Convergence of continuous descent methods

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Abstract—**We consider continuous descent methods for the minimization of Lipschitzian functions defined on a general Banach space. We present several convergence theorems for those methods which are generated by regular vector fields. Since the complement of the set of regular vector fields is σ-porous, we conclude that our results apply to most vector fields in the sense of Baire's categories.**

Keywords— **Complete metric space, descent method, Lipschitzian function, porous set, regular vector field.**

I. Introduction

THE study of discrete and continuous descent methods is an important topic in optimization theory and in dynamical systems. See, for example, [4, 9, 11, 12, 13]. Given a continuous convex function f on a Banach space X, we associate with f a complete metric space of vector fields $V : X \to X$ such that $f^0(x,Vx) \le 0$ for all $x \in X$. Here $f^0(x,h)$ is the right-hand derivative of f at x in the direction $h \in X$. To each such vector field there correspond two gradient-like iterative processes. In two recent papers [12, 13] it is shown that for most of the vector fields in this space, both iterative processes generate sequences $\{x_n\}_{n=1}^{\infty}$ such that the sequences $\{f(x_n)\}_{n=1}^{\infty}$ tend to $\inf(f)$ as $n \to \infty$. In [15] the convergence of the values of the function f to its infimum along the trajectories of an analogous continuous dynamical system governed by such vector fields was studied. In this paper we consider the situation for Lipschitzian functions which are not necessarily convex. We also discuss continuous descent methods for Lipschitzian functions which satisfy the Palais-Smale condition.

When we say that most of the elements of a complete metric space Y enjoy a certain property, we mean that the set of points which have this property contains a G_δ everywhere dense subset of Y. In other words, this property holds generically. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y, has already been successfully applied in many areas of Analysis [5-7, 10, 18].

We now recall the concept of porosity [2, 6, 7, 13, 14, 16, 17] which enables us to obtain even more refined results.

Let (Y,d) be a complete metric space. We denote by $B_d(y,r)$ the closed ball of center $y \in Y$ and radius $r > 0$. We say that a subset $E \subset Y$ is porous in (Y,d) if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0,r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B_d(z,\alpha r) \subset B_d(y,r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y,d) if it is a countable union of porous subsets in (Y,d).

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Other notions of porosity have been used in the literature [2, 16]. We use the rather strong notion which appears in [5, 6, 7, 13, 14].

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space R^n, then σ-porous sets are of Lebesgue measure 0. The existence of a non-σ-porous set $P \subset R^n$, which is of the first Baire category and of Lebesgue measure 0, was established in [16]. It is easy to see that for any σ-porous set $A \subset R^n$, the set $A \cup P \subset R^n$ also belongs to the family $\mathcal{E}$ consisting of all the non-σ-porous subsets of R^n which are of the first Baire category and have Lebesgue measure 0. Moreover, if $Q \in \mathcal{E}$ is a countable union of sets $Q_i \subset R^n$, $i = 1,2,\ldots$, then there is a natural number j for which the set Q_j is non-σ-porous. Evidently, this set Q_j also belongs to $\mathcal{E}$. Thus one sees that the family $\mathcal{E}$ is quite large. Also, every complete metric space without isolated points contains a closed nowhere dense set which is not σ-porous [17].

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z,s) \subset B_d(y,r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0,1)$ is a constant which depends only on E.

Our paper is organized as follows. In Section 2 we consider a function f which is Lipschitzian on bounded subsets of a Banach space X, but not necessarily convex. We introduce a class of vector fields associated with such a function and present a porosity result for this class. We also discuss briefly the convergence of discrete descent methods for the minimization of such functions. In Section 3 we present convergence results for continuous descent methods. The last section is devoted to functions which satisfy a Palais-Smale type condition.

II. Lipschitzian functions

Let $(X,||\cdot||)$ be a Banach space, $(X^*,||\cdot||_*)$ its dual space, and let $f : X \to R^1$ be a function which is bounded from below and Lipschitzian on bounded subsets of X. Recall that for each pair of sets $A,B \subset X^*$,

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}||x-y||_*,\ \sup_{y\in B}\inf_{x\in A}||x-y||_*\}$$

is the Hausdorff distance between A and B. For each $x \in X$, let

$$f^0(x,h) = \limsup_{t\to 0^+,\, y\to x}\, [f(y+th)-f(y)]/t,\ h \in X, \tag{1}$$

be Clarke's generalized directional derivative of f at the point x and let

$$\partial f(x) = \{l \in X^*:\ f^0(x,h) \ge l(h) \text{ for all } h \in X\} \tag{2}$$

be Clarke's generalized gradient of f at x. We also define

$$\Xi(x) = \inf\{f^0(x,h):\ h \in X \text{ and } ||h|| = 1\}. \tag{3}$$

It is well known that the set $\partial f(x)$ is nonempty and bounded. Set

$$\inf(f) = \inf\{f(x) : x \in X\}.$$

From now on, we denote by $\mathcal{A}$ the set of all mappings $V : X \to X$ such that V is bounded on every bounded subset of X, and for each $x \in X$, $f^0(x, Vx) \le 0$. We denote by $\mathcal{A}_c$ the set of all continuous $V \in \mathcal{A}$ and by $\mathcal{A}_b$ the set of all $V \in \mathcal{A}$ which are bounded on X. Finally, let $\mathcal{A}_{bc} = \mathcal{A}_b \cap \mathcal{A}_c$. Next, we endow the set $\mathcal{A}$ with two metrics, ρ_s and ρ_w. To define ρ_s, we set, for each $V_1, V_2 \in \mathcal{A}$,

$$\tilde{\rho}_s(V_1, V_2) = \sup\{||V_1x - V_2x|| : x \in X\}$$

and

$$\rho_s(V_1, V_2) = \tilde{\rho}_s(V_1, V_2)(1 + \tilde{\rho}_s(V_1, V_2))^{-1}. \tag{4}$$

(Here we use the convention that $\infty/\infty = 1$.) Clearly, $(\mathcal{A}, \rho_s)$ is also a complete metric space. To define ρ_w, we set, for each $V_1, V_2 \in \mathcal{A}$ and each integer $i \ge 1$,

$$\rho_i(V_1, V_2) = \sup\{||V_1x - V_2x|| : x \in X \text{ and } ||x|| \le i\} \tag{5}$$

and

$$\rho_w(V_1, V_2) = \sum_{i=1}^{\infty} 2^{-i}[\rho_i(V_1, V_2)(1 + \rho_i(V_1, V_2))^{-1}]. \tag{6}$$

Clearly, $(\mathcal{A}, \rho_w)$ is a complete metric space. It is also not difficult to see that the collection of the sets

$$E(N, \varepsilon) = \{(V_1, V_2) \in \mathcal{A} \times \mathcal{A} : ||V_1x - V_2x|| \le \varepsilon,\ x \in X,\ ||x|| \le N\},$$

where $N, \varepsilon > 0$, is a base for the uniformity generated by the metric ρ_w. It is easy to see that

$$\rho_w(V_1, V_2) \le \rho_s(V_1, V_2) \text{ for all } V_1, V_2 \in \mathcal{A}.$$

The metric ρ_w induces on $\mathcal{A}$ a topology which is called the weak topology and ρ_s induces a topology which is called the strong topology Clearly, $\mathcal{A}_c$ is a closed subset of $\mathcal{A}$ with the weak topology while $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ are closed subsets of $\mathcal{A}$ with the strong topology. We consider the subspaces $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ with the metrics ρ_s and ρ_w which induce the strong and the weak topologies, respectively.

To minimize a convex function f, one usually looks for a sequence $\{x_i\}_{i=1}^{\infty}$ which tends to a minimum point of f (if such a point exists) or at least such that $\lim_{i\to\infty} f(x_i) = \inf(f)$. If f is not necessarily convex, but X is finite-dimensional, then we expect to construct a sequence which tends to a critical point z of f, namely a point z for which $0 \in \partial f(z)$. If f is not necessarily convex and X is infinite-dimensional, then the problem is more difficult and less understood because we cannot guarantee, in general, the existence of a critical point and a convergent subsequence. To partially overcome this difficulty, we have introduced the function $\Xi : X \to R^1$. Evidently, a point z is a critical point of f if and only if $\Xi(z) \ge 0$. Therefore we say that z is ε-critical for a given $\varepsilon > 0$ if $\Xi(z) \ge -\varepsilon$. In [14] we looked for sequences $\{x_i\}_{i=1}^{\infty}$ such that either $\liminf_{i\to\infty} \Xi(x_i) \ge 0$ or at least $\limsup_{i\to\infty} \Xi(x_i) \ge 0$. In the first case, given $\varepsilon > 0$, all the points x_i, except possibly a finite number of them, are ε-critical, while in the second case this holds for a subsequence of $\{x_i\}_{i=1}^{\infty}$.

In [14] it was shown, under certain assumptions on f, that for most (in the sense of Baire's categories) vector fields $W \in \mathcal{A}$, the discrete iterative processes defined in Section 2 yield sequences with the desirable properties. Moreover, it was shown there that the complement of the set of "good" vector fields is not only of the first category, but also σ-porous. In this section we will use porosity with respect to a pair of metrics, a concept which was introduced in [18].

Recal that when (Y, d) is a metric space we denote by $B_d(y, r)$ the closed ball of center $y \in Y$, and radius $r > 0$. Assume that Y is a nonempty set and $d_1, d_2 : Y \times Y \to [0, \infty)$ are two metrics which satisfy $d_1(x, y) \le d_2(x, y)$ for all $x, y \in Y$.

A subset $E \subset Y$ is called porous with respect to the pair (d_1, d_2) (or just porous if the pair of metrics is fixed) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there is $z \in Y$ for which $d_2(z, y) \le r$ and

$$B_{d_1}(z, \alpha r) \cap E = \emptyset.$$

A subset of the space Y is called σ-porous with respect to (d_1, d_2) (or just σ-porous if the pair of metrics is understood) if it is a countable union of porous (with respect to (d_1, d_2)) subsets of Y.

Note that if $d_1 = d_2$, then by Proposition 1.1 of [18] our definitions reduce to those in [5-7, 13]. We use porosity with respect to a pair of metrics because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, it is shown in [18, Proposition 1.2] that if a subset $E \subset Y$ is porous with respect to (d_1, d_2), then E is porous with respect to any metric which is weaker than d_2 and stronger than d_1. For each subset $E \subset X$, we denote by $cl(E)$ the closure of E in the norm topology. The results of [14] were established in any Banach space and for those functions which satisfy the following two assumptions.

B(i) For each $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that

$$cl(\{x \in X : \Xi(x) < -\varepsilon\}) \subset \{x \in X : \Xi(x) < -\delta\};$$

B(ii) For each $r > 0$, the function f is Lipschitzian on the ball $\{x \in X : ||x|| \le r\}$.

We will say that a mapping $V \in \mathcal{A}$ is regular if for any natural number n there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying $||x|| \le n$ and $\Xi(x) < -1/n$, we have $f^0(x, Vx) \le -\delta(n)$.

We denote by $\mathcal{F}$ the set of all regular vector fields $V \in \mathcal{A}$.

The following result was established in [14].

Theorem 2.1: Assume that both B(i) and B(ii) hold. Then $\mathcal{A} \setminus \mathcal{F}$ (respectively, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_b \setminus \mathcal{F}$ and $\mathcal{A}_{bc} \setminus \mathcal{F}$) is a σ-porous subset of the space $\mathcal{A}$ (respectively, $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$) with respect to the pair (ρ_w, ρ_s).

In the sequel we will also make use of the following assumption:

B(iii) For each integer $n \ge 1$ there exists $\delta > 0$ such that for each $x_1, x_2 \in X$ satisfying $||x_1||, ||x_2|| \le n$, $\min\{\Xi(x_i) : i = 1, 2\} \le -1/n$, and $||x_1 - x_2|| \le \delta$, the following inequality holds: $H(\partial f(x_1), \partial f(x_2)) \le 1/n$.

III. Continuous descent methods for Lipschitzian functions

Throughout this paper we let $x \in W^{1,1}(0,T;X)$, i.e. (see, e.g., [3]),

$$x(t) = x_0 + \int_0^t u(s)ds,\ t \in [0,T],$$

where $T > 0$, $x_0 \in X$ and $u \in L^1(0,T;X)$. Then $x : [0,T] \to X$ is absolutely continuous and $x'(t) = u(t)$ for a.e. $t \in [0,T]$.

Recall that the function $f : X \to R^1$ is Lipschitzian on bounded subsets of X. Thus the restriction of f to the set $\{x(t) : t \in [0,T]\}$ is Lipschitzian. Hence the function $(f \cdot x)(t) := f(x(t))$, $t \in [0,T]$, is absolutely continuous. It follows that for almost every $t \in [0,T]$, both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist:

$$x'(t) = \lim_{h\to 0} h^{-1}[x(t+h) - x(t)],$$

$$(f \cdot x)'(t) = \lim_{h\to 0} h^{-1}[f(x(t+h)) - f(x(t))].$$

The next proposition was proved in [1].

Proposition 3.1: Assume that $t \in [0,T]$ and that both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist. Then

$$(f \cdot x)'(t) = \lim_{h\to 0} h^{-1}[f(x(t) + hx'(t)) - f(x(t))].$$

Now we are ready to state three convergence theorems which have been proved in [1].

Theorem 3.1: Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular and let

$$x \in W^{1,1}_{loc}([0,\infty);X).$$

Assume that

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty)$$

and that the function $x(t)$, $t \in [0,\infty)$, is bounded. Then for each $\varepsilon > 0$,

$$\lim_{T\to\infty} \mu(\{t \in [T,\infty) :\ \Xi(x(t)) < -\varepsilon\}) = 0.$$

Theorem 3.2: Let $V \in \mathcal{A}$ be regular, let B(i), B(ii) and B(iii) hold, and let $x \in W^{1,1}_{loc}([0,\infty);X)$ be a bounded function which satisfies

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then

$$\liminf_{t\to\infty} \Xi(x(t)) \geq 0.$$

Theorem 3.3: Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular, and suppose that

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let K_0 and ε be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $T \geq N_0$, each $W \in \mathcal{U}$, and each mapping $x \in W^{1,1}(0,T;X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,T],$$

the following inequality holds:

$$\mu\{t \in [0,T] :\ \Xi(x(t)) < -\varepsilon\} \leq N_0.$$

Corollary 3.1: Let B(i) and B(ii) hold, let $V \in \mathcal{A}$ be regular, and suppose that

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let K_0, ε be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $W \in \mathcal{U}$ and each mapping $x \in W^{1,1}_{loc}([0,\infty);X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = Wx(t) \text{ for a.e. } t \in [0,\infty)$$

the following inequality holds:

$$\mu\{t \in [0,\infty) :\ \Xi(x(t)) < -\varepsilon\} \leq N_0.$$

This corollary, which is an extension of Theorem 3.1, follows immediately from Theorem 3.3.

IV. A Palais-Smale type condition

In this section $f : X \to R^1$ is a locally Lipschitzian function which is bounded from below. We begin with the following proposition [1].

Proposition 4.1: For each $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that

$$f(x_\varepsilon) \leq \inf(f) + \varepsilon \text{ and } \Xi(x_\varepsilon) \geq -\varepsilon.$$

In our setting we say that the function f satisfies the Palais-Smale (P-S) condition if each sequence $\{x_n\}_{n=1}^\infty \subset X$ such that

$$\sup\{|f(x_n)| :\ n = 1,2,\ldots\} < \infty$$

and $\limsup_{n\to\infty} \Xi(x_n) \geq 0$ has a norm convergent subsequence.

Note that this is a generalization of the classical Palais-Smale condition to locally Lipschitzian functions.

Define

$$\mathrm{Cr}(f) = \{x \in X :\ \Xi(x) \geq 0\}.$$

Proposition 4.2: Let $\{x_n\}_{n=1}^\infty \subset X$ be such that $\lim_{n\to\infty} x_n = x$ and

$$\liminf_{n\to\infty} \Xi(x_n) \geq 0.$$

Then $\Xi(x) \geq 0$.

Propositions 4.1 and 4.2 imply the following fact.

Proposition 4.3: Assume that the function f satisfies the (P-S) condition. Then $\mathrm{Cr}(f) \neq \emptyset$.

Proposition 4.4: Assume that the function f is bounded on bounded subsets of X and satisfies the (P-S) condition. Then for each $r > 0$, the set

$$\{x \in X :\ ||x|| \leq r\} \cap \mathrm{Cr}(f)$$

is compact in the norm topology.

For each $x \in X$ and $A \subset X$, set

$$d(x,A) = \inf\{||x - y|| :\ y \in A\}.$$

Proposition 4.5: Let $r, \varepsilon > 0$, and let f be bounded on bounded subsets of X and satisfy the (P-S) condition. Then there is $\delta > 0$ such that if $x \in X$ satisfies

$$||x|| \leq r \text{ and } \Xi(x) \geq -\delta,$$

then $d(x, \mathrm{Cr}(f)) \leq \varepsilon$.

The next three theorems have also been proved in [1].

Theorem 4.1: Let f satisfy B(i), B(ii) and the (P-S) condition, let $V \in \mathcal{A}$ be regular and let $x \in W^{1,1}_{loc}([0,\infty);X)$ be a bounded mapping which satisfies

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then for each $\varepsilon > 0$,

$$\lim_{T\to\infty} \mu(\{t \in [0,\infty) : d(x(t), \mathrm{Cr}(f)) > \varepsilon\}) = 0.$$

Theorem 4.2: Let f satisfy the (P-S) condition, let $V \in \mathcal{A}$ be regular, let B(i), B(ii) and B(iii) hold, and let $x \in W^{1,1}_{loc}([0,\infty);X)$ be a bounded mapping which satisfies

$$x'(t) = V(x(t)) \text{ for a.e. } t \in [0,\infty).$$

Then

$$\limsup_{t\to\infty} d(x(t), \mathrm{Cr}(f)) = 0.$$

Theorem 4.3: Let f satisfy the (P-S) condition, B(i) and B(ii), and suppose that

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let $V \in \mathcal{A}$ be regular, and let K_0 and γ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $T \geq N_0$, each $W \in \mathcal{U}$, and each mapping $x \in W^{1,1}(0,T;X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = Wx(t) \text{ for a.e. } t \in [0,T],$$

the following inequality holds:

$$\mu(\{t \in [0,T] : d(x(t), \mathrm{Cr}(f)) > \gamma\} \leq N_0.$$

Corollary 4.1: Let f satisfy the (P-S) condition, B(i) and B(ii), and suppose that

$$\lim_{||x||\to\infty} f(x) = \infty.$$

Let $V \in \mathcal{A}$ be regular, and let K_0 and γ be positive numbers. Then there exist $N_0 > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ with the weak topology such that for each $W \in \mathcal{U}$ and each mapping $x \in W^{1,1}_{loc}([0,\infty);X)$ satisfying

$$||x(0)|| \leq K_0$$

and

$$x'(t) = W(x(t)) \text{ for a.e. } t \in [0,\infty),$$

the following inequality holds:

$$\mu\{t \in [0,\infty) : d(x(t), \mathrm{Cr}(f)) > \gamma\} \leq N_0.$$

This corollary, which is an extension of Theorem 4.1, is a consequence of Theorem 4.3.

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