INS AIDING USING OPTICAL FLOW - THEORY *

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Abstract

The reduction of the navigation error in an inertial navigation system by optically tracking a ground object is investigated. Multiple observations of the ground object are used, however the location of the ground object is assumed unknown. The analysis of the measurement situation on hand reveals that by optically tracking an unknown ground object using passive, *bearings-only* measurements, the aircraft's angle of attack and sideslip angle can be measured. Thus, two new independent measurement equations featuring the aircraft's angular navigation variables $\psi, \theta, \phi, \gamma$ and H are obtained and inertial navigation system aiding is possible. Moreover, simultaneously, and in parallel, the estimation algorithm also updates the aircraft's positional navigation variables and the geo-location of the ground object. The theory allows inclusion of information on the coordinates of the ground object. Thus, the theory is sufficiently general to encompass the conventional methods of inertial navigation system updating where the coordinates of the ground object are known.

1 INTRODUCTION

An *Inertial Navigation System* (INS) is a self-contained and autonomous navigation instrument wherein all the required measurements are obtained without the aid of external sources. Acceleration and angular rate information necessary for dead reckoning navigation are measured by on board accelerometers and gyroscopes, which are non-jammable and non-radiating. Unfortunately, INS instruments suffer from *drift*, a degradation in accuracy of the position and velocity estimates over time. Since the accuracy of an INS deteriorates over time, it needs to be updated periodically.

Optical bearing measurements of an unknown landmark have been suggested in [3], where INS aiding using bearings-only measurements of an unknown (lunar) landmark was considered for the *Apollo* mission. The tracker envisioned in this paper consists of a precision telescope mounted on a gimbal system. This allows the telescope to remain pointed to the ground object independent of vehicle motion. The direction of the Line of Sight (LOS) relative to the aircraft body axes is measured by pickups attached to the gimbals. In addition, the inertial angular rate $\vec{\omega}$ of the LOS is measured.

The paper is organized as follows. In Sect. 2, the optical flow kinematic measurement situation is introduced. The measurement equation derived from the optical flow is developed in Sect. 3. The INS aiding scheme is developed in Sect. 4, followed by concluding remarks in Sect. 5.

2 ANALYSIS

We consider the plane **P** formed by the aircraft's velocity vector **V** and the point P on the ground - see, e.g., Fig 1. The inertial reference frame is X, Y, Z. A local frame of reference x, y, z is also introduced. Its origin is collocated with the aircraft's initial position X_0, Y_0, Z_0, the x-axis is aligned with the aircraft's inertial velocity vector **V**, the y-axis is in the plane **P**, normal to the x-axis, and points in the direction of the point P, and the z-

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axis complements the right handed axes system. The aircraft's body axes are x_b, y_b, and z_b. The

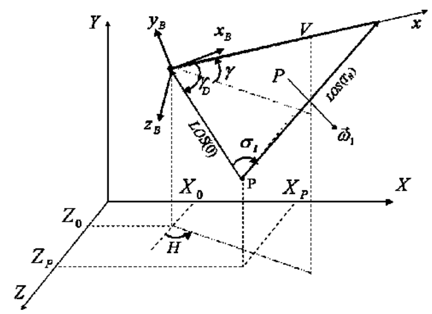

Figure 1: The measurement situation in 3-D case

bearings-only kinematic measurement scenario is illustrated in Fig.1.The raw measurements are

1. The time instants t_k, $k = 0, 1, ..., N$.

2. The LOS rate with respect to inertial space, i.e., σ_k, $k = 0, 1, ..., N$.

3. The angle of depression of the initial LOS, θ_D; it is the angle included between the aircraft's x-body axis and the initial LOS to the ground object P.

We'll show in Sect. 3 that the following holds.
Theorem 1 Consider the kinematic measurement scenario shown in Fig 1 where bearing measurements on a ground object, whose position is not known, are taken over time. It is then possible to estimate the angles which specify the direction of the aircraft's inertial velocity vector $\mathbf{V}$ relative to the aircraft's body axes, viz., α' and β', where

$$\alpha' = \arctan(\frac{w}{u}) \ , \quad \beta' = \arctan(\frac{v}{u}) \ ,$$

and where u, v, and w are the components of the inertial velocity vector $\mathbf{V}$ resolved in the body axes.
Thus, we realize that:
Corollary 2 An optical flow sensor measures the angles α' and β' which specify the direction of the aircraft's inertial velocity vector relative to the aircraft's body axes.

We also realize that - see, e.g., Sect 3.3:
Proposition 4 The angles α' and β' are related to the aircraft's angular navigation variables $\psi,\ \theta,\ \phi,\ \gamma$ and H.

Theorem 1 and Proposition 4 are exploited to lay the foundation for INS - aiding using bearings-only measurements, as stated in:
Theorem 5 The kinematic measurement scenario which entails bearing measurements over time on a ground object whose position is not known, yields two new independent measurement equations featuring the aircraft's angular navigation variables $\psi,\ \theta,\ \phi,\ \gamma$ and H.

Indeed, the proposed INS aiding concept is a modern mechanization of a $driftmeter$ [4] . In other words, the gist of the paper is an investigation into INS aiding using a modern $driftmeter$, a.k.a., optical flow.

3 ESTIMATION

3.1 The Regressor H

In the sequel we'll confine our attention to flight in the vertical plane (X, Y) $(\equiv \mathbf{P})$. The navigation variables are $X_0, Y_0, V, \gamma, \theta$, and the ground object coordinates are X_P, Y_P.
The ground object, referred to as the point P, has the coordinates (x, y) in the local frame in the plane $\mathbf{P}$. The point P is located at the intersection of the $N+1$ circles C_k - see, e.g, Fig. 2b.
After some algebraic manipulations, a linear homogeneous system, compactly written in matrix form as

$$H\theta = 0 \tag{1}$$

is obtained, where the parameter vector

$$\theta = \left[\begin{array}{ccc} x & y & V \end{array} \right]^T$$

and the $N \times 3$ regressor matrix $H =$

$$\left[\begin{array}{ccc} 2T_1 - t_1 - t_0 & t_1 \cot\sigma_1 - t_0 \cot\sigma_0 & -T_1(T_1 - t_1) \\ \vdots & \vdots & \vdots \\ 2T_k - t_k - t_0 & t_k \cot\sigma_k - t_0 \cot\sigma_0 & -T_k(T_k - t_k) \\ \vdots & \vdots & \vdots \\ 2T_N - t_N - t_0 & t_N \cot\sigma_N - t_0 \cot\sigma_0 & -T_N(T_N - t_N) \end{array} \right]_{N \times 3}$$

3.2 The γ_D Formula

As shown in Fig. 3, the navigation variables consist of 3 positional variables and 2 angular variables. They are X_0, Y_0, V, and γ, θ, respectively.

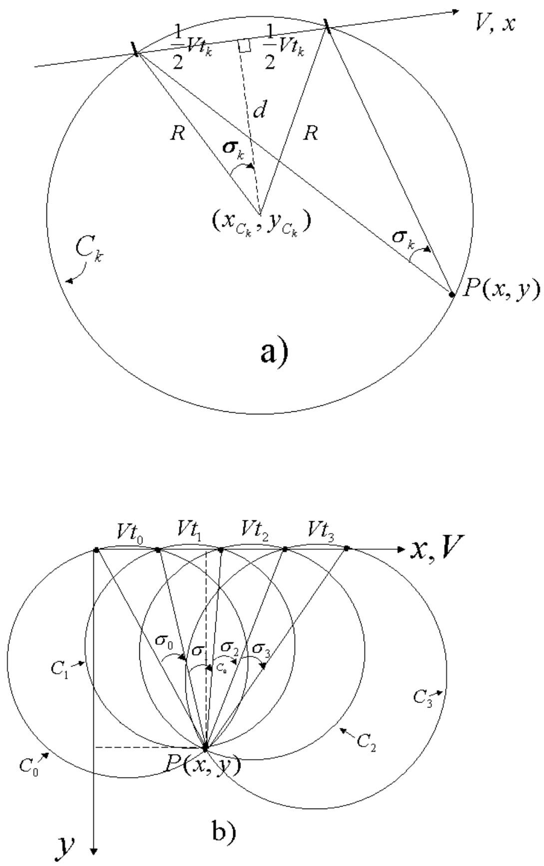

Figure 2: Geometry of bearings-only measurements - a) k th circle, b)Four circles (N=3)

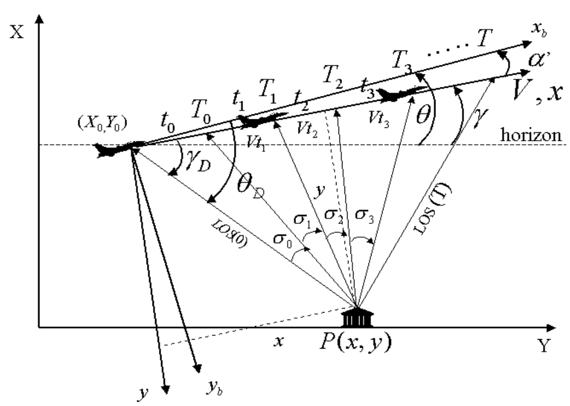

Figure 3: The measurement situation in the 2-D case - the plane **P**

The navigation variables are assumed constant during the short measurement interval.
The raw measurements are pre-processed and the regressor H is formed according to the development in Section 3.1. The angle γ_D included between the velocity vector $\mathbf{V}$ and the initial LOS is computed as follows.
Performing a Singular Value Decomposition (SVD) of the noise corrupted regressor H yields

$$H = U\Sigma V^T$$

where U and V are $N \times 3$ and $3 \times N$ matrices, respectively. The 3×3 diagonal matrix Σ has 3 singular values. The third singular value is much smaller than the first two singular values. Set the third singular value to 0 and only the nonzero elements of Σ are preserved in a reduced 2×2 matrix $\underline{\Sigma}$. This allows us to replace the original linear homogeneous system of N equations with the reduced linear homogeneous system of 2 independent equations in 3 unknowns

$$K\theta = 0 \qquad (2)$$

Thus, eq. (2) is a set of 2 linear equations in the 3 unknowns x, y and V, viz.,

$$\begin{aligned} K_{1,1}x + K_{1,2}y + K_{1,3}V &= 0 \\ K_{2,1}x + K_{2,2}y + K_{2,3}V &= 0 \end{aligned} \qquad (3)$$

This yields the solution

$$\begin{aligned} x &= K_x V \\ y &= K_y V \end{aligned} \qquad (4)$$

where the "gains"

$$\begin{aligned} K_x &= \frac{K_{1,2}K_{2,3} - K_{2,2}K_{1,3}}{K_{1,1}K_{2,2} - K_{1,2}K_{2,2}} \\ K_y &= \frac{K_{2,1}K_{1,3} - K_{1,1}K_{2,3}}{K_{1,1}K_{2,2} - K_{1,2}K_{2,2}} \end{aligned} \qquad (5)$$

Evidently, x and y are homogeneous in V.
Now, the angle $\gamma_D = \arctan(\frac{y}{x})$.
The SVD yields the "gains" K_x and K_y and, in view of eqs. (4), we calculate

$$\gamma_D = \arctan(\frac{K_y}{K_x})$$

where the "clean" K_x and K_y parameters are the result of the SVD. The above calculated angle γ_D included between the A/C velocity vector and the initial LOS to P is referred to as $\gamma_{D_{meas}}$.

3.3 The Angle θ_D

The angle included between the A/C body axis and the initial LOS to P,

$$\theta_D = \gamma_D + \alpha \tag{6}$$

where α, the angle included between the aircraft body axis x_b and flight path axis x, is the aircraft's Angle Of Attack (AOA). The θ_D measurement is generated as follows

$$\theta_{D_{meas}} = \theta_D + v_2, \qquad v_2 = \mathcal{N}(0, \sigma^2_{\theta_D}) \tag{7}$$

At the same time, α is the difference between θ and γ, which are the INS measured pitch angle and flight path angle, respectively:

$$\theta = \gamma + \alpha \tag{8}$$

The difference between γ_D and θ_D is also equal to α - see, e.g., eq.(6). The γ_D and θ_D measurements are provided by the optical arrangement described above. Thus, subtracting equation (6) from equation (8) yields the equation which relates the optical bearing measurements to the angular navigation variables, as stated in Preposition 4,

$$\theta_D - \gamma_D = \theta - \gamma \tag{9}$$

Equation (9) makes INS-aiding using bearings-only measurements of a ground object, possible.
The development in this section constitutes the proofs of Theorem 1 and Proposition 4 for the simplified two dimensional case.

4 INS Aiding

4.1 Phase 1 - Angular Navigation Variables

The measurement equation used for INS-aiding is based on the development leading to Eq. (9).
We define the measurement

$$z = \theta_{D_{meas}} - \gamma_{D_{meas}} \tag{10}$$

where

$$\gamma_{D_{meas}} = \gamma_D + v_3, \qquad v_3 = \mathcal{N}(0, \sigma^2_{\gamma_D}) \tag{11}$$

Combining eqs. (7), and (9)-(11) yields the measurement equation for INS aiding:

$$\begin{aligned} z &= \theta_D + v_2 - \gamma_D - v_3 \\ &= \theta - \gamma + v_6 \end{aligned} \tag{12}$$

where $v_6 = v_2 - v_3$, the measurement noise, is

$$v_6 = \mathcal{N}(0, \sigma^2_{\theta_D} + \sigma^2_{\gamma_D}) \tag{13}$$

On the other hand, at the time instant of INS updating, the stand alone INS provides the prior estimates of θ and γ, $\hat{\theta}^-$ and $\hat{\gamma}^-$, respectively:

$$\hat{\theta}^- = \theta + v_4 \tag{14}$$

$$\hat{\gamma}^- = \gamma + v_5 \tag{15}$$

where v_4 and v_5 are white Gaussian noise with the INS error statistics $v_4 = \mathcal{N}(0, \sigma^2_\theta)$, $v_5 = \mathcal{N}(0, \sigma^2_\gamma)$.
To eqs. (14) and (15), one appends the measurement equation (12), thus obtaining a linear regression in γ and θ. Hence, we can write the linear regression for INS aiding as follows:

$$\underbrace{\begin{bmatrix} \hat{\theta}^- \\ \hat{\gamma}^- \\ z \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \theta \\ \gamma \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{V} \tag{16}$$

The linear regression (16) is solved using the Weighted Least Squares / Minimum Variance (MV) formulae [1]:

$$\hat{X}^+ = [H^T R^{-1} H]^{-1} H^T R^{-1} Z \tag{17}$$

$$P^+ = [H^T R^{-1} H]^{-1} \tag{18}$$

where $\hat{X}^+$ is the minimum variance parameter estimate, P^+ is the predicted parameter estimation error covariance matrix, and the weighting matrix R is the equation error covariance

$$R = \begin{bmatrix} \sigma^2_{v_4} & 0 & 0 \\ 0 & \sigma^2_{v_5} & 0 \\ 0 & 0 & (\sigma^2_{v_2} + \sigma^2_{v_3}) \end{bmatrix} \tag{19}$$

The solution (17) and (18) of the linear regression (16) yields the improved angular navigational variables estimates $\hat{\theta}$ and $\hat{\gamma}$, thus accomplishing phase 1 of INS aiding.

4.2 Phase 2 - Positional Navigation Variables

In Phase 1, we obtained the improved angular navigation variables' estimates $\hat{\theta}^+$ and $\hat{\gamma}^+$. We have the estimates

$$\gamma_D = \mathcal{N}(\hat{\gamma}_D, \sigma^2_{\gamma_D}), \qquad \gamma = \mathcal{N}(\hat{\gamma}, \sigma^2_{\gamma}) \tag{20}$$

In order to update the positional variables, we need to include additional measurements.

4.2.1 Basic Linear Regression

We recognize that

$$x = R\ cos\gamma_D, \qquad y = R\ sin\gamma_D \tag{21}$$

where R is the initial slant range to the point P. Replacing in eqs. (21) γ_D with $\hat{\gamma}_D$, and using eqs (20), we obtain

$$\begin{aligned} x &\approx R\ cos\hat{\gamma}_D - R\ sin\hat{\gamma}_D.v_{\gamma_D} \\ y &\approx R\ sin\hat{\gamma}_D + R\ cos\hat{\gamma}_D.v_{\gamma_D} \end{aligned}$$

Furthermore, we recall the equalities

$$x = K_x V, \qquad y = K_y V$$

where, the gains K_x and K_y obtained after the application of the SVD algorithm, are "clean". Hence, we declare R and V the primary parameters and write the linear regression in R and V:

$$\begin{bmatrix} 0 \\ 0 \\ V_m \end{bmatrix} = \begin{bmatrix} cos\hat{\gamma}_D & -K_x \\ sin\hat{\gamma}_D & -K_y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R \\ V \end{bmatrix} + \begin{bmatrix} -R\ sin\hat{\gamma}_D & 0 \\ R\ cos\hat{\gamma}_D & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{\gamma_D} \\ v_V \end{bmatrix} \tag{22}$$

We also have the coordinate transformation equations

$$\begin{aligned} x\ cos\gamma + y\ sin\gamma &= X_P - X_0 \\ x\ sin\gamma - y\ cos\gamma &= Y_P - Y_0 \end{aligned} \tag{23}$$

Inserting eqs. (21) into the coordinate transformation equations (23), we obtain

$$\begin{aligned} R\ cos(\gamma_D - \gamma) &= X_P - X_0 \\ R\ sin(\gamma_D - \gamma) &= Y_0 - Y_P \end{aligned} \tag{24}$$

Eqs. (20) yield

$$\gamma_D - \gamma = \hat{\gamma}_D - \hat{\gamma} + v_{\gamma_D} - v_\gamma$$

and thus, linearization yields

$$\begin{aligned} cos(\gamma_D - \gamma) &\approx cos(\hat{\gamma}_D - \hat{\gamma}) - sin(\hat{\gamma}_D - \hat{\gamma})v_{\gamma_D} \\ &+ sin(\hat{\gamma}_D - \hat{\gamma})v_V\gamma \\ sin(\gamma_D - \gamma) &\approx sin(\hat{\gamma}_D - \hat{\gamma}) + cos(\hat{\gamma}_D - \hat{\gamma})v_{\gamma_D} \\ &- cos(\hat{\gamma}_D - \hat{\gamma})v_\gamma \end{aligned} \tag{25}$$

Inserting eqs. (25) into eqs. (24), we obtain the linear regression in the parameter (R, X_0, Y_0, X_P, Y_P):

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} cos(\hat{\gamma}_D - \hat{\gamma}) & 1 & 0 & -1 & 0 \\ sin(\hat{\gamma}_D - \hat{\gamma}) & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R \\ X_0 \\ Y_0 \\ X_P \\ Y_P \end{bmatrix} \\ &+ \begin{bmatrix} -sin(\hat{\gamma}_D - \hat{\gamma}) & sin(\hat{\gamma}_D - \hat{\gamma}) \\ cos(\hat{\gamma}_D - \hat{\gamma}) & -cos(\hat{\gamma}_D - \hat{\gamma}) \end{bmatrix} \begin{bmatrix} v_{\gamma_D} \\ v_\gamma \end{bmatrix} \end{aligned} \tag{26}$$

We have the additional non-linear equality constraint:

$$\sqrt{(X_0 - X_P)^2 + (Y_0 - Y_P)^2} - R = 0 \tag{27}$$

The linearization of eq. (27) about a prior parameter estimate yields an additional linear regression equation.
At this point, we use the INS provided measurements V_m, X_{o_m}, Y_{o_m}, the linear regression equations (22) and (26), and the linear constraint which were derived so far, and we augment the linear regression by including the complete prior information on the ground object. Thus, the additional "measurement" equations are included

$$Y_{P_m} = Y_P + v_{Y_P} \tag{28}$$

$$X_{P_m} = X_P + v_{X_P} \tag{29}$$

The ensuing augmented linear regression $Z = H\theta + \Gamma V$ is given in eq. (31), where the superscript "-" denotes a prior estimate of the parameter. The 10×10 equation error covariance matrix is

$$R = \Gamma \begin{bmatrix} \sigma_V^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{X_0}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{Y_0}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\gamma_D}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{Y_P}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{X_P}^2 \end{bmatrix} \Gamma^T \tag{30}$$

and rank(R)=7. The technicalities of dealing with a singular equation error covariance matrix are dealt with in [2]. The solution of the linear regression (31) yields improved A/C and ground object position estimates.

$$\begin{bmatrix} V_m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ X_{o_m} \\ Y_{o_m} \\ Y_{P_m} \\ X_{P_m} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -K_x & cos\hat{\gamma}_D & 0 & 0 & 0 & 0 \\ -K_y & sin\hat{\gamma}_D & 0 & 0 & 0 & 0 \\ 0 & cos(\hat{\gamma}_D - \hat{\gamma}) & 1 & 0 & -1 & 0 \\ 0 & sin(\hat{\gamma}_D - \hat{\gamma}) & 0 & -1 & 0 & 1 \\ 0 & 1 & \sqrt{1-\left(\frac{\hat{Y}_P^- - \hat{Y}_0^-}{\hat{R}^-}\right)^2} & \frac{\hat{Y}_P^- - \hat{Y}_0^-}{\hat{R}^-} & -\sqrt{1-\left(\frac{\hat{Y}_P^- - \hat{Y}_0^-}{\hat{R}^-}\right)^2} & \frac{\hat{Y}_0^- - \hat{Y}_P^-}{\hat{R}^-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V \\ R \\ X_0 \\ Y_0 \\ X_P \\ Y_P \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{R}^- sin\hat{\gamma}_D & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{R}^- cos\hat{\gamma}_D & 0 & 0 \\ 0 & 0 & 0 & sin(\hat{\gamma}_D - \hat{\gamma}) & -sin(\hat{\gamma}_D - \hat{\gamma}) & 0 & 0 \\ 0 & 0 & 0 & -cos(\hat{\gamma}_D - \hat{\gamma}) & cos(\hat{\gamma}_D - \hat{\gamma}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_V \\ v_{X_0} \\ v_{Y_0} \\ v_\gamma \\ v_{\gamma_D} \\ v_{Y_P} \\ v_{X_P} \end{bmatrix} \tag{31}$$

5 CONCLUSIONS

INS - aiding by optically tracking a stationary ground object over time is investigated. A careful analysis of the parsimonious optical measurement scheme reveals that the LOS measurements are conducive to a stand alone estimate of the angles α' and β' included between the aircraft's inertial velocity vector **V** and the aircraft body axes. The measured α' and β' angles are related to the aircraft's attitude, heading, and flight path angle angular navigation variables. Thus, the α' and β' angle measurements provided by the optical sensor can be used for INS aiding. In other words, the gist of the paper is an investigation into INS aiding using a modern *driftmeter*, a.k.a., optical flow.

The INS-aiding process entails two phases. We exclusively address the angular navigational variables in Phase 1 and update the INS provided estimates of θ and γ using the optically provided α' measurement. The results of Phase 1 are used in Phase 2 to improve the INS provided position estimates. In order to improve the estimates of the positional navigation variables, information on the position of the ground object is used. Accurate own ship position estimation and accurate geo-location are possible, using prior ground object position and altitude information.

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