Stability of Switching Controlled Chained Systems Based on Time-State Control Form

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Abstract - **It is well-known that the non-linear systems transformed into chained form can not be stabilized by a time-invariant static state feedback control. That is why many approaches have been proposed to avoid this problem. One of them makes it possible to design a nonlinear control law based on the linear control theory by using time-state control form. We proposed a periodic switching control law and showed the necessary and sufficient conditions for stabilization.**

In this paper, the conditions for stabilization with arbitrary switching intervals are derived by focusing the switching time. These conditions are closely related to the results for stochastic jumped system. The relatively simple and convenient conditions are expressed by using the norm of transition matrix of linear system. The conditions derived in this paper are rather tight as compared with the case of periodic switching law. However, the chained system can be stabilized if the two linear feedback controllers are switched with arbitrary intervals greater than the lower limit derived in this paper.

Keywords - **chained system, switching control, stability, hybrid system.**

I. Introduction

The chained form is one of the canonical forms that describe nonlinear systems such as car systems or space robots. However, it is well known that there is no continuous time-invariant state feedback stabilizing controller for chained systems [1]. Therefore, many stabilizing controllers have been proposed by using discontinuous feedback controllers or time-variant inputs [2-8]. In one of them, the control method based on time-state controlled form was proposed [2-4]. By using this method, a nonlinear system is transformed into a combination of controllable canonical formed linear system and first-order linear system. Then, it is possible to design a stabilizing controller easily by applying the linear control theory. Since one of the states is regarded as time in this method, the input switches are required in order to settle this state to 0. Therefore, this controlled system is one of the hybrid systems and the input switching law and its condition have not been discussed well.

In this paper, the conditions for stabilizing this switching controlled chained system are discussed. We proposed a periodic switching law and showed the necessary and sufficient conditions for stabilization [12][13]. This paper shows the conditions for stabilization by using the switching controller with arbitrary switching intervals. The conditions are closely related to stochastic jumped system [14]. For chained system based on time-state control form, the sufficient conditions for arbitrary switching control law to stabilize the system were described in [15]; they have been derived based on Lyapunov equation of continuous time system [18][19]. However, these conditions are very tight, and it sounds to be conservative compared with the case of periodic switching. The conditions derived in this paper are relatively relaxed by setting the mean of switching intervals or the lower limit of switching intervals.

In section 2, the chained system with switching controller based on time-state control form is expressed as a hybrid system. The influence of the switching control on the transition of the states is clarified as a certain matrix.

The stability of the chained system is discussed in two stages in section 3. Firstly, we discuss the convergence of states of the hybrid system. The conditions for stabilizing this system are shown in cases of arbitrary switching and periodic switching, respectively. In case of arbitrary switching whose intervals have a lower limit, the conditions are comparatively tight as compared with periodic switching law. However, the existence of a stabilizing controller is guaranteed. These conditions are expressed by using the norm of transition matrix. On the other hand, in case of periodic switching, the conditions are expressed as eigenvalues of transition matrix. These conditions are derived by method of analyzing sampled-data systems focusing on the switching time. Secondly, we show switching laws for settling a remaining state to zero; it plays the role of time in this switching control method.

In section 4, proposed switching control methods are applied to a simple car system that is an example of chained system. The relations between the convergence speed of states and the switching intervals are examined.

II. Time-state Control Form

A. Time-state control form and Hybrid system

Consider a chained system given by the following differential equation [9],

$$\frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 , \tag{1}$$

where x_0, x_1, ...,and x_n are states, u_1 and u_2 are control inputs.

Consider an input transformation given as

$$u_1 = \mu_1 ,\ \ u_2 = \mu_1 \mu_2 . \tag{2}$$

It is possible to handle a state $x_0(t)$ as time if $x_0(t)$ is an increasing function of time t, i.e., input μ_1 has the positive sign. The time-state control formed system[2] is obtained as

$$\frac{dx_0}{dt} = \mu_1 > 0 \tag{3a}$$

$$\frac{d}{dx_0}\begin{bmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\mu_2 . \tag{3b}$$
$$= \boldsymbol{Ax} + \boldsymbol{b}\mu_2$$

The system (3b) is a linear system expressed by the controllable canonical form. Therefore, it is easy to obtain a control input μ_2 for stabilizing the state $\boldsymbol{x} = [x_n, ..., x_2, x_1]^T$ by using the linear control theory. Consequently, the system(3b) can be stabilized by a state feedback controller with appropriate $k_j > 0$,

$$\mu_2 = -\boldsymbol{kx} ,\ \ \boldsymbol{k} = [k_n, k_{n-1}, \cdots, k_1] . \tag{4}$$

However, if the input μ_1 is positive, then evidently the state $x_0(t)$ keeps increasing. For settling the state x_0 to zero, the sign of input μ_1 has to become negative at appropriate time.

If the input μ_1 is negative and $x_0(t)$ decreases, we have to consider $x_0' = -x_0$ as time axis instead of state x_0. Then the time-state control formed system for $\mu_1 < 0$ is obtained by the same way as that system(3) is derived [12].

$$\frac{dx_0}{dt} = \mu_1 < 0 \tag{5a}$$

$$\frac{d}{dx_0'}\begin{bmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}\mu_2 \tag{5b}$$
$$= E_n \boldsymbol{A} E_n \boldsymbol{x} + (-1)^n E_n \boldsymbol{b} \mu_2$$

where $E_n \in \mathbf{R}^n$ is a diagonal matrix such that

$$\begin{aligned} E_n &= E_n^{-1} \\ &= diag(1, -1, \cdots, (-1)^{n-1}) \end{aligned} . \tag{6}$$

This matrix satisfies that $E_n^{2k+1} = E_n$ and $E_n^{2k} = I_n$ (Identity matrix).

It is also possible to stabilize the system(5b) by a linear feedback input. When the sign of the input μ_1 is negative, we choose input μ_2,

$$\mu_2 = -\boldsymbol{k}'\boldsymbol{x} ,\ \ \begin{aligned} \boldsymbol{k}' &= (-1)^n \boldsymbol{k} E_n \\ &= [(-1)^n k_n, (-1)^{n-1} k_{n-1}, \cdots, -k_1] \end{aligned} . \tag{7}$$

By substituting this input(7) to (5b), the system matrix of closed loop system is obtained as,

$$E_n \boldsymbol{A} E_n - (-1)^n E_n \boldsymbol{b} (-1)^n \boldsymbol{k} E_n = E_n (\boldsymbol{A} - \boldsymbol{bk}) E_n . \tag{8}$$

By switching the feedback input(4) and input(7) according to the sign of μ_1, the closed loop systems are expressed as the following hybrid system.

$$\frac{d}{d\tau} \boldsymbol{z}(\tau) = (A - \boldsymbol{bk}) \boldsymbol{z}(\tau) \tag{9a}$$

$$\boldsymbol{z}(\tau^+) = \begin{cases} E_n \boldsymbol{z}(\tau), & \mu_1(\tau^+) \neq \mu_1(\tau) \\ \boldsymbol{z}(\tau), & \mu_1(\tau^+) = \mu_1(\tau) \end{cases} \tag{9b}$$

$$\boldsymbol{x}(\tau) = \begin{cases} E_n \boldsymbol{z}(\tau), & \mu_1(\tau) < 0 \\ \boldsymbol{z}(\tau), & \mu_1(\tau) > 0 \end{cases} \tag{9c}$$

where $\tau = \int |\mu_1| dt$, $\mu_1(\tau^+)$ denotes the successor value of $\mu_1(\tau)$, i.e., it is the right-hand limit[18]. By replacing matrix E_n with identity matrix I_n, this equation expresses a linear system with no input switching. Therefore, it can be said that the matrix E_n represents the characteristic of input switching.

In the following section, first, we discuss the conditions for stabilizing this hybrid system(9) asymptotically. Note that $\boldsymbol{x} = [x_n, ... x_2, x_1]$ in (9), although the states of the chained system are $\boldsymbol{x}$ and x_0. Therefore, secondly, a method of settling x_0 to zero is discussed in order to stabilize the chained system(1).

Our goal in this paper is to stabilize the chained system in the sense of the following definition.

Definition The chained system is stable if there exists a finite time T such that

$$|x_0(t)| + \|\boldsymbol{x}(t)\| < \varepsilon \tag{10}$$

for $\forall \varepsilon > 0$, $\forall t > T$ and arbitrary initial states $x_0(0)$ and $\boldsymbol{x}(0)$.

B. Transition of the State by Input Switches

Let t_i be the i-th switching time when the sign of μ_1 changes, i.e., $\mu_1(t_i^+) \neq \mu_1(t_i)$. Now, we define switching intervals as

$$\begin{aligned} \Delta X_i &= |x_0(t_{i+1}) - x_0(t_i)| \\ &= \int_{t_i}^{t_{i+1}} |\mu_1| dt \end{aligned} . \tag{11}$$

By using ΔX_i, the state $\boldsymbol{x}$ after the m-th switch of the sign of μ_1 is described as

$$
\begin{aligned}
z(t) &= e^{(A-\boldsymbol{bk})|x_0(t)-x_0(t_m)|}E_n \cdots \\
&\qquad e^{(A-\boldsymbol{bk})\Delta X_2}E_n e^{(A-\boldsymbol{bk})\Delta X_1}E_n e^{(A-\boldsymbol{bk})\Delta X_0}\boldsymbol{z}(t_0) \\
&= e^{(A-\boldsymbol{bk})|x_0(t)-x_0(t_m)|}\prod_{i=1}^{m}E_n e^{(A-\boldsymbol{bk})\Delta X_{m-i}}\boldsymbol{x}(t_0) \quad . \qquad (12) \\
&\qquad , t_m \le t < t_{m+1}
\end{aligned}
$$

Especially, at input switching time t_m, the following relation is obtained.

$$
\boldsymbol{z}(t_m) = \prod_{i=1}^{m}(E_n e^{(A-\boldsymbol{bk})\Delta X_{m-i}})\,\boldsymbol{z}(t_0) \qquad (13)
$$

Hence, by noting the state $\boldsymbol{z}(t_i)$ at input switching time t_i, the system is regarded as a discrete time system. The state of this discrete time system is redefined as $\boldsymbol{z}(i)=\boldsymbol{z}(t_i)$.Then, (13) is rewritten as

$$
\boldsymbol{z}(i+1) = E_n e^{(A-\boldsymbol{bk})\Delta X_i}\boldsymbol{z}(i)\,. \qquad (14)
$$

III. Switching Control Laws

A. Arbitrary Switching Control Law

From the above formulation(14), we can derive the following theorem for stabilization of (9) on the basis of the discrete version of Lyapunov inequality.

Theorem 1 Consider switching the inputs arbitrarily with intervals ΔX_i. The system(9) is asymptotically stable if there is a positive definite matrix P such that

$$
e^{(A-\boldsymbol{bk})^T\Delta X_i}E_n P E_n e^{(A-\boldsymbol{bk})\Delta X_i} - P < 0 \qquad (15)
$$

for all ΔX_i, i=0,1,2,...

Proof: Assume that (15) is held. Then, $V(i)= \boldsymbol{z}(i)^T P\boldsymbol{z}(i)$ decreases by input switching with ΔX_i, i.e., the state $\boldsymbol{z}(t_i)$ converges to **0** as $t_i \to \infty$. From (9c) and

$$
\boldsymbol{z}(t) = e^{(A-\boldsymbol{bk})|x_0(t)-x_0(t_i)|}\boldsymbol{z}(t_i) \quad , t_i \le t < t_{i+1}\,,
$$

$\boldsymbol{x}(t)\to\mathbf{0}$ if $\boldsymbol{z}(t_i) \to \mathbf{0}$.

Remark Because of the matrix E_n**,** the condition(15) can not be straightly simplified such as the condition that A-$\boldsymbol{bk}$ is stable. This difficulty is closely concerned with the tight condition in [15]. Their conditions require that the continuous version of Lyapunov equation has a positive definite matrix such that $P=E_nPE_n$.

However, it is sure that there exists a switching control law for stabilizing the system(9) if A-$\boldsymbol{bk}$ is stable. We show sufficient but convenient conditions of switching intervals ΔX_i and feedback gain $\boldsymbol{k}$.

Lemma 1 The system(9) is asymptotically stable if switching intervals ΔX_i satisfy that

$$
\|E_n e^{(A-\boldsymbol{bk})\,\Delta X_i}\| < 1, \quad \text{for } \forall \Delta X_i > \Delta X_{min}\,. \qquad (16)
$$

where $\|\cdot\|$ denotes a certain appropriate matrix norm.

Proof: The transition matrix $\Phi(t_m)$ at switching time is obtained from (13).

$$
\Phi(t_m) = \prod_{i=1}^{m}(E_n e^{(A-\boldsymbol{bk})\Delta X_{m-i}}) \qquad (17)
$$

The arbitrary norm of this transition matrix is bounded such as

$$
\|\Phi(t_m)\| \le \prod_{i=0}^{m-1}(\|E_n e^{(A-\boldsymbol{bk})\Delta X_i}\|)\,. \qquad (18)
$$

Therefore, if (16) is held, $\Phi(t_i)\to 0$ and the state $\boldsymbol{z}(t_i)$ converges to **0** as $t_i\to\infty$.

If we choose a norm such that $\|E_n\|$=1, (16) is rewritten as $\|e^{(A-\boldsymbol{bk})\,\Delta X_i}\| < 1$. Euclidean norm is an example of such norm. In this case, the left side of inequality(16) is the maximum singular value of transition matrix. This condition is the same as (15) in case of $P=I_n$.

When the condition of Lemma 1 is satisfied, the norm of state $\boldsymbol{z}(t_i)$ decreases monotonously for every switching time. However, it is not necessary that (16) is satisfied every intervals. It is sufficient to be satisfied on the average. Therefore, consider the mean value of switching intervals.

Lemma 2 There is the mean value of switching intervals $\Delta\overline{X}$ such that the system(9) is asymptotically stable.

Proof: If matrix A-$\boldsymbol{bk}$ is stable, there exists a constant $\alpha > 0$, $\lambda > 0$ for $\forall\tau > 0$ such that

$$
\|e^{(A-\boldsymbol{bk})\tau}\| \le \alpha\, e^{-\lambda\tau}\,. \qquad (19)
$$

where $\|\cdot\|$ denotes an arbitrary matrix norm [10].

By applying (19) into (18), the norm of this matrix is bounded such as

$$
\begin{aligned}
\|\Phi(t_m)\| &\le \prod_{i=0}^{m-1}(\|E_n\|\|e^{(A-\boldsymbol{bk})\Delta X_i}\|) \\
&\le \prod_{i=0}^{m-1} a e^{-\lambda\Delta X_i} \qquad\qquad . \qquad (20) \\
&= (a e^{-\lambda\Delta\overline{X}})^m
\end{aligned}
$$

Consequently, the asymptotic stability of controlled system is guaranteed if the mean of the switching intervals satisfies

$$
a e^{-\lambda\Delta\overline{X}} < 1\,. \qquad (21)
$$

From lemma2, even if the inputs are switched with such short intervals that the condition(16) is broken, $\boldsymbol{x}(t)$ converges to **0** if one sufficient long interval exists.

Next corollaries show that there exist feedback gains and switching intervals for satisfying these conditions.

Corollary 1 Consider the feedback gain $\boldsymbol{k}$ such that matrix A-$\boldsymbol{bk}$ is stable. Then, there exist finite values ΔX_{min} and $\Delta\overline{X}$ which satisfy lemma 1 and 2, respectively.

Proof: In (19), there exists a constant $a > 0$, $\lambda > 0$ for $\forall\tau > 0$ because matrix A-$\boldsymbol{bk}$ is stable. If a certain ΔX_{min} satisfies $ae^{-\lambda\Delta X_{min}} < 1$, then $\boldsymbol{x}(t) \to 0$ when all $\Delta X_i > \Delta X_{min}$.

Corollary 2 Assume a certain value ΔX_{min} or $\Delta\overline{X}$ >0. There exist feedback gains $\boldsymbol{k}$ for satisfying lemma 1.

Proof: Consider the feedback gain $\boldsymbol{k}$ that assigns eigenvalues of A-$\boldsymbol{bk}$ into $\lambda_1, \lambda_2, \ldots \lambda_n$ ($\lambda_i \neq \lambda_j$). We introduce a nonsingular matrix T such that

$$e^{(A-\boldsymbol{bk})\Delta X} = T \begin{bmatrix} e^{\lambda_1 \Delta X} & 0 & \cdots \\ 0 & e^{\lambda_2 \Delta X} & \\ \vdots & & \ddots \end{bmatrix} T^{-1} \cdot \tag{22}$$

This matrix T is Vandermonde matrix of which the elements are the (n-1)-th order monomial of λ_i at the most [11][14]. So the elements of T^{-1} are rational functions of λ_i. Therefore, by taking note of the fact that $\lambda^k e^{\lambda \Delta X} \to 0$ as $\lambda \to -\infty$,

$$\lim_{\lambda_i \to -\infty} e^{(A-\boldsymbol{bk})\Delta X} = \mathbf{0} \tag{23}$$

is verified. This proves existences of stabilizing feedback gains which satisfy (16) and (21) respectively.

Concerning the convergence of $\boldsymbol{x}(t)$, we can choose any combination of switching intervals as long as the condition(16) or (21) is satisfied. Therefore, it is easy to settle the last state x_0 to zero. We propose a switching method for stabilizing the chained systems(1).

Switching method 1:

- ***step1*** Switch the sign of μ_1 with an appropriate switching period ΔX_i satisfying lemma1 or 2.
- ***step2*** If $\|\boldsymbol{x}(T)\| < \varepsilon'$, $\varepsilon' < \varepsilon(\max_{\tau} \| e^{(A-\boldsymbol{bk})\tau} \|)^{-1}$ is satisfied, let $\mu_1 < 0$ if $x_0 > 0$ and let $\mu_1 > 0$ if $x_0 < 0$.
- ***step3*** If $|x_0(t)| + \|\boldsymbol{x}(t)\| < \varepsilon$ then let $\mu_1 \equiv 0$.

If the input μ_1 is 0, then the time x_0 is fixed. Therefore, the value of the state $\boldsymbol{x}$ is also fixed.

Theorem 2 Consider switching the sign of μ_1 according to the proposed switching method 1. The chained system(1) is stable if and only if the system(9) is asymptotically stable.

Proof: If and only if the system(9) is asymptotically stable, then there exists a finite time T' such that $\|\boldsymbol{x}(T')\| < \varepsilon'$, $\varepsilon' \leq \varepsilon(\max_{\tau} \|e^{(A-\boldsymbol{bk})\tau}\|)^{-1}$. In step 2, $|x_0| \to 0$ and $\|\boldsymbol{x}(t)\|$is bounded such as,

$$\begin{aligned} \|\boldsymbol{x}(t)\| &\leq \|z(t)\| \\ &= \left\| e^{(A-\boldsymbol{bk})(x_0(t)-x_0(T'))} \boldsymbol{x}(T') \right\| . \\ &\leq \max_{\tau} \left\| e^{(A-\boldsymbol{bk})\tau} \right\| \|\boldsymbol{x}(T')\| \end{aligned}$$

From step2, x_0 becomes 0 at $^{\exists}T > T'$. Therefore, (10) is satisfied at $^{\exists}t > T'$.

B. Periodic Switching Control Law

As a special case of former results, we consider switching the input periodically. The switching intervals ΔX_i are taken to be constant, i.e.,

$$\Delta X_i = \Delta X \quad (=const.)\,, \ ^{\forall}i > 0. \tag{24}$$

and ΔX is called as a switching period. By applying (24) to (15), it becomes a time-invariant discrete version of Lyapunov Inequality. Therefore, the following theorem is obtained [12].

Theorem 3 Consider switching the sign of μ_1 alternately with period ΔX. The system(9) is asymptotically stable if and only if $|\lambda_i(E_n e^{(A-\boldsymbol{bk})\Delta X})| < 1$ is satisfied for i=1,2,...n, where $\lambda_i(M)$ is the eigenvalue of matrix M.

Proof: It is evident from (14) and the proof of theorem 1.

Generally, $|\lambda_i(E_n e^{(A-\boldsymbol{bk})\Delta X})| < 1$ is not always satisfied, even if matrix A-$\boldsymbol{bk}$ is a stable matrix; we can find such counter examples easily[12]. However, existence of feedback gain and switching period for stabilizing the system can be verified. The following relation between eigenvalues and matrix norms is generally satisfied [10].

$$\max_i \left| \lambda_i(E_n e^{(A-\boldsymbol{bk})\tau}) \right| \leq \left\| E_n e^{(A-\boldsymbol{bk})\tau} \right\| \leq \left\| e^{(A-\boldsymbol{bk})\tau} \right\| \tag{25}$$

Therefore, the condition of theorem3 for periodic switching control is satisfied if eq.(16) concerned with arbitrary switching is held. That is, from corollaries 1 and 2, there exist feedback gain $\boldsymbol{k}$ and switching period ΔX satisfying theorem 3.

The matrix A is a controllable canonical form. By utilizing this property, it is confirmed in case of n=2 that $|\lambda_i(E_2 e^{(A-\boldsymbol{bk})\Delta X})| < 1$ is satisfied if and only if matrix (A-$\boldsymbol{bk}$) is stable [12].

Now, consider the following switching control law on the basis of periodic switches in order to settle x_0 into zero.

Switching method 2:

- ***step1*** Until x_0 satisfies

$$x_0(t_0') < 0 < x_0(t_0') + \Delta X, \tag{26}$$

  let $\mu_1 < 0$ if $x_0 > 0$ and let $\mu_1 > 0$ if $x_0 < -\Delta X$.
- ***step2*** Switch the sign of μ_1 periodically with an appropriate switching period ΔX. Because of step 1, the state x_0 crosses 0 in each switching period.
- ***step3*** If $|x_0(t)| + \|\boldsymbol{x}(t)\| < \varepsilon$ is satisfied, then let $\mu_1 \equiv 0$.

By using this switching method, the chained system is stabilized.

Theorem 4 Consider switching the sign of μ_1 according to the proposed switching method 2. The chained system(1) is stable if and only if the system(9) is asymptotically stable.

Proof: It is similar to the proof of theorem 2.

Remark The absolute value of input μ_1 is independent of convergence condition of $\boldsymbol{x}$. For example, we can set $\mu_1 = -kx_0$ in the last period for asymptotic convergence of x_0; it is similar to some conventional controllers, e.g., in [16][17].

IV. Simulations

Consider the four-wheeled car described in Fig.1. The state equation of this nonlinear system is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 , \tag{27}$$

where (x, y) and θ are positions and direction of the car, and ϕ is an angle of the steering. Control inputs are forward velocity of the car u_1 and angular velocity u_2. Parameter l=1.0 is the length between the front and rear wheel shaft. It is known that this system can be transformed into a chained form by using coordinate and input transformations [7].

The input transformations,

$$\mu_1 = u_1 \cos\theta, \mu_2 = \frac{u_2 + \frac{3}{l} u_1 \sin^2\phi \tan\theta}{l u_1 \cos^2\phi \cos^4\theta} \tag{28}$$

and coordinate transformations,

$$x_0 = x, \quad x_1 = \frac{\tan\phi}{l\cos^3\theta}, \quad x_2 = \tan\theta, \quad x_3 = y \tag{29}$$

are applied to (27). Then, the system is expressed by time-state control form under the condition of $|\theta|$<90[deg].

$$\frac{dx_0}{dt} = \mu_1 \tag{30a}$$

$$\frac{d}{dx_0}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mu_2 \tag{30b}$$

We set the parameters $\boldsymbol{k}$ = [1,3,3], $|\mu_1|$ =1.0 and ε=0.01. The change of the eigenvalues of $E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$ in changing the switching period ΔX is shown as a solid-line in Fig. 2. The eigenvalues are 1 and -1 when ΔX = 0. As the switching period ΔX is longer, all eigenvalues are closer to 0. In this case, all absolute values $|\lambda_i(E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X})|$ are smaller than 1 for all $\Delta X > 0$. These eigenvalues of $E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$ are not the same as the eigenvalue of $e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$; they are described as a dashed-line in Fig.2 with multiplicity 3. In this example, $\max|\lambda(E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X})|$ is larger than $|\lambda(e^{(A\text{-}\boldsymbol{bk})\,\Delta X})|$ for all $\Delta X > 0$. This difference expresses the difficulty of guaranteeing the stability of the switching controlled system.

The trajectories of the car in x-y plane are shown in Fig.3. The simulations start with $x(0)$ = 0.0, $y(0)$ = 3.0, $\phi(0)$ =0 and $\theta(0)$ = 1.0[rad]. The trajectories with ΔX = 6.0, 4.0, and 2.0 are plotted in Fig.3. Maximum eigenvalues of $E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$ are 0.1546, 0.4862 and 0.9299, respectively. At the second switching time with ΔX =6.0, state y is quite close to 0. On the other hand, the controller with ΔX=4.0 requires to switch the sign of μ_1 more than ten times until y becomes as small as the former. In case of ΔX=2.0, the input switches 200 times until $|x_0(t)|+\|(x_1, x_2, x_3)^T\|<0.01$ is satisfied. The behavior of the state x_0 is more oscillatory because one of eigenvalues of $E_3e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$ is close to 1.

Fig. 4 shows the changes of singular values of $e^{(A\text{-}\boldsymbol{bk})\,\Delta X}$. The condition(16) is held if ΔX_i>2.493, and for example, $\|e^{(A\text{-}\boldsymbol{bk})\,\Delta X}\|_2<2.15\ e^{-0.3\Delta X_i}$. In Fig.5, a solid line shows a trajectory of the car when switching intervals change at random in $5>\Delta X_i>2.5$ and $\Delta\overline{X} = 3.75$. The sum of squares $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$, is plotted in Fig.6. Dashed lines in Figs.5 and 6 show the results in case of $5>\Delta X_i>0.2$ and $\Delta\overline{X} = 2.51$. In Fig.6, white diamonds indicate $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ at the switching time. In case of ΔX_i>2.5, $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ decreases for every switching. In case of ΔX_i>0.2, $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ increases at 2nd switching time and so on.

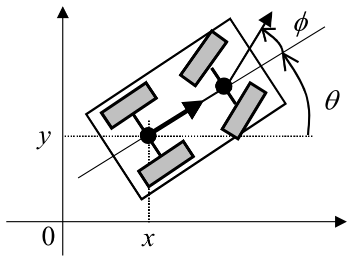

Fig.1 Four-wheeled car

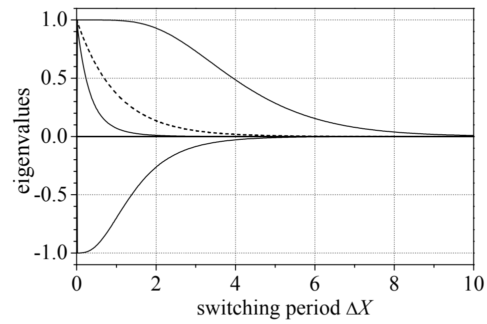

Fig.2 Eigenvalues of $E_3e^{A\Delta X}$(solid line) and $e^{A\Delta X}$(dashed line)

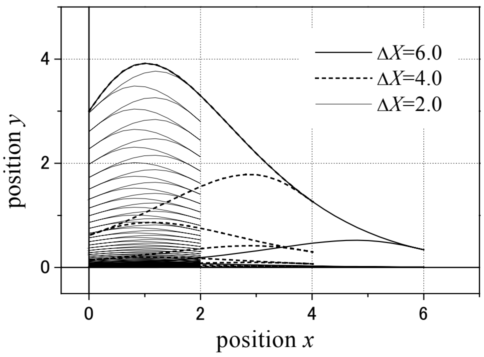

Fig.3 Trajectories of the four-wheeled car with the periodic switching control laws, ΔX=2.0, 4.0, 6.0

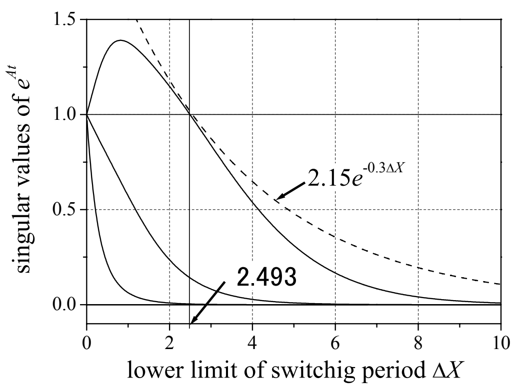

Fig.4 Singular values of $e^{A\Delta X}$; these singular values are smaller than $2.15e^{-0.3\Delta X}$

However, the system is stable in this case because $2.15e^{-0.3*2.51}<1$. It takes t=44.2 and 49.6 for satisfying the convergence conditions respectively. Short interval causes many switches and late convergence.

For example, consider that the inputs are switched with $\Delta X_i \geq 1.0$. If all poles of A-$\boldsymbol{bk}$ are chosen as -2 then $\|e^{(A-\boldsymbol{bk})1.0}\| \leq 0.998$ and the condition(16) is satisfied. On the other hand, concerning a periodic switching law with ΔX_i =1.0, we can choose the feedback gain such that the poles are -1. In this case, maximum absolute value of eigenvalue of $E_n e^{(A-\boldsymbol{bk})1.0}$is 0.997<1 and the stability of the system is guaranteed. The former condition requires large feedback gains, but it admits relaxed switching intervals of the input.

Thin lines in Figs.5 and 6 show the results in case of $2.4>\Delta X_i>0.2$. These intervals is not guaranteed theoretically, but the condition of convergence(10) is satisfied at t=63.0 in simulation. That is why the conditions are sufficient conditions.

V. Conclusion

In this paper, we discussed the stability of the chained systems controlled with switching controllers based on the time-state control form.

The conditions for stabilization with arbitrary switching intervals were derived by focusing the states at switching time. The relatively simple and convenient conditions are expressed by using matrix norms of transition matrix of linear system. The conditions are rather tight as compared with the case of periodic switching law. However, the chained system can be stabilized if the two linear feedback controllers are switched with arbitrary intervals greater than the lower limit derived in this paper. That is, it admits shifting the timing of switches.

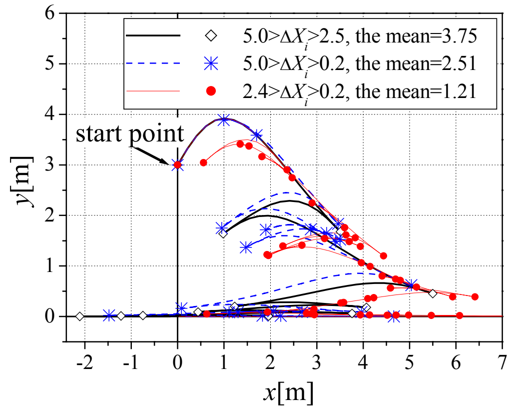

Fig.5 The transition of state x_i, with arbitrary switching intervals

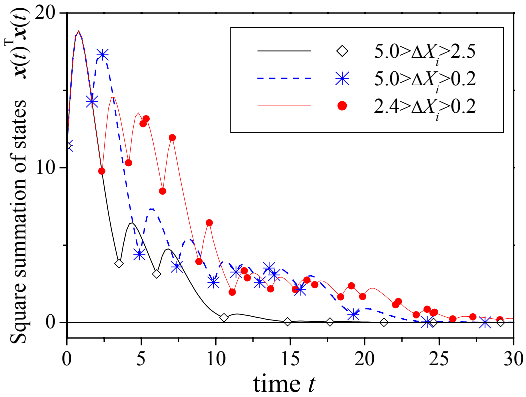

Fig.6 Sum of squares of states $\boldsymbol{x}^T\boldsymbol{x}$. The transition of state x_i, with ΔX=4.0 and l=1.0

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