A New Approach to Stochastic Causal System Continualization

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***Abstract*— Necessity to study primarily discrete-time systems follows from new axioms of a recently submitted alternative system theory which introduces newly reviewed causality law into basic system definitions. The system definitions are based on quite new system paradigms stemming from attentive observations and resulting in an axiomatic system theory with correctly and uniquely defined notions. Continuous-time systems are then regarded as limit cases of suitable sequences of discrete-time systems. This limit process is called the continualization procedure. The new approach to system theory provides various cybernetic problems with surprisingly clear and easy solutions.**

I. Introduction

The continuous-time systems should be studied only as limit cases of suitable sequences of previously defined discrete-time systems as follows from the system paradigms of recently submitted new approach to system theory [5]. This is due to the fact that we are able to obtain only a finite number of independent observations while studying the system properties in the real world. When a continuous-time system is given in the context of a cybernetic task, to exhaustively describe its properties is necessary to find such a sequence of discrete-time systems together with all causal dependencies, that converges (in a particular sense) to the given continuous-time system. This extension to infinite sets cannot be based on observations but has to be postulated by appropriate limit process called the continualization procedure.

The most general cybernetic system is believed to be the stochastic causal system. However, since the terms of causality law, causal probability, causal function easily defined in the discrete-time domain are losing their meanings in the continuous-time domain, full attention has to be given to the process of system continualization.

II. Stochastic causal system

The system trajectory is according to real observations generated in a sequence of certain segments, which are determined by an ordered decomposition $\mathcal{D}$ of the definition domain D of the system trajectory s (see [5]). Each segment $s \mid D^{(k,l)}$ is according to the principle of causality law generated (not necessarily in the deterministic way) by its comprehensive immediate cause $s \mid C^{(k,l)}$, whereas for each of these relations the causality law is required to hold. Such an approach assures an unambiguous system trajectory description. The causal system $\boldsymbol{\mathcal{CS}}$ is then defined as an ordered triplet

$$\boldsymbol{\mathcal{CS}} = (T, V, \mathcal{C}), \tag{1}$$

where the set of all system causal relations $\mathcal{C}$ was added to the general abstract system $\boldsymbol{\mathcal{S}} = (T, V)$ defined in [5].

If we admit an axiom that each segment $s \mid D_{k,l}$ of the system trajectory $s \in \Omega$ (Ω is a set of all system trajectories) is generated by its comprehensive immediate cause $s \mid C^{(k,l)}$ in a stochastic way, we can extend the causal system with a set $\mathcal{P}$ of all probabilistic mappings $P^{(k,l)}$ and define the stochastic causal system as an ordered quadruplet

$$\boldsymbol{\mathcal{PCS}} = (T, V, \mathcal{C}, \mathcal{P}), \tag{2}$$

where the set $\mathcal{P}$ consists of parametric probabilities

$$\begin{gathered} P^{(k,l)}(s \mid D^{(k,l)} : s \mid C^{(k,l)}), \\ k = 0, 1, 2, \cdots, e, \; l = 1, 2, \cdots, m. \end{gathered} \tag{3}$$

$P^{(k,l)}$ is called the causal probability of the system $\boldsymbol{\mathcal{PCS}}$, see [5] for more details.

III. Continualization of discrete systems

The extension (continualization) of discrete-time models of real systems to uncountable infinite sets, mostly given in terms of continuous real-number intervals, is considerable only if the necessarily missing knowledge of the system properties can be amended to these larger sets in a suitable fashion. The time continuity of either the real system model or its trajectory is thus only a hypothesis but not an experimentally proved fact of the matter. The cybernetic continuous-time system is therefore to be derived from a discrete-time system.

A. Extension of the time-points set

Let us suppose that there is given a general causal system $\boldsymbol{\mathcal{CS}_0}$ defined on a time-points set T_0

$$T_0 = \{0, h_0, 2h_0, 3h_0, \cdots, n_0 h_0\}, \tag{4}$$

where

$$h_0 = t_i - t_{i-1}, \qquad i = 1, 2, \cdots, e \tag{5}$$

and

$$n_0 h_0 = t_e. \tag{6}$$

The time-points set can be extended by inserting a new time point between each two ones of T_0 in the middle of their distance - see figure 1. In this way the equidistance of time points is protected also in a new time-points set T_1

$$T_1 = \{0,\ \tfrac{h_0}{2},\ 2\tfrac{h_o}{2},\ 3\tfrac{h_o}{2},\ \cdots\ , n_1\tfrac{h_0}{2}\} = \tag{7}$$
$$= \{0,\ h_1,\ 2h_1,\ 3h_1,\ \cdots\ , n_1 h_1\}, \tag{8}$$

distance between elements of which is

$$h_1 = \frac{h_0}{2}\ , \tag{9}$$

where obviously

$$n_1 h_1 = t_e. \tag{10}$$

For an arbitrary $k-th$ step of the extension of the original time points set T_0, a sequence of time points sets is clearly defined as T_k,

$$T_k = \{0, h_k, 2h_k, 3h_k,\ \cdots\ , n_k h_k\}, \quad k = 1, 2, \cdots \tag{11}$$

where

$$h_k = \frac{h_{k-1}}{2} = \frac{h_0}{2^k} \tag{12}$$

and again

$$n_k h_k = t_e. \tag{13}$$

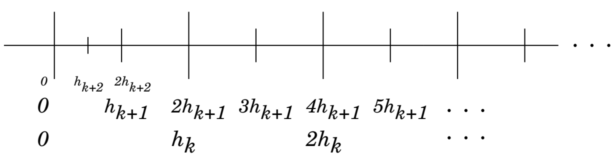

Fig. 1. Extension of the time-points set

Such a sequence of sets is increasing for every $h_0 > 0$. If $k \to \infty$ then the limit of an infinite ascending sequence of sets is the unification,

$$\lim_{k\to\infty} T_k = \bigcup_{k=0}^{\infty} T_k, \tag{14}$$

as shown in [1].

It is obvious from the construction of the sets T_k that for $\lim_{k\to\infty} T_k$ from the equation (14) it holds

$$\textstyle\bigcup_{k=0}^{\infty} T_k\ \subset\ \langle 0\ ;\ t_e\rangle, \tag{15}$$
$$\textstyle\bigcup_{k=0}^{\infty} T_k \quad \text{is a dense set in } \langle 0\ ;\ t_e\rangle. \tag{16}$$

(Topological) closure of the set $\bigcup_{k=0}^{\infty} T_k$ is then to be

$$\overline{\bigcup_{k=0}^{\infty} T_k} \quad = \langle 0\ ;\ t_e\rangle, \tag{17}$$

as the declaration (16) holds. The sequence of sets T_k from the equation (11) converges to a dense subset of the continuous interval $\langle 0\ ,\ t_e\rangle$. A dense subset is "large" enough to ensure correctness of the following abstract thoughts. Let J is an interval in $\mathbb{R}$ and H is a dense subset of J. If f, g are two continuous-time functions on J then $f(x) = g(x)$ for $x \in H$ implies $f = g$ on J. If f is a uniformly continuous function on H then there exists the one and only extension of f to the whole J.

B. *Causal system continualization*

Suppose that a causal system $\mathcal{CS}_0$ is given by the equation (1). Our aim is to propose such a sequence of causal systems

$$\mathcal{CS}_k = (T_k, V_k, \mathcal{C}_k), \qquad k = 0, 1, 2, \cdots, \tag{18}$$

that will converge for $k \to \infty$ to a continuous-time system

$$\mathcal{CS} = (T, V, \mathcal{C}). \tag{19}$$

As for the set of time-points T_k, it is given by equation (11). According to the causal system definition [5], for each $k = 1, 2, 3, \cdots$ it is necessary to define the whole set of system attributes A_k, state variables $s_k(t),\ t \in T_k$ together with the definition domain V_k as well as the set Ω_k of all system trajectories and the set $\mathcal{S}_k$ of all system events.

Naturally, the original system variables (e.g. system attributes, state variables) corresponding to the system $\mathcal{CS}_0$ should be tried to be preserved including their definition domains and the decomposition of the set I_0 (see [5] for more details) for each arbitrary $\mathcal{CS}_k,\ k = 1, 2, \cdots$. However, in some special cases it is unavoidable to change the original system variables as shown e.g. in [3] and being discussed later in this paper.

The time-points set T_k contains T_0 as a subset and therefore the definition domain D_0 of the system trajectory s_0 must be redefined to the definition domain D_k of the system trajectory s_k

$$D_k = T_k \times I_k, \tag{20}$$

including its decomposition from $\mathcal{D}_0$ to

$$\mathcal{D}_k = \{D_k^{(i,j)} \mid D_k^{(i,j)} = T_k^{(i)} \times I_k^{(j)}, T_k^{(i)} \in \mathcal{T}_k, I_k^{(j)} \in \mathcal{I}_k\}, \tag{21}$$
$$i = 0, 1, 2, \ldots, e,$$
$$j = 1, 2, 3, \cdots, m,$$

where $\mathcal{I}_k$ is a partition of the set I_k and $\mathcal{T}_k$ is an ordered decomposition of the set T_k ([5] for more details). The system $\mathcal{CS}_k$ can be now written as an ordered triplet from the equation (18).

In this way we can proceed for any $k = 1, 2, \cdots$, whereupon a sequence of causal system is obtained. The limit case of such a sequence is for $k \to \infty$ the continuous-time system from the equation (19) defined on a dense subset of the interval $\langle\ 0\ ;\ t_e\ \rangle$. However, there arise some problems with the cause-effect relation of the dynamic state variables definition in the continuous-time domain because there is no time instant $t',\quad t' \neq t > 0$ which is immediately preceding to a time instant $t,\ \ t \in \langle 0\ ;\ t_e\rangle$. Therefore, a higher attention should be payed to stochastic properties of the stochastic causal system during the process of continualization. On the other hand, there is no difficulty with static state variables and, consequently, there is no difficulty with the structural terms definition of continuous-time systems either.

C. Sequence of linear stochastic causal system

The task of linear stochastic causal system continualization has previously been solved as a part of the general system theory [5], [6], [7]. However, it faced some terminology problems with random variables convergence [5], [8], [3]. This paper deals with a new approach based on the convergence (in distribution) of cumulative probability distribution functions, which completely describe properties of stochastic causal systems. Generally, there is also a possibility that the limit to a given sequence of systems does not exist. On the one hand, it means that there is no diffusion system corresponding to the given sequence of discrete-time systems. On the other hand, at least one sequence of discrete-time systems can be found for each diffusion system. Consequently, the set of discrete-time systems can be, in a certain way, regarded as ”richer” than the set of continuous-time systems.

At this point we can link the continualization procedure of stochastic systems to the continualization of causal systems. Suppose that there is a stochastic causal system $\boldsymbol{\mathcal{PCS}}_0$ given according to its definition by an ordered quadruplet

$$\boldsymbol{\mathcal{PCS}_0} = (T_0, V_0, \mathcal{C}_0, \mathcal{P}_0) \tag{22}$$

properties of which are described by causal probability density function

$$\begin{aligned} &f_0(s_0(t+h_0) : s_0(t)) = \\ &= \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{detQ_0}} \cdot \\ &\cdot e^{-\frac{1}{2}(s_0(t+h_0)-A_0 \cdot s_0(t))^T \cdot Q_0^{-1} \cdot (s_0(t+h_0)-A_0 \cdot s_0(t))}, \\ &t, t+h_0 \in T_0, \end{aligned} \tag{23}$$

with a probabilistic initial condition

$$\begin{aligned} f_0(s_0(0)) = &\frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{detQ_0(0)}} \cdot \\ &\cdot e^{-\frac{1}{2}(s_0(0)-m_0(0))^T \cdot Q_0^{-1}(0) \cdot (s_0(0)-m_0(0))}, \end{aligned} \tag{24}$$

where $s_0(t) \in \mathbb{R}^N$ is a state vector, $t \in T_0$, A_0 is a real square matrix of the corresponding dimension, Q_0 is a conditional covariance matrix of the system state $s_0(t+h_0)$ conditioned by the state vector $s_0(t)$, m_0 is a vector of the mean and $Q_0(0)$ is a covariance matrix of the generally stochastic initial condition of the system $\boldsymbol{\mathcal{PCS}}_0$.

In any arbitrary $k-th$ step of general causal system continualization a set T_k was defined as well as the sets V_k and C_k were. Now, the set $\mathcal{P}_0$ is to be redefined to $\mathcal{P}_k$ in such a way that any system $\boldsymbol{\mathcal{PCS}}_k$ has equal stochastic properties to the original system $\boldsymbol{\mathcal{PCS}}_0$ for each $k = 1, 2, \cdots$ and $t \in T_0$. From this reason we demand that

$$\begin{aligned} f_k(s_k(t+2^k h_k)|s_k(t)) =& f_k(s_k(t+h_0)|s_k(t)) = \\ =& f_0(s_0(t+h_0) : s_0(t)), \end{aligned} \tag{25}$$

$$h_k = \frac{h_0}{2^k} \implies 2^k h_k = h_0 \tag{26}$$

$$t, t+h_0 \in T_0; \;\; k = 1, 2, \cdots \;,$$

or consequently

$$\begin{aligned} f_k(s_k(t+n_k h_k)|s_k(t)) = f_k(s_k(t_2)|s_k(t_1)) = \\ = f_0(s_0(t+n_0 h_0)|s_0(t)) = f_0(s_0(t_2)|s_0(t_1)), \end{aligned} \tag{27}$$

$$n_k h_k = n_0 h_0 \implies n_k = 2^k n_0 \tag{28}$$

$$n_0, n_k > 0; \;\; t, t+n_0 h_0 \in T_0,$$

with equal initial conditions

$$f_k(s_k(0)) = f_0(s_0(0)), \tag{29}$$

$$\begin{aligned} f_k(s_k(0)) =& \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{detQ_k(0)}} \cdot \\ &\cdot e^{-\frac{1}{2}(s_k(0)-m_k(0))^T \cdot Q_k^{-1}(0) \cdot (s_k(0)-m_k(0))} \end{aligned} \tag{30}$$

where m_k is a vector of the mean and $Q_k(0)$ is a covariance matrix of generally stochastic initial condition of the system $\boldsymbol{\mathcal{PCS}}_k$. In consequence of (25), the equality of unconditioned probability density functions must also hold

$$f_k(s_k(t)) = f_0(s_0(t)), \quad t \in T_0. \tag{31}$$

From the above formulated demands expressed in equations (25), (27), (29) and (31) it is subsequently possible to find out (by comparing right sides of probability density functions in the referred equations) parameters A_k and Q_k of the causal probability density function

$$\begin{aligned} &f_k(s_k(t+h_k) : s_k(t)) = \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{detQ_k}} \cdot \\ &\cdot e^{-\frac{1}{2}(s_k(t+h_k)-A_k \cdot s_k(t))^T \cdot Q_k^{-1} \cdot (s_k(t+h_k)-A_k \cdot s_k(t))}, \\ &t, t+h_k \in T_k, \end{aligned} \tag{32}$$

which fully describes properties of the system $\boldsymbol{\mathcal{PCS}}_k$ for each $k = 1, 2, \cdots$.

1) Derived probability density functions: Using the Markovian properties of the system state we can write

$$\begin{aligned} &f_k(s_k(t+n_k h_k)|s_k(t)) = \\ &= \int_{s_k(t+(n_k-1)h_k)} f_k(s_k(t+n_k h_k) : s_k(t+(n_k-1)h_k)) \cdot \\ &\quad \cdot f_k(s_k(t+(n_k-1)h_k)|s_k(t)) \mathrm{d}s_k(t+(n_k-1)h_k), \end{aligned} \tag{33}$$

in order to find conditional probability density function $f_k(s_k(t+n_k h_k)|s_k(t))$, $t, t+2h_k \in T_k$. After the generalized form of the integral from equation (33) has been found the derived probability density function $f_k(s_k(t+n_k h_k)|s_k(t))$ can be rewritten in the following form

$$\begin{aligned} &f_k(s_k(t+n_k h_k)|s_k(t)) = \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{det \sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i}} \cdot \\ &\cdot e^{-\frac{1}{2}\left(s_k(t+n_k h_k)-A_k^{n_k} s_k(t))^T \left(\sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i\right)^{-1} \cdot (s_k(t+n_k h_k)-A_k^{n_k} s_k(t))\right)}, \end{aligned} \tag{34}$$

$$t, t+n_k h_k \in T_k; \;\; n_k h_k = h_0; \;\; n_k > 0.$$

Using the equation (30) it is possible to find the unconditional probability density function $f_k(s_k(t))$

$$
\begin{aligned}
&f_k(s_k(t)) = f_k\left(s_k(n_k h_k)\right) = \\
&= \int_{s_k(0)} f_k(s_k(n_k h_k)|s_k(0)) \cdot f_k(s_k(0)) \mathrm{d}s_k(0) = \\
&= \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{det\ \mathbf{M}}} \cdot \\
&\cdot e^{-\frac{1}{2}\left(s_k(n_k h_k) - A_k^{n_k} m_k(0)\right)^T \cdot (\mathbf{M})^{-1} \cdot \left(s_k(n_k h_k) - A_k^{n_k} m_k(0)\right)}, \qquad (35)\\
&\mathbf{M} = \left(A_k^{n_k} Q_k(0)(A_k^T)^{n_k}\right) + \sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i \\
&n_k > 0\ ;\ t = n_k h_k \in T_k.
\end{aligned}
$$

2) Parameters of system $\mathcal{PCS}_k$: It follows from the equation (29) that

$$m_k(0) = m_0(0) \tag{36}$$

and

$$Q_k(0) = Q_0(0) \tag{37}$$

for every $k = 1, 2, \cdots$. In ensuring the equality of equations (23) and (34) (as a special case of $n_k = 2^k$), as demanded in the equation (25) for $t \in T_0$, it is possible to find out parameters A_k and Q_k of the causal probability density functions $f_k(s_k(t + h_k) : s_k(t))$ of the systems $\mathcal{PCS}_k$ by comparing right sides of the mentioned equations. From the comparison we obtain

$$A_k^{2^k} = A_0 \tag{38}$$

and

$$\sum_{i=0}^{2^k-1} A_k^i Q_k (A_k^T)^i = Q_0; \tag{39}$$

similarly, from the equation (27) it follows that for $t \in T_0$ it holds

$$A_k^{n_k} = A_0^{n_0} \tag{40}$$

and

$$\sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i = \sum_{j=0}^{n_0-1} A_0^j Q_0 (A_0^T)^j, \tag{41}$$

where $n_k h_k = n_0 h_0$. Equations (38) and (39) are obviously special cases of the equations (40) and (41) for $n_0 = 1$ and $n_k = 2^k n_0$. Furthermore, from the equation (31) and $t \in T_0$ it follows that

$$A_k^{n_k} m_k(0) = A_0^{n_0} m_0(0), \tag{42}$$

and

$$
\begin{aligned}
&A_k^{n_k} Q_k(0)(A_k^T)^{n_k} + \sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i = \\
&= A_0^{n_0} Q_0(0)(A_0^T)^{n_0} + \sum_{j=0}^{n_0-1} A_0^j Q_0 (A_0^T)^j, \qquad (43)
\end{aligned}
$$

where $n_k h_k = n_0 h_0 = t$. The equation (42) can be rewritten to the form of the equation (40) using the equation (36). As the implication (28) holds we can imply

$$A_k^{n_k} = A_k^{2^k n_0} \Longrightarrow A_k^{2^k n_0} = A_0^{n_0} \tag{44}$$

$$\Longrightarrow A_k^{2^k} = A_0 \Longrightarrow A_k = A_0^{\frac{1}{2^k}}. \tag{45}$$

Considering the powers $(\cdot)^{\frac{1}{n_0}}$ and $(\cdot)^{\frac{1}{2^k}}$ in implications (44) and (45), matrices A_0 and A_k, $k = 1, 2, \cdots$ must be positively semidefinite.If the matrix A_0 is not positively semidefinite, there is either no diffusion system corresponding to the originally given linear stochastic causal system or it is necessary to change the state variables of the system $\mathcal{PCS}_1$ as shown e.g. in [3]. If we redefine the original state variables then we must also redefine the set of their domain V_1, the set of all system trajectories Ω_1, the set of all events $\mathcal{S}_1$, domain D_1 of the system trajectory s_1 as well as the set of all causal relations $\mathcal{C}_1$ and the set of all probabilistic mapping $\mathcal{P}_1$. Further, we can either proceed in the continualization procedure as if there was no change of the system variables or we can regard the system $\mathcal{PCS}_1$ as an original system $\mathcal{PCS}_0^{(1)}$ of a new system sequence $\mathcal{PCS}_k^{(1)}$, $k = 1, 2, \cdots$.

The equation (43) can be transformed to the same form as the equation (41) because it holds

$$
\begin{aligned}
&A_k^{n_k} Q_k(0)(A_k^T)^{n_k} = A_k^{2^k n_0} Q_k(0)(A_k^T)^{2^k n_0} = \\
&= A_0^{n_0} Q_k(0)(A_0^T)^{n_0} = A_0^{n_0} Q_0(0)(A_0^T)^{n_0}. \qquad (46)
\end{aligned}
$$

The matrix A_k from the causal probability density function $f_k(s_k(t + h_k) : s_k(t))$ of the system $\mathcal{PCS}_k$ can be found directly from the implication (45)

$$A_k = A_0^{\frac{1}{2^k}}. \tag{47}$$

However, before finding Q_k it is useful to introduce a vector $\mathrm{v}Q_i$ in order to evaluate the sums $\sum_{j=0}^{n_0-1} A_0^j Q_0 (A_0^T)^j$ and $\sum_{i=0}^{n_k-1} A_k^i Q_k (A_k^T)^i$ for any arbitrary n_0, $n_k = 2^k n_0$. $\mathrm{v}Q_i$ is an equivalent notation to a covariance matrix Q_i of the causal probability density function $f_i(s_i(t+h_i) : s_i(t))$ of the system $\mathcal{PCS}_i$, $i = 0, 1, 2, \cdots$ as it is given by the following scheme

$$
\begin{aligned}
\mathrm{v}Q_i = [&Q_i^{(1,1)}, Q_i^{(1,2)}, \ \cdots\ , Q_i^{(1,N)}, \\
&Q_i^{(2,1)}, Q_i^{(2,2)}, \ \cdots\ , Q_i^{(2,N)}, \\
&\cdots\cdots\ , Q_i^{(N,N)}]. \qquad (48)
\end{aligned}
$$

Using the above constructed vector $\mathrm{v}Q_k$ and the Kronecker's multiplication of matrices [2] it is possible to rewrite the equation (41) into the form

$$\sum_{i=0}^{n_k-1} {A_{k(KR)}}^i \cdot \mathrm{v}Q_k = \sum_{j=0}^{n_0-1} {A_{0(KR)}}^j \cdot \mathrm{v}Q_0, \tag{49}$$

where generally

$$
\begin{aligned}
&A_{i(KR)} = A_i \otimes A_i \qquad \Longrightarrow \qquad {A_{i(KR)}}^j = A_i^j \otimes A_i^j, \\
&i = 0, 1, 2, \cdots . \qquad (50)
\end{aligned}
$$

and $A_i \otimes A_i$ means the Kronecker's multiplication of the matrices A_i and A_i. From the equation (49) we obtain

$$
\begin{aligned}
&({A_{k(KR)}}^{n_k} - I)(A_{k(KR)} - I)^{-1} \cdot \mathrm{v}Q_k = \\
&= ({A_{0(KR)}}^{n_0} - I)(A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0,
\end{aligned} \tag{51}
$$

because it is generally known that

$$
\sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} = ({A_{k(KR)}}^{n_k} - I) \cdot (A_{k(KR)} - I)^{-1}, \tag{52}
$$

provided $(A_{k(KR)} - I)$ is a regular matrix. Properties of the matrix A_k, $k = 1, 2, \cdots$ are discussed in detail e.g. in [3]. If $(A_{k(KR)} - I)$ is regular then $({A_{0(KR)}}^{n_0} - I)$ is also regular and it holds

$$
\mathrm{v}Q_k = (A_{k(KR)} - I)(A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0, \tag{53}
$$

because

$$
{A_{k(KR)}}^{n_k} = {A_{0(KR)}}^{n_0}. \tag{54}
$$

The vector $\mathrm{v}Q_k$ is obviously of the $N^2 \times 1$ order. However, according to the scheme (48) and thanks to the fact that Q_0 is the covariance matrix, it can be easily rewritten as a real, symmetrical, positively semidefinite square matrix Q_k, which can be the covariance matrix of the state $s_k(t + h_k)$ conditioned by given state $s_k(t)$ from the causal probability density function $f_k(s_k(t+h_k) \; : \; s_k(t))$ of the system $\boldsymbol{\mathcal{PCS}}_k$, $k = 0, 1, 2, \cdots$.

The above derived parameters A_k and Q_k were not determined uniquely, which follows from the solution to the equation (38) (or (47)). Nonetheless, they satisfy all conditions demanding equivalent stochastic properties of the system $\boldsymbol{\mathcal{PCS}}_0$ and $\boldsymbol{\mathcal{PCS}}_k$ for each $k = 0, 1, 2, \cdots$ on the time-points set T_0 because their behavior is on T_0 described by equivalent probability density functions. A class of systems $\boldsymbol{\mathcal{PCS}}_k$, which is determined by a class of sets $\mathcal{P}_k$, was derived for different solutions A_k and Q_k. The class of sets $\mathcal{P}_k$ is according to the definition (see [5]) defined as sets of all probabilistic mappings determined by causal probability density functions $f_k(s_k(t + h_k) : s_k(t))$ with corresponding solutions A_k, Q_k of the system $\boldsymbol{\mathcal{PCS}}_k$ in above stated equations. These classes of systems form for $k = 0, 1, 2, \cdots$ an infinite sequence and therefore there generally exist more than one sequence of stochastic causal systems $\boldsymbol{\mathcal{PCS}}_k$ converging to the same diffusion system.

Without any loss of generality we can confine ourselves to finding only one sequence of systems. Consequently, it is possible to find only one solution of A_k and Q_k for each k.

It is also sometimes useful if it holds that

$$
\begin{aligned}
f_k(s_k(t_2)|s_k(t_1)) = f_l(s_l(t_2)|s_l(t_1)),& \\
t_2 = t_1 + n_k h_k = t_1 + n_l h_l& \\
k > l, \; k = 1, 2, \cdots, l = 0, 1, 2, \cdots,& \\
t_1, t_2 \in T_l&
\end{aligned} \tag{55}
$$

for systems from the sequence $\boldsymbol{\mathcal{PCS}}_k$ converging to a particular diffusion system.

D. Linear stochastic causal system continualization

Suppose that a sequence of linear stochastic causal systems is given. Furthermore, we derive an explicit solution to the causal probability density function $f_k(s_k(t+h_k) \; : \; s_k(t))$ of the system $\boldsymbol{\mathcal{PCS}}_k$, $k = 0, 1, 2, \cdots$ parameters of which have just been determined. Hence, the function from the equation (34) is to be written as follows

$$
\begin{aligned}
&f_k(s_k(t+n_k h_k)|s_k(t)) = f_k(s_k(t_2) \mid s_k(t_1)) = \\
&\frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{det \sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} \cdot \mathrm{v}Q_k}} \cdot \\
&\cdot e^{-\frac{1}{2}\left(s_k(t+n_k h_k) - A_k^{n_k} s_k(t)\right)^T \left(\sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} \cdot \mathrm{v}Q_k\right)^{-1} \cdot \left(s_k(t+n_k h_k) - A_k^{n_k} s_k(t)\right)}, \\
&t_1 = t; \; t_2 = +n_k h_k, \\
&t, t+n_k h_k \in T_k; \; n_k h_k = h_0; \; n_k > 0,
\end{aligned} \tag{56}
$$

where we suppose that the vector $\sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} \cdot \mathrm{v}Q_k$ is rewritten into the covariance matrix according to the scheme in equation (48) as discussed above. If we substitute the equation (53) to the last equation the sum $\sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} \cdot \mathrm{v}Q_k$ can be expressed as

$$
\begin{aligned}
\sum_{i=0}^{n_k-1} {A_{k(KR)}}^{i} \cdot \mathrm{v}Q_k &= ({A_{k(KR)}}^{n_k} - I) \cdot (A_{k(KR)} - I)^{-1} \cdot \\
&\cdot (A_{k(KR)} - I) \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0 = \\
&= ({A_{k(KR)}}^{n_k} - I) \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0 = \\
&= ({A_{0(KR)}}^{\frac{t_2 - t_1}{h_0}} - I) \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0,
\end{aligned} \tag{57}
$$

where we can write by analogy to the implication (54)

$$
{A_{k(KR)}}^{n_k} = {A_{0(KR)}}^{\frac{n_k}{2^k}} = {A_{0(KR)}}^{\frac{t_2 - t_1}{h_0}} \tag{58}
$$

and where obviously

$$
n_k h_k = t_2 - t_1. \tag{59}
$$

Now, the causal probability density function $f_k(s_k(t_2) \; : \; s_k(t_1))$ can be rewritten

$$
\begin{aligned}
&f_k(s_k(t+n_k h_k)|s_k(t)) = f_k(s_k(t_2) \mid s_k(t_1)) = \\
&\frac{1}{(2\pi)^{\frac{N}{2}} \cdot \sqrt{det \, \mathbf{M}}} \cdot \\
&\cdot e^{-\frac{1}{2}\left(\left(s_k(t_2) - {A_0}^{\frac{t_2 - t_1}{h_0}} s_k(t_1)\right)^T \cdot (\mathbf{M})^{-1} \cdot \left(s_k(t_2) - {A_0}^{\frac{t_2 - t_1}{h_0}} s_k(t_1)\right)\right)},
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
&\mathbf{M} = ({A_{0(KR)}}^{\frac{t_2 - t_1}{h_0}} - I) \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0, \\
&t_1 = t; \; t_2 = t + n_k h_k, \\
&t, t + n_k h_k \in T_k; \; n_k h_k = h_0; \; n_k > 0.
\end{aligned}
$$

If $t_2 > t_1$ then the limit case of the probability distribution function corresponding to the equation (56) is (according to the convergence in distribution [2]) trivial as A_0 and hence also $A_{0(KR)}$ are constant matrices. The case when $t_2 \to t_1$ for $k \to \infty$ ($h_k \to 0^+$) is more interesting. From the probability density function in equation (56) follows that the random variables $s_k(\cdot)$, or in fact their probability distribution functions, do converge in distribution to a "constant value" $s(t_2) = s(t_1)$ if $t_2 \to t_1$, $k \to \infty$, i.e. to a degenerate random variable cumulative probability distribution function of which is

$$F(s(t_2)) = 0 \quad \text{if} \quad s(t_2) \leq s(t_1), \tag{61}$$

$$F(s(t_2)) = 1 \quad \text{if} \quad s(t_2) > s(t_1). \tag{62}$$

It is desired to describe the probabilistic properties of continuous-time stochastic systems by conditioned probability density functions $f(s(t + \mathrm{d}t)|s(t))$ using a differential of the state $\mathrm{d}s(t)$. Such systems are then called diffusion systems if they satisfy some additional conditions [4]. Parameters of $f(s(t + \mathrm{d}t)|s(t))$ can be derived from the sequence of stochastic causal systems $\boldsymbol{\mathcal{PCS}}_k$, $k \to \infty$ using the vector forward Kolmogorov's partial differential equation (Fokker-Planck's equation) [4], [9]

$$\frac{\partial f(s(t_2), t_2|s(t_1), t_1)}{\partial t_2} = -\sum_{i=1}^{N} \frac{\partial[\alpha_i(s(t_2), t_2) \cdot f(s(t_2), t_2|s(t_1), t_1)]}{\partial s_i(t_2)} + \sum_{i,j=1}^{N} \frac{1}{2} \cdot \frac{\partial^2[\beta_{i,j}(s(t_2), t_2) \cdot f(s(t_2), t_2|s(t_1), t_1)]}{\partial s_i(t_2) \partial s_j(t_2)}, \tag{63}$$

where $f(s(t_2), t_2|s(t_1), t_1)$ is the limit case of the equation (60) for $t_2 = t_1 + \Delta t$, $\Delta t > 0$, infinitesimal parameter $\alpha(s(t_2), t_2)$ is called the drift coefficient and infinitesimal parameter $\beta(s(t_2), t_2)$ is called the diffusion coefficient if $\Delta t \to 0$. Both $\alpha(s(t_2), t_2)$ and $\beta(s(t_2), t_2)$ can be determined by comparing left and right sides of the equation (63) after the partial derivatives were found. Hence we obtain

$$\alpha(s(t_2), t_2) = \lim_{\Delta t \to 0} \frac{1}{h_0} \cdot A_0^{\frac{t_2 - t_1}{h_0}} \cdot lnA_0 \cdot s(t_1) = \frac{1}{h_0} \cdot lnA_0 \cdot s(t_1) \tag{64}$$

because

$$A_{0(KR)}{}^{\frac{t_2 - t_1}{h_0}} \to I \quad \text{as} \quad \Delta t \to 0 \tag{65}$$

and

$$\beta(s(t_2), t_2) = \frac{1}{h_0} \cdot \mathbf{M}^{-1} \cdot A_{0(KR)}{}^{\frac{t_2 - t_1}{h_0}} \cdot lnA_{0(KR)} \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0 \cdot \mathbf{M}, \tag{66}$$

where

$$\mathbf{M} = (A_{0(KR)}{}^{\frac{t_2 - t_1}{h_0}} - I) \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0 \tag{67}$$

and the vector $\mathbf{M}$ is supposed to be in the form of corresponding symmetric square matrix according to the scheme (48). The equation (66) can be simplified using the relation

$$\beta(s(t_2), t_2) = \frac{\partial}{\partial t_2} \left(\ln \mathbf{M}(t_2)\right)^T \cdot \mathbf{M}(t_2) \tag{68}$$

into form

$$\beta(s(t), t) = \lim_{\Delta t \to 0} \beta(s(t + \Delta t), t + \Delta t) = \frac{1}{h_0} \cdot \ln A_{0(KR)} \cdot (A_{0(KR)} - I)^{-1} \cdot \mathrm{v}Q_0. \tag{69}$$

The properties of the derived diffusion system are then described by the equation (63) and by coefficients $\alpha(s(t_2), t_2)$ and $\beta(s(t_2), t_2)$ from the equations (64) and (68), respectively.

IV. Conclusions

A new approach to the process of continualization was presented in this paper in terms of new approach to system theory. All the system variables were, from principal reasons, defined on finite sets only. Finally, under carefully chosen continuity hypothesis the extension to infinite sets based on convergence in distribution is straightforward and brings important results in adequate description of continuous-time systems.

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