|q_0 - q_0^*| \leq r$, $q_i^* > r/w_i$ $(i = 0,1,2,3)$.

Similarly, the following eight vertex polynomials of $D_w^3(s)$ are as follows :

$$p_{w1}(s) = q_3^* s^3 + q_2^* s^2 + q_1^* s + (q_0^* + r/w_0),$$

$$p_{w2}(s) = q_3^* s^3 + q_2^* s^2 + q_1^* s + (q_0^* - r/w_0),$$

$$p_{w3}(s) = q_3^* s^3 + q_2^* s^2 + (q_1^* + r/w_1)s + q_0^*,$$

$$p_{w4}(s) = q_3^* s^3 + q_2^* s^2 + (q_1^* - r/w_1)s + q_0^*,$$

$$p_{w5}(s) = q_3^* s^3 + (q_2^* + r/w_2)s^2 + q_1^* s + q_0^*,$$

$$p_{w6}(s) = q_3^* s^3 + (q_2^* - r/w_2)s^2 + q_1^* s + q_0^*.$$

$$p_{w7}(s) = (q_3^* + r/w_3)s^3 + q_2^* s^2 + q_1^* s + q_0^*,$$

$$p_{w8}(s) = (q_3^* - r/w_3)s^3 + q_2^* s^2 + q_1^* s + q_0^*.$$

Now, we have

$$H_2^3(p_{w1}) = q_2^* q_1^* - q_3^*(q_0^* + r/w_0),$$

$$H_2^3(p_{w2}) = q_2^* q_1^* - q_3^*(q_0^* - r/w_0),$$

$$H_2^3(p_{w3}) = q_2^*(q_1^* + r/w_1) - q_3^* q_0^*,$$

$$H_2^3(p_{w4}) = q_2^*(q_1^* - r/w_1) - q_3^* q_0^*,$$

$$H_2^3(p_{w5}) = (q_2^* + r/w_2)q_1^* - q_3^* q_0^*,$$

$$H_2^3(p_{w6}) = (q_2^* - r/w_2)q_1^* - q_3^* q_0^*,$$

$$H_2^3(p_{w7}) = q_2^* q_1^* - (q_3^* + r/w_3)q_0^*,$$

$$H_2^3(p_{w8}) = q_2^* q_1^* - (q_3^* - r/w_3)q_0^*.$$

Then, the following relations hold.

$$H_2^3(p_{w2}) - H_2^3(p_{w1}) = 2rq_3^*/w_0 \;\; > 0,$$

$$H_2^3(p_{w3}) - H_2^3(p_{w4}) = 2rq_2^*/w_1 \;\; > 0,$$

$$H_2^3(p_{w5}) - H_2^3(p_{w6}) = 2rq_1^*/w_2 \;\; > 0,$$

$$H_2^3(p_{w8}) - H_2^3(p_{w7}) = 2rq_0^*/w_3 \;\; > 0.$$

If $H_2^3(p_{w1}), H_2^3(p_{w4}), H_2^3(p_{w6}), H_2^3(p_{w7})$ are all positive, then $H_2^3(p_{w2}), H_2^3(p_{w3}), H_2^3(p_{w5}), H_2^3(p_{w8})$ are also positive. Thus, if $p_{w1}(s)$, $p_{w4}(s)$, $p_{w6}(s)$ and $p_{w7}(s)$ are Hurwitz stable, it follows from Lemma 3 that $p_{w2}(s)$, $p_{w3}(s)$, $p_{w5}(s)$ and $p_{w8}(s)$ are also Hurwitz stable, that is, if we assume that the weights w_i $(i = 0,1,2,3)$ satisfy condition (i) in Theorem 1, then $D_w^3(s)$ is Hurwitz stable if and only if four vertex polynomials $p_{w1}(s)$, $p_{w4}(s)$, $p_{w6}(s)$ and $p_{w7}(s)$ are Hurwitz stable.

The case of n =4.

Consider the following 4-th order weighted diamond of polynomials $D_w^4(s)$ given by

$$p(s,q) := q_4 s^4 + q_3 s^3 + q_2 s^2 + q_1 s + q_0,$$

where $p(s,q^*) := q_4^* s^4 + q_3^* s^3 + q_2^* s^2 + q_1^* s + q_0^*$, $q^* := [q_4^*, q_3^*, q_2^*, q_1^*, q_0^*] \in \boldsymbol{R}^5$, $w_4|q_4 - q_4^*| + w_3|q_3 - q_3^*| + w_2|q_2 - q_2^*| + w_1|q_1 - q_1^*| + w_0|q_0 - q_0^*| \leq r$, $q_i^* > r/w_i$ $(i = 0,1,2,3,4)$.

Similarly, the following eight vertex polynomials of $D_w^4(s)$ are as follows :

$$p_{w1}(s) = q_4^* s^4 + q_3^* s^3 + q_2^* s^2 + q_1^* s + (q_0^* + r/w_0),$$

$$p_{w2}(s) = q_4^* s^4 + q_3^* s^3 + q_2^* s^2 + q_1^* s + (q_0^* - r/w_0),$$

$$p_{w3}(s) = q_4^* s^4 + q_3^* s^3 + q_2^* s^2 + (q_1^* + r/w_1)s + q_0^*,$$

$$p_{w4}(s) = q_4^* s^4 + q_3^* s^3 + q_2^* s^2 + (q_1^* - r/w_1)s + q_0^*,$$

$$p_{w5}(s) = q_4^* s^4 + (q_3^* + r/w_3)s^3 + q_2^* s^2 + q_1^* s + q_0^*,$$

$$p_{w6}(s) = q_4^* s^4 + (q_3^* - r/w_3)s^3 + q_2^* s^2 + q_1^* s + q_0^*.$$

$$p_{w7}(s) = (q_4^* + r/w_4)s^4 + q_3^* s^3 + q_2^* s^2 + q_1^* s + q_0^*,$$

$$p_{w8}(s) = (q_4^* - r/w_4)s^4 + q_3^* s^3 + q_2^* s^2 + q_1^* s + q_0^*.$$

Now, we have

$H_1^4(p_{w1}) = H_1^4(p_{w2}) = H_1^4(p_{w3}) = H_1^4(p_{w4}) = H_1^4(p_{w7}) = H_1^4(p_{w8}) = q_3^* \;\; > 0$, $H_1^4(p_{w5}) = q_3^* + r/w_3 \;\; > 0$, $H_1^4(p_{w6}) = q_3^* - r/w_3 \;\; > 0$.

Further, we have

$$H_3^4(p_{w1}) = q_3^* q_2^* q_1^* - q_4^* {q_1^*}^2 - {q_3^*}^2(q_0^* + r/w_0),$$

$$H_3^4(p_{w2}) = q_3^* q_2^* q_1^* - q_4^* {q_1^*}^2 - {q_3^*}^2(q_0^* - r/w_0),$$

$$H_3^4(p_{w3}) = q_3^* q_2^*(q_1^* + r/w_1) - q_4^*(q_1^* + r/w_1)^2 - {q_3^*}^2 q_0^*,$$

$$H_3^4(p_{w4}) = q_3^* q_2^*(q_1^* - r/w_1) - q_4^*(q_1^* - r/w_1)^2 - {q_3^*}^2 q_0^*,$$

$$H_3^4(p_{w5}) = (q_3^* + r/w_3)q_2^* q_1^* - q_4^* {q_1^*}^2 - q_3^* + r/{w_3}^2 q_0^*,$$

$$H_3^4(p_{w6}) = (q_3^* - r/w_3)q_2^* q_1^* - q_4^* {q_1^*}^2 - q_3^* - r/{w_3}^2 q_0^*,$$

$$H_3^4(p_{w7}) = q_3^* q_2^* q_1^* - (q_4^* + r/w_4){q_1^*}^2 - {q_3^*}^2 q_0^*,$$

$$H_3^4(p_{w8}) = q_3^* q_2^* q_1^* - (q_4^* - r/w_4){q_1^*}^2 - {q_3^*}^2 q_0^*,$$

Then, the following relations hold.

$$H_3^4(p_{w2}) - H_3^4(p_{w1}) = 2r{q_3^*}^2/w_0 \ > 0,$$

$$H_3^4(p_{w8}) - H_3^4(p_{w7}) = 2r{q_1^*}^2/w_4 \ > 0. \qquad (1)$$

First, we note that $H_1^4(p_{wi})$ $(i = 1, \cdots, 8)$ are all positive. If $H_3^4(p_{w1})$ and $H_3^4(p_{w7})$ are positive in (1), then $H_3^4(p_{w2})$ and $H_3^4(p_{w8})$ are also positive. Thus, if $p_{w1}(s)$ and $p_{w7}(s)$ are Hurwitz stable, it follows from Lemma 3 that $p_{w2}(s)$ and $p_{w8}(s)$ are also Hurwitz stable, that is, if we assume that the weights w_i $(i = 0, 1, 2, 3, 4)$ satisfy condition (ii) in Theorem 1, then $D_w^4(s)$ is Hurwitz stable if and only if six vertex polynomials $p_{w1}(s)$, $p_{w3}(s)$, $p_{w4}(s)$, $p_{w5}(s)$, $p_{w6}(s)$ and $p_{w7}(s)$ are Hurwitz stable.

The case of $n \geq 5$.

It can be easily shown that the eight vertex polynomials can not be reduced for $D_w^n(s)$ $(n \geq 5)$ to be stable.

The obtained results are summarized as the following theorem which is a result of simplifications for low order polynomials in Theorem 1.

Theorem 4:

(i) $D_w^1(s)$ and $D_w^2(s)$ are always Hurwitz stable.

(ii) Assume that $w_0 w_3 = w_1 w_2$. Then, $D_w^3(s)$ is Hurwitz stable if and only if the four vertex polynomials $p_{w1}(s)$, $p_{w4}(s)$, $p_{w6}(s)$ and $p_{w7}(s)$ are Hurwitz stable.

(iii) Assume that $\sqrt[4]{w_0}\sqrt[2]{w_3} = \sqrt[2]{w_1}\sqrt[4]{w_4}$, $\sqrt[2]{w_0}\sqrt[4]{w_4} \leq \sqrt[4]{w_0}\sqrt[2]{w_2}$. Then, $D_w^4(s)$ is Hurwitz stable if and only if the six vertex polynomials $p_{w1}(s)$, $p_{w3}(s)$, $p_{w4}(s)$, $p_{w5}(s)$, $p_{w6}(s)$ and $p_{w7}(s)$ are Hurwitz stable. ■

The following corollary is a result of simplifications for low order polynomials in Theorem 2.

Corollary 5:

(i) $D^1(s)$ and $D^2(s)$ are always Hurwitz stable.

(ii) $D^3(s)$ is Hurwitz stable if and only if the four vertex polynomials $p_1(s)$, $p_4(s)$, $p_6(s)$ and $p_7(s)$ are Hurwitz stable.

(iii) $D^4(s)$ is Hurwitz stable if and only if the six vertex polynomials $p_1(s)$, $p_3(s)$, $p_4(s)$, $p_5(s)$, $p_6(s)$ and $p_7(s)$ are Hurwitz stable. ■

IV. Conclusions

In this paper, the results on the stability of a weighted diamond of polynomials which was studied by Kharitonov and Tempo[4] were simplified for low-order polynomials. Specifically, for n =3 and 4, it was shown that the number of polynomials required to be checked the stability is four and six, respectively, instead of eight (see TABLE I). Further, the results on the stability of a diamond of polynomials which was studied by Barmish et al.[2] were also simplified for low-order polynomials.

TABLE I

The numbers of necssary vertex polynomials

degree n	necessary polynomials	numbers
1	always stable	0
2	always stable	0
3	p_{wi}, $(i = 1, 4, 6, 7)$	4
4	p_{wi}, $(i = 1, 3, 4, 5, 6, 7)$	6
$n \geq 5$	p_{wi}, $(i = 1, 2, 3, 4, 5, 6, 7, 8)$	8

References

[1] S.D.O. Anderson, E. I. July and M. Mansour, *On Robust Hurwitz Polynomials*, IEEE Transactions on Automatic Control, Vol. AC-32, No.10, pp. 909–913, 1987.

[2] B. R. Barmish, R. Tempo, C. V. Hollot and H. I. Kang, *An Extreme point result for robust stability of a diamond of polynomials*, IEEE Transactions on Automatic Control, Vol. AC-37, No.9, pp. 1460–1462, 1992.

[3] V. L. Kharitonov, *Asymptotic stability of an equilibrium position of a family of systems of linear differential equations*, Differential Equations, Vol. 14, pp. 1483–1485, 1979.

[4] V. L. Kharitonov and R. Tempo, *On the stability of a weighted diamond of real polynomials*, Systems & Control Letters, Vol. 22, pp. 5–7,1994.

[5] M. Marden, *Geometry of Polynomials*, Mathematical Surveys and Monographs, No.3, American Mathematical Society, 1989.