Zero Structures of n-D Systems

A.C. Pugh & E.M.O. EL-Nabrawy
Department of Mathematical Sciences, Loughborough University
Loughborough, Leicestershire,
LE11 3TU, UK

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Abstract

A definition of the zero structure of an n-D polynomial matrix is developed from previous work by Zerz, and the nature of its invariance is established.

1 Introduction

Structural (algebraic) invariants play a crucial role in control systems analysis, since such an invariant frequently encapsulates some particular system property. Thus in classical multivariable theory [8] the zeros of a polynomial matrix are basic to the study of, for example, system stability, controllability and observability. The analysis of a system is frequently aided by transforming it to some simpler (reduced) but equivalent form, and in this the extent to which structural invariants are invariant under such transformations is an important issue. The relevant indicators of a system's behaviour must not be changed by such action, if the conclusions of the reduced system's behaviour are to remain valid for the original system.

In contrast to classical multivariable systems theory, the analysis of mutidimensional (n-D) systems is much less developed, in part due to dislocation in the interpretation of invariants, in the strict algebraic sense, as indicators of specific system properties. Zerz [11], however, has made solid the connection between the zero structure of an n-D polynomial matrix, and the controllabiltiy and observability (in a behavioral setting) of the system it represents. This paper refines this definition of zero structure [11], and presents results concerning its invariance with repect to a transformation which has particular relevance to the n-D systems setting [6], [7].

2 Preliminary Definitions

Let F be a field. The ideal generated by the polynomials $f_1, \dots, f_s \in F[x_1, \dots, x_n]$, denoted by $I = \langle f_1, \dots, f_s \rangle$, is defined as

$$\langle f_1, \dots, f_s \rangle \triangleq \{\sum_{i=1}^{s} u_i f_i | u_i \in F[x_1, \dots, x_n], i = 1, \dots, s\}$$

$S = \{f_1, \dots, f_s\}$ is then called a generating set for I.

Definition 1. *The VARIETY defined by the ideal $I = \langle f_1, \dots, f_s \rangle$, denoted $V_{\bar{F}}(I)$, is defined to be the set of all solutions in $\bar{F}$ (where $\bar{F}$ denotes the algebraic closure of F) of the system of algebraic equations*

$$f_1 = 0, f_2 = 0, \dots, f_s = 0 \tag{1}$$

$$\text{i.e. } V_{\bar{F}}(I) \triangleq \{a = (a_1, \dots, a_n) \in \bar{F}^n; \; f_i(a) = 0, i = 1, \dots, s\}$$

Any $a \in V_{\bar{F}}(I)$ is called a ZERO of $f_1, \dots, f_s$.

Invariably the fields F we shall use are $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}$ is algebraically closed, while $\bar{\mathbb{R}} = \mathbb{C}$. It is noted that $V_{\bar{F}}(\langle f_1, \dots, f_s \rangle) = \bigcap_{i=1}^{s} V_{\bar{F}}(\langle f_i \rangle)$.

Lemma 1. *$V_{\bar{F}}(I) = \emptyset$ if and only if $1 \in G$, where G is some Gröbner basis for the ideal I [1]. (i.e. given polynomials $f_1, \dots, f_s$, then there are no solutions to the system $f_1 = 0, f_2 = 0, \dots, f_s = 0$ in $\bar{F}^n$ if and only if $\{1\}$ is the reduced Gröbner basis for I.)*

Definition 2. *The set $S = \{f_1, \dots, f_s\}$ is said to be zero coprime if there exists no value $a = (a_1, \dots, a_n) \in \bar{F}^n$ such that $f_1, \dots, f_s$ are identically zero.*

It follows that $S = \{f_1, \dots, f_s\}$ is zero coprime iff $V_{\bar{F}}(I) = \emptyset$ where $I \triangleq \langle f_1, \dots, f_s \rangle$.

Definition 3. *$S = \{f_1, \dots, f_s\}$ is said to be factor coprime if there exists no $g \in F[x_1, \dots, x_n]$, which is a common divisor of $f_1 \dots, f_s$.*

Generally in the case of polynomials in more than one indeterminate, factor coprimeness does not imply zero coprimeness,. i.e. it is only in the case $n = 1$ that the variety defined by the ideal generated by a set of 1-D polynomials is empty coincides precisely with the set of the 1-D polynomials being factor coprime. In general where $n > 1$, even if the set of polynomials is factor coprime, the variety defined by this set of multivariate polynomials might not be empty, unless the polynomials are additionally zero coprime. A result used in the sequel is

Lemma 2. *Let $g, h \in F[x_1, \ldots, x_n]$ and suppose that g divides $f_i h$, written $g|f_i h$, for $i = 1, \ldots, s$ where $S = \{f_1, \ldots, f_s\}$ is a factor coprime set of polynomials, then $g|h$.*

3 Zero Structures of Matrices

Let $P(x)$ be a $p \times q$ n-D polynomial matrix, where x denotes the n-tuple $(x_1, \ldots, x_n)$. For ease of presentation we shall assume that $P(x)$ is of full rank and its rank r is $\min(p, q)$.

There are various zero structures one can define for $P(x)$, but all definitions are based on the property that a zero is associated with a rank reduction of the matrix.

Definition 4. *The i^{th} DETERMINANTAL DIVISOR $d_i(x)$ of the matrix $P(x)$ is the greatest common divisor of the i^{th} order minors of $P(x)$. The zeros of $d_i(x)$, $i = 1, \ldots, r$, are called the i^{th} DETERMINANTAL ZEROS of $P(x)$.*

This definition is the direct extension of the 1-D case where it characterises exactly the situation in which a 1-D matrix loses rank. The simple example $P(x) = (x_1 \ x_2)$ has $d_1(x) = 1$ and so has no determinantal divisors. Nevertheless $P(x)$ loses rank for $x_1 = x_2 = 0$. We require a more complete definition.

For any $p \times q$ n-D polynomial matrix $P(x)$, let $m_{(i,j)}$ denote the $i \times i$ minors of $P(x)$ where $j = 1, \ldots, k_i = \frac{p!}{i!(p-i)!}\frac{q!}{i!(q-i)!}$. Denote the ideals generated by the $i \times i$ minors of $P(x)$ by $I_i^{[P]}$ and write $I_i^{[P]} = d_i J_i^{[P]}$, where $J_i^{[P]}$ is the ideal generated by the set of polynomials which result from the $i \times i$ minors of $P(x)$ when the i^{th} determinantal divisor $d_i(x)$ is removed. Clearly each ideal $J_i^{[P]}$ is generated by a set of factor coprime polynomials. This set may not be additionally zero coprime which is the distinctive feature of n-D ($n > 1$), and the situation which the previous simple example illustrates. These considerations lead us to the following definitions which are developed from Zerz [11].

Definition 5. *The i^{th} ORDER INVARIANT ZEROS of an n-D polynomial matrix $P(x)$, are the elements of $V_{\bar{F}}(I_i^{[P]})$ (the variety defined by the ideal $I_i^{[P]}$), $i = 1, \ldots, r$.*

Definition 6. *The ALGEBRAIC MULTIPLICITY of an invariant zero $a = (a_1, \ldots, a_n) \in \bar{F}^n$ is the non-negative integer $n(a)$ defined as*

$$n(a) \triangleq r - \text{rank } P(a)$$

Rather loosely we say

Definition 7. *The i^{th} GEOMETRIC MULTIPLICITY $\delta_i(a)$ of an invariant zero $a = (a_1, \ldots, a_n)$ is the number of times a occurs in the variety $V_{\bar{F}}(I_i^{[P]})$.*

The fact that every $i \times i$ minor can be written as a linear combination of $(i-1) \times (i-1)$ minors has a number of consequences which are summarised in 1-D by the relationships

$$d_i(x)|d_{i+1}(x), \ i = 1, \ldots, r-1 \tag{2}$$

In n-D the implications are

Theorem 1. *If If $P(x)$ is a $p \times q$ matrix then*

$$I_r^{[P]} \subseteq \cdots \subseteq I_1^{[P]}$$

$$V_{\bar{F}}(I_r^{[P]}) \supseteq \cdots \supseteq V_{\bar{F}}(I_1^{[P]})$$

Proof. Clearly since any $i \times i$ minor can be expressed in terms of $(i-1) \times (i-1)$ minors then

$$m_{(i,j)} \in I_{i-1}^{[P]}$$

Hence $I_i^{[P]} \subseteq I_{i-1}^{[P]}$ and the rest follows. □

Corollary 1. *If $d_i(x)$ denotes the i^{th} determinantal divisor of $P(x)$ then*

$$\langle d_r \rangle \subseteq \cdots \subseteq \langle d_1 \rangle$$

$$V_{\bar{F}}(\langle d_r \rangle) \supseteq \cdots \supseteq V_{\bar{F}}(\langle d_1 \rangle)$$

Example 1. *Consider the 3-D polynomial matrix given by*

$$P(x, y, z) = \begin{bmatrix} x^2 & 0 & xy \\ 0 & z^2 & xz \end{bmatrix}$$

The ideals generated by the first and second order minors are:

$$I_2^{[P]} = \langle x^3 z, x^2 z^2, -xyz^2 \rangle \tag{3}$$

$$I_1^{[P]} = \langle x^2, xy, xz, z^2 \rangle \tag{4}$$

It is clear that $I_2 \subset I_1$, as every element of I_2 can be written in terms of elements in I_1. Note that $d_2 = xz, d_1 = 1$ and so $\langle d_2 \rangle \subseteq \langle d_1 \rangle$. It follows then that

$$J_2^{[P]} = \langle x^2, xz, -yz \rangle \tag{5}$$
$$J_1^{[P]} = \langle x^2, xy, xz, z^2 \rangle \tag{6}$$

Now, for example $-yz \in J_2^{[P]}$, but $\notin J_1^{[P]}$, since it clearly cannot be expressed as a linear combination of the elements of the generating set for $J_1^{[P]}$. Therefore $J_2^{[P]} \nsubseteq J_1^{[P]}$. Also $xy \in J_1^{[P]}$ but $\notin J_2^{[P]}$, for the same reason. Thus $J_2^{[P]} \nsupseteq J_1^{[P]}$ This illustrates the non-existence of a definite inclusion between the ideals $J_i^{[P]}$.

4 The Invariance of Zeros

Whichever approach is adopted, say the clasical one of [8] or the behavioral one of [9] one is lead invariably to the study of polynomial matrices as a basic element in much of the analysis of linear systems. The invariant zeros of such matrices have implications in the study of problems such as, for example, stability in n-D systems theory [3], and controllability and observability of n-D systems [11]. We wish to determine the nature of the invariance of such zeros.

Definition 8. [10] *Two $p \times q, p \times l$ (respectively $p \times q, m \times q$) n-D polynomial matrices $T(x), U(x)$ (resp. $T(x), V(x)$) are said to be ZERO LEFT (resp. RIGHT) COPRIME, written zlc (resp. zrc) in case*

$$\begin{gathered}\operatorname{rank}\begin{pmatrix}T(x) & U(x)\end{pmatrix} = p \\ \left(\textit{resp. } \operatorname{rank}\begin{pmatrix}T^T(x) & V^T(x)\end{pmatrix}^T = q\right)\end{gathered} \tag{7}$$

$\forall\ x \in \bar{F}^n$.

An immediate consequence of the previous definition is

Theorem 2. *[11] The $p \times q, p \times l$ n-D polynomial matrices $T(x), U(x)$ are zlc iff $\begin{pmatrix}T(x) & U(x)\end{pmatrix}$ possesses no invariant zeros, with a similar statement holding for zrc.*

For many reasons it is necessary to transform a polynomial (system) matrix to a simpler but equivalent form. One fundamental equivalence transformation in the n-D context, with rich properties from this point of view, is the following [2], [6], [7].

Definition 9. *Denote the class of $(s+p) \times (s+q)$ n-D polynomial matrices by $\mathcal{P}(p, q)$, where $s > -min(p, q)$. $P_1(x), P_2(x) \in \mathcal{P}(p, q)$ are said to be ZERO COPRIME EQUIVALENT (ZC-E) in case $\exists$ polynomial matrices $L(x), R(x)$ of appropriate dimensions such that*

$$L(x)P_1(x) = P_2(x)R(x) \tag{8}$$

with L, P_2 zlc, and P_1, R zrc.

One main result which is proved here for full rank matrices (for simplicity) is

Theorem 3. *Suppose that $P_i(x) \in \mathcal{P}(p, q)$, with dimensions $p_i \times q_i, i = 1, 2$ where $p_1 - q_1 = p_2 - q_2 (= p - q)$ are ZC-E according to the relation*

$$M(x)P_2(x) = P_1(x)N(x) \tag{9}$$

then

$$I_{h_1-i}^{[P_1]} = I_{h_2-i}^{[P_2]},\ i = 0, \ldots, h \tag{10}$$

where $h = \min(h_1 - 1, h_2 - 1), h_1 = \min(p_1, q_1),\ h_2 = \min(p_2, q_2)$, and where $I_j^{[\cdot]}$ denotes the ideal generated by the $j \times j$ minors of the indicated polynomial matrix. For any $i > h$, $I_{h_1-i}^{[P_1]} = \langle 1 \rangle$ or $I_{h_2-i}^{[P_2]} = \langle 1 \rangle$ in case $h_1 - i > 0$ or $h_2 - i > 0$.

Proof. (Sketch) Assume the matrices P_1, P_2 have full rank. From the coprimeness requirements of ZC-E $\exists$ polynomial matrices $X(x), Y(x), W(x), Z(x)$ of appropriate dimensions such that

$$\begin{aligned} MX + P_1Y &= I_{p_1} \\ WP_2 + ZN &= I_{q_2} \end{aligned} \tag{11}$$

From (9) and (11) it follows that

$$\begin{pmatrix} W & -Z \\ M & P_1 \end{pmatrix} \begin{pmatrix} P_2 & X \\ -N & Y \end{pmatrix} = \begin{pmatrix} I_{q_2} & J \\ 0 & I_{p_1} \end{pmatrix} \tag{12}$$

where $J = WX - ZY$.

For any matrix Q let $Q_{j_1,\ldots,j_k}^{i_1,\ldots,i_k}$ denote the $k \times k$ submatrix formed from rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$. Consider then the following equation formed from ,(12)

$$\begin{aligned} &\underbrace{\begin{pmatrix} E^{i_1,\ldots,i_k} & 0 \\ M & P_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} P_{2_{j_1,\ldots,j_k}} & X \\ -N_{j_1,\ldots,j_k} & Y \end{pmatrix}}_{B} \\ &\qquad = \begin{pmatrix} P_{2_{j_1,\ldots,j_k}}^{i_1,\ldots,i_k} & X_{j_1,\ldots,j_k}^{i_1,\ldots,i_k} \\ 0 & I_{p_1} \end{pmatrix} \end{aligned} \tag{13}$$

where $k = h_2, \ldots, 1$ if $h_1 > h_2$, and $k = h_2, \ldots, h_2 - h_1 + 1$ if $h_2 > h_1$, and $E^{i_1,\ldots,i_k}$ is that matrix whose r, s^{th} element is 1 if $s = i_r$, and zero otherwise.

Take determinants of of both sides of (13), and use the Cauchy-Binet theorem to expand the left hand side. This gives

$$\sum_m |A^{1,\dots,p_1+k}_{m_1,\dots,m_{p_1+k}}||B^{m_1,\dots,m_{p_1+k}}_{1,\dots,p_1+k}| = |P_2{}^{i_1,\dots,i_k}_{j_1,\dots,j_k}| \quad (14)$$

Now the form of A indicates that any factor of A of the type occurring in the left hand side of (14) for which $\{i_1,\dots,i_k\}$ is not a subset of $\{m_1,\dots,m_{p_1+k}\}$ is zero. Thus all minors of A which occur in the left hand side of (14) contain the columns $\{i_1,\dots,i_k\}$. Such a factor is then expressible via Laplace expansion in terms of products of minors of M and P_1. The smallest minor of P_1 occuring in this Laplace expansion is of order $p_1 + k - p_2 (= h_1 + k - h_2)$. Thus $|P_2{}^{i_1,\dots,i_k}_{j_1,\dots,j_k}|$ is expressible as a linear combination of minors of P_1 of order h_1+k-h_2 and greater. Since any minor can be expanded in terms of lower order minors, it follows that $|P_2{}^{i_1,\dots,i_k}_{j_1,\dots,j_k}|$ can be written as a linear combination of the h_1+k-h_2 order minors of P_1. It thus follows (on writing $i = h_2 - k$) that

$$I^{[P_2]}_{h_2-i} \subset I^{[P_1]}_{h_1-i},\ i = 0,\dots,h \quad (15)$$

By the symmetry property of the ZC-E relation

$$M'P_1 = P_2N'$$

where $M'(x), N'(x)$ are $p_2\times p_1, q_2\times q_1$ polynomial matrices, and $M'(x), P_2(x)$ are zero left coprime, $P_1(x)$, $N'(x)$ are zero right coprime. Applying the same procedure as above gives:

$$I^{[P_1]}_{h_1-i} \subset I^{[P_2]}_{h_2-i}$$

where $i = 0,\dots,h$, and so the theorem follows. □

The theorem has a number of corollaries.

Corollary 2. *If $V(I)$ is the variety generated by the ideal I, then under the conditions of Theorem 3*

$$V(I^{[P_1]}_{h_1-i}) = V(I^{[P_2]}_{h_2-i}) \text{ for } i = 0,1,\dots,h \quad (16)$$

and for any $i > h$, $V(I^{[P_1]}_{h_1-i}) = \emptyset$ or $V(I^{[P_2]}_{h_2-i}) = \emptyset$ in case $h_1 - i > 0$ or $h_2 - i > 0$. Further if $a \in \bar{F}$ is an invariant zero of P_1 with algebraic multiplicity $n(a)$ and geometric multilicities $\delta_{h_1-n(a)+1}(a),\dots,\delta_{h_1}(a)$ then a is a zero of P_2 with identical algebraic and geometric multiplicities.

Theorem 3 taken together with Lemma 2 gives

Corollary 3. *Let $d^{[P_1]}_i$, $d^{[P_2]}_i$ denote the i^{th} determinantal divisors of the matrices P_1, P_2 respectively, related as in theorem 3. Let $V(\langle d^{[P_1]}_i\rangle)$, $V(\langle d^{[P_2]}_i\rangle)$ denote the varieties defined by the ideals generated by $d^{[P_1]}_i, d^{[P_2]}_i$ then*

$$d^{[P_1]}_{h_1-i} = c_i d^{[P_2]}_{h_2-i}$$
$$V(\langle d^{[P_1]}_{h_1-i}\rangle) = V(\langle d^{[P_2]}_{h_2-i}\rangle)$$

where $i = 0,\dots,h$, and $c_i \in \mathbb{R}\backslash\{0\}$.

Suppose the ideal I_i generated by the $i\times i$ minors of the n-D matrix P is written $d_iJ^{[P]}_i$. Although we have seen that no particular relation of inclusion holds between the ideals $J^{[P]}_i$ there is something that can be said about the corresponding ideals of matrices P_1 and P_2 related by ZC-E

Corollary 4. *Let the matrices P_1 and P_2 be related as in theorem 3,and the ideals $J^{[P_1]}_i$ and $J^{[P_2]}_i$ be as defined above then*

$$J^{[P_1]}_{h_1-i} = J^{[P_2]}_{h_1-i}$$
$$V(J^{[P_1]}_{h_1-i}) = V(J^{[P_2]}_{h_2-i})$$

where $i = 0,\dots,h$.

Proof. By theorem 3 and corollary 3, we have:

$$d^{[P_1]}_{h_1-i}J^{[P_1]}_{h_1-i} = c_i d^{[P_2]}_{h_2-i}J^{[P_2]}_{h_2-i} \quad (17)$$

for $i = 0,\dots,h$ and hence the result follows □

5 Implications for MFDs

The above results have an immediate implication for matrix fraction descriptions of an n-D transfer function matrix. For given a $p\times q$, n-D transfer function matrix $G(x)$, it is known that if there exist a (left or right) zero coprime MFD, then all coprime MFD's (left and right) of $G(x)$ are zero coprime [2]. Hence we have

Theorem 4. *Let $G(x)$ be an n-D rational matrix which possesses a zero coprime MFD. If*

$$G(x) = N_1(x)D_1^{-1}(x) \quad (18)$$
$$= D_2^{-1}(x)N_2(x)$$

are any coprime factorisations of $G(x)$ then

$$I^{[D_1]}_{q-i} = I^{[D_2]}_{p-i} \text{ for } i = 0,\dots,\min(p-1,q-1) \quad (19)$$

where, for $i > \min(p-1,q-1)$, $I^{[D_1]}_{q-i} = \langle 1\rangle, I^{[D_2]}_{p-i} = \langle 1\rangle$ in case $q-i > 0$ or $p-i > 0$.

$$I^{[N_1]}_k = I^{[N_2]}_k \text{ for } k = 1,\dots,\min(p,q) \quad (20)$$

Generally when an n-D transfer function matrix, $G(x)$, has an MFD which is other than zero coprime then only restricted statements are available. The main result is that the coprimeness type of the MFD is latent in its handedness. That is, for example if $G(x)$ has one minor (resp. factor) coprime left MFD then all coprime left MFDs are of this coprimeness type. Nothing can be inferred about the coprimeness of right MFDs of $G(x)$ which may be either (but not both) minor or factor coprime. The main result above has utilised the Bezout characterisation of zero coprimeness, and only a restricted version of this is available in the case of minor coprimeness, while no such characteristaion exists at all for factor coprimeness. In case $G(x)$ has both left and right MFDs of the minor coprimeness type then the previous theorem may be extended to the following

Theorem 5. *Let $G(x)$ be a $p \times q$ n-D rational matrix which possesses minor coprime left and right MFDs. If*

$$\begin{aligned} G(x) &= N_1(x)D_1^{-1}(x) \quad (21) \\ &= D_2^{-1}(x)N_2(x) \end{aligned}$$

are any coprime factorisations of $G(x)$ then

$$I_{q-i}^{[D_1]} \supseteq \Psi_j(x_j)^q I_{p-i}^{[D_2]} \quad (22)$$

$$I_{p-i}^{[D_2]} \supseteq \Phi_j(x_j)^p I_{q-i}^{[D_1]} \quad (23)$$

where $i = 0, 1, \dots, h$, $h = \min(p-1, q-1)$, $j = 1, \dots, n$, and where $I_n^{[\cdot]}$ denotes the ideal generated by the $n \times n$ minors of the indicated polynomial matrix. Here $\Psi_j(x_j)$, $\Phi_j(x_J)$ $j = 1, \dots, n$ are the n-D polynomials, in which the indicated indeterminate is absent, given in the Bezout minor coprime characterisation [10],

$$N_1X_j + D_1Y_j = \Psi_j(x_j)I_{p_1} \quad (24)$$

$$W_jD_2 + Z_jN_2 = \Phi_j(x_j)I_{q_2} \quad (25)$$

for $j = 1, \dots, n$

Clearly analogous results hold for the ideals generated by the minors of N_1, N_2.

6 Conclusions

The study of polynomial matrices lies at the heart of much of linear systems analysis. Particularly the zero structure of such matrices plays a vital role in the 1-D theory, and the work of Zerz, for example, highlights its importance in n-D theory. Here a certain zero structure is developed from [11] and is shown to be invariant under the transformation of ZC-E. This is an important result since ZC-E has been established ([6], [7]) as being a fundamental transformation for the reduction of linear n-D systems. It also has immediate implications for the structure of certain matrix fraction descriptions of an n-D transfer function matrix, as has been described.

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