Optimal open-loop tracking using sampled-data system with preview

K.Yu. Polyakov, E.N. Rosenwasser, and B.P. Lampe

Abstract—. **An optimal tracking problem is considered for an open-loop SISO sampled-data system. It is assumed that the reference signal is known in advance over an interval τ. A rigorous frequency-domain solution is presented on basis of the Laplace transform in continuous time. Formulae for the degrees of numerator and denominator of the optimal controller are given. The dependence of the cost function on the preview interval is investigated and a lower bound is obtained for the performance criterion.**

Keywords— **Sampled-data system, tracking system, Laplace transform, direct design, preview control**

I. Introduction

In some problems, for instance, in control of robot manipulators and terrain following for flying vehicles, the desired trajectory of the system motion is known in advance over an interval τ. In this case, the information on "future" values of the input (called preview) can be exploited for decreasing the tracking error.

Previously, the problem of preview control was considered mostly for time invariant continuous and discrete-time systems (see [1-5] and references therein). Much less attention was paid to analogous problems for sampled-data systems. A similar problem of delayed signal reconstruction for open-loop sampled-data systems was considered in [6-7] where the system performance was evaluated by the $\mathcal{H}_\infty$-norm of the corresponding operator. These papers demonstrated great problems in constructing equivalent discrete state-space models for sampled-data systems with arbitrary delays and preview.

Below it will be shown that the problem of preview tracking reduces to a computational scheme which incorporates elements with transfer functions of the form $\mathrm{e}^{s\tau}$. Thus, when such a sampled-data system is investigated in continuous time, it becomes infinite-dimensional. This fact leads to great difficulties in its analysis and design. Such well-known techniques of direct sampled-data system design as "lifting" [8] and "FR-operator" [9] do not cope with this task, because so far they have not been adapted for time-delay systems.

This paper presents a rigorous solution of an open-loop preview tracking problem for SISO sampled-data system on basis of the frequency-domain theory of digital systems [10], which makes it possible to take into account networks like $\mathrm{e}^{s\tau}$ without any approximation [11-12]. As distinct from [6-7], we use the $\mathcal{L}_2$-norm of the tracking error, i.e., the integral quadratic error between desired and actual outputs, as an optimality criterion. The solution is based on the Wiener-Hopf method for sampled-data systems [10]. The proposed method takes into account restrictions on control power and holds for arbitrary τ rather than for an integer multiple of the sampling period.

The paper is organized as follows. A formal statement of the problem is given in Sec. 3. There is constructed an equivalent computational scheme for the preview tracking problem. This system incorporates pure delay networks rather than a preview unit and provides for the same value of the cost function.

In Sec. 4 an equivalent discrete model is constructed for the original sampled-data optimization problem.

In Sec. 5 a complete solution to the $\mathcal{L}_2$-optimal sampled-data tracking problem is presented, including an algorithm for synthesis of the optimal digital filter, formulas for determination of its order and minimal value of the cost function.

In Sec. 6 the dependence of the cost function on the value of the preview interval τ is investigated, including limiting properties as $\tau \to \infty$. An explicit expression is given for the lower bound for the $\mathcal{L}_2$-norm of the tracking error. As distinct from the stationary discrete-time case [4], [13], it is impossible to attain a zero cost function in the limit as $\tau \to \infty$, even without restrictions imposed on control power. This is caused by the fact that digital filters in sampled-data systems can use only the values of the input signal at the sampling instants rather than the complete signal history.

II. Notation

Let T be the sampling period, $\zeta \overset{\triangle}{=} e^{-sT}$ the unit delay operator and $\omega \overset{\triangle}{=} 2\pi/T$ the sampling frequency.

The asterisk denotes the Hermitian conjugate function such that, for the scalar case,

$$F^*(s) \overset{\triangle}{=} F(-s)\,, \quad \mathcal{F}^*(\zeta) = \mathcal{F}(\zeta^{-1})\,.$$

Real rational functions in s and ζ will be called stable, if they are analytic in $\mathbf{Re}\, s \geq 0$ and $|\zeta| \leq 1$, respectively. Also, polynomials in ζ are called stable if they have no roots inside the closed unit disk.

Introduce, for any real rational function $F(s)$, the *displaced pulse-frequency response*

$$\Phi_F(s,t) \overset{\triangle}{=} \frac{1}{T} \sum_{k=-\infty}^{\infty} F(s+kj\omega)\, \mathrm{e}^{kj\omega t} \tag{1}$$

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and the *discrete Laplace transform* [10]:

$$D_F(s,t) \triangleq \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s+kj\omega)\,\mathrm{e}^{(s+kj\omega)t} \qquad (2)$$

$$D_F(\zeta,t) \triangleq D_F(s,t)\,|_{exp(-sT)=\zeta}\ .$$

Denote the monic denominator of the function $D_F(\zeta,t)$ by $d_F(\zeta)$ and its degree by $\delta(F) = \deg d_F$.

By a quasipolynomial we mean a rational function in ζ free of poles except for, possibly, $\zeta = 0$.

III. Statement of the problem

The block-diagram of the open-loop sampled-data system under consideration is shown in Fig. 1. The system includes a plant with transfer function $F(s)$, actuator $H(s)$ and digital controller $C(\zeta)$ with a prefilter $F_0(s)$. The block $\mathrm{e}^{-s\tau_1}$ simulates the pure delay in continuous networks and the computational delay.

The transfer function of the hold element (not shown in Fig. 1 for space economy) will be denoted as $G_h(s)$. The given solution is valid for any form of modulated impulse.

Let a reference signal $r(t)$ be given as

$$r(t) = \begin{cases} 0, & t < 0 \\ r_0(t), & t \ge 0 \end{cases}$$

where $r_0(t)$ is a function, and let it have the Laplace transform $R(s)$. Assume that the input signal is known in advance over the interval τ, and the input of the sampled-data system is acted upon by "future" values of the signal $r(t)$ with preview τ, i.e., $r(t+\tau)$.

The task of the system is to restore, as close as possible, some linear transformation $\hat{y}(t)$ of the reference signal $r(t)$ given as a unit with stable transfer function $Q(s)$.

In order to restrict the control signal, let us introduce an ideal control signal $\hat{u}(t)$, which is the result of a transformation of the reference signal $r(t)$ by a linear network with stable transfer function $Q_u(s)$ [10-15].

Then, the cost function may include the weighted sum of integral square errors with respect to outputs and cotnrol:

$$\begin{aligned} J &= \int_{-\infty}^{\infty} [y(t)-\hat{y}(t)]^2 + \rho^2 [u(t)-\hat{u}(t)]^2\, dt \\ &= \int_{-\tau}^{\infty} [y(t)-\hat{y}(t)]^2 + \rho^2 [u(t)-\hat{u}(t)]^2\, dt \qquad (3) \end{aligned}$$

where ρ^2 is a nonnegative weighting coefficient. The problem consists in constructing a transfer function of an optimal stable digital controller $C(\zeta)$ which ensures the minimal value of the cost function J.

In [10] a constructive method was proposed to use the Laplace transform for analysis and optimal synthesis of sampled-data systems under deterministic disturbances. But the problem is that the system at hand incorporates a physically non-realizable unit performing the preview of the input signal. Hence, even if the system has zero initial energy, there are non-zero signals in the sampled-data system for $-\tau \le t < 0$, so it is impossible to use the ordinary (unilateral) Laplace transform. Nevertheless, changing the zero point of time, it is possible to construct an equivalent physically realizable system which provides for the same value of the cost function.

Indeed, let $\tilde{t} = t + \sigma T$, with

$$\tau = \sigma T - \theta \qquad (4)$$

where σ is an integer and $0 \le \theta < T$. Then, for $\tilde{t} < 0$ all signals in the system are zero. Moreover, the criterion (3) transforms to the form

$$J = \int_{\theta}^{\infty} \left[y(\tilde{t})-\hat{y}(\tilde{t})\right]^2 + \rho^2\left[u(\tilde{t})-\hat{u}(\tilde{t})\right]^2 d\tilde{t}\,.$$

Since all signals are zero for $\tilde{t} < \theta$, we obtain

$$J = \int_{0}^{\infty} \left[y(\tilde{t})-\hat{y}(\tilde{t})\right]^2 + \rho^2\left[u(\tilde{t})-\hat{u}(\tilde{t})\right]^2 d\tilde{t}\,. \qquad (5)$$

Then, let us transform the system in Fig. 1 using the new independent variable $\tilde{t}$. Since we changed the time zero-point by an integer multiple of the sampling period and the digital system is T-periodic, this operation causes no phase shift of the sampling unit in the new system.

Using (4), we find that the unit Q is acted upon by the signal $r(\tilde{t}-\sigma T)$, while the sampled-data system gets $r(\tilde{t}-\theta)$. These signals can be viewed as a result of the passage of the signal $r(\tilde{t})$ through pure delay networks with transfer functions $\mathrm{e}^{-s\sigma T}$ and $\mathrm{e}^{-s\theta}$, respectively. The block-diagram of an equivalent system is shown in Fig. 2. Here Σ denotes the sampled-data system shown in the dashed box in Fig. 1.

Note that the system in Fig. 2 contains two pure delay units instead of a preview block. All parts are now physically realizable and all signals are zero for $\tilde{t} < 0$. Hereinafter we will work only with the equivalent system shown in Fig. 2. For brevity, we write t instead of $\tilde{t}$ everywhere.

It should be noted that the construction of the equivalent system can also be justified on basis of the bilateral Laplace transform in the corresponding strip of convergence [10].

IV. Equivalent discrete problem

Using Parseval's formula, the cost function (5) can be written as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (Y-\hat{Y})(Y-\hat{Y})^* + \rho^2(U-\hat{U})(U-\hat{U})^*\, ds \quad (6)$$

where $Y(s)$, $U(s)$, $\hat{Y}(s)$, and $\hat{U}(s)$ denote the Laplace images of the corresponding signals. Optionally, frequency-dependent weighting functions can be used in (6) [10].

Using the technique of [10], Laplace transforms of the signals $y(t)$ and $u(t)$ in the equivalent system shown in Fig. 2 can be found as

$$\begin{aligned} Y(s) &= FH\mathrm{e}^{-s\tau_1} G_h D_{F_0R}(s,-\theta)C \\ U(s) &= H\mathrm{e}^{-s\tau_1} G_h D_{F_0R}(s,-\theta)C \end{aligned}$$

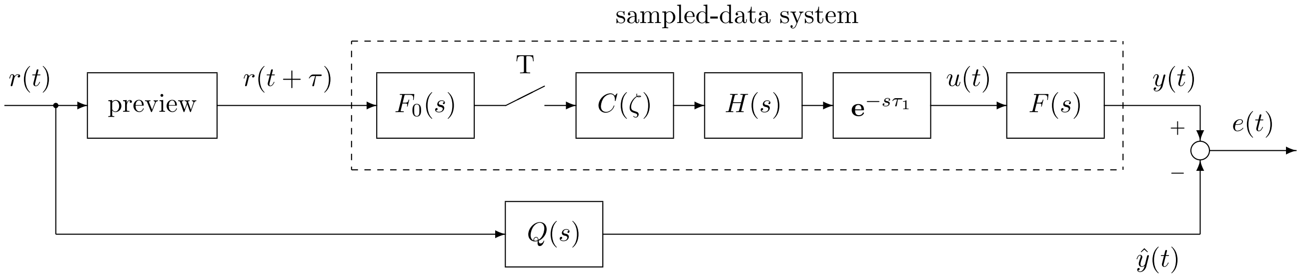

Fig. 1. Sampled-data open-loop tracking system

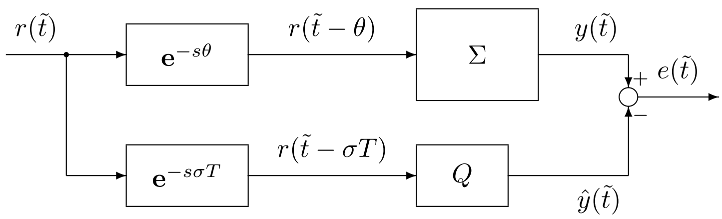

Fig. 2. Block diagram of an equivalent system

Images of the ideal signals are calculated in a classical way as

$$\hat{Y}(s) = QR\mathrm{e}^{-s\sigma T}\,, \qquad \hat{U}(s) = Q_u R\mathrm{e}^{-s\sigma T}\,.$$

Decompose τ and τ_1 as

$$\begin{aligned}\tau &= \gamma T + \psi = \sigma T - \theta\\ \tau_1 &= \gamma_1 T + \psi_1 = \sigma_1 T - \theta_1\end{aligned}$$

where γ, γ_1, σ and σ_1 are integers such that

$$\begin{aligned}0 \le \theta < T\,, \qquad & 0 \le \theta_1 < T\\ 0 \le \psi < T\,, \qquad & 0 \le \psi_1 < T\,.\end{aligned}$$

Note that

$$\sigma = \gamma + \mu\,, \qquad \sigma_1 = \gamma_1 + \mu_1 \tag{7}$$

with the notation

$$\mu = \begin{cases} 0 & , \quad \theta = 0\\ 1 & , \quad \theta \ne 0\end{cases} \qquad \mu_1 = \begin{cases} 0 & , \quad \theta_1 = 0\\ 1 & , \quad \theta_1 \ne 0\,.\end{cases} \tag{8}$$

Then,

$$\begin{aligned}\mathrm{e}^{-s\tau_1} &= \mathrm{e}^{-s\sigma_1 T}\mathrm{e}^{s\theta_1}\\ D_{F_0R}(s,-\theta) &= D_{F_0R}(s,(\gamma-\sigma)T+\psi)\\ &= \mathrm{e}^{(\gamma-\sigma)T}D_{F_0R}(s,\psi)\,.\end{aligned}$$

Hence,

$$\begin{aligned}Y\hat{Y}^* &= \mathrm{e}^{s(\gamma-\sigma_1)T}FH\mathrm{e}^{s\theta_1}R^*Q^*G_hD_{F_0R}(s,\psi)C\\ U\hat{U}^* &= \mathrm{e}^{s(\gamma-\sigma_1)T}H\mathrm{e}^{s\theta_1}R^*Q_u^*G_hD_{F_0R}(s,\psi)C\,.\end{aligned}$$

Using the above relations, after discretization and a passage to the variable ζ [10] we find

$$J = \frac{1}{2\pi j}\oint_\Gamma (ACC^* - BC - B^*C^* + E)\,\frac{d\zeta}{\zeta} \tag{9}$$

where Γ is the unit circle passed anti-clockwise and

$$\begin{aligned}A(\zeta) &= D_{A_0G_hG_h^*}(\zeta,0)D_{F_0R}(\zeta,\psi)D_{F_0R}(\zeta^{-1},\psi)\\ B(\zeta) &= \zeta^{\sigma_1-\gamma}D_{B_0G_h}(\zeta,\theta_1)D_{F_0R}(\zeta,\psi)\\ E(\zeta) &= D_{E_0}(\zeta,0)\\ A_0(s) &= (FF^*+\rho^2)HH^*\\ B_0(s) &= (FQ^*+\rho^2Q_u^*)HR^*\\ E_0(s) &= (QQ^*+\rho^2Q_uQ_u^*)RR^*\,.\end{aligned}$$

In the general case, Q_u and Q may have different poles. Let $d_Q(\zeta)$ and $d_{Q_u}(\zeta)$ be the monic denominators of $D_Q(T,\zeta,0)$ and $D_{Q_u}(T,\zeta,0)$, respectively. Then, denote by $d_{Q_1}(\zeta)$ the least common multiple of $d_Q(\zeta)$ and $d_{Q_u}(\zeta)$. Let also $\delta_1 = \deg\, d_{Q_1}$.

V. Controller design

Introduce the following assumptions:

A1. The functions R, F, H, F_0, Q, and Q_u are free of poles in $\mathbf{Re}\, s > 0$.

A2. The functions F_0R, QR, and Q_uR are strictly proper.

A3. The functions H and FH are proper (or strictly proper).

Notice that from *A2* and *A3* it follows that A_0 is proper, while B_0 and E_0 are strictly proper.

Calculating the discrete Laplace transforms $D_{A_0G_hG_h^*}(T,\zeta,0)$, $D_{F_0R}(T,\zeta,\psi)$, and $D_{B_0G_h}(T,\zeta,\theta_1)$, we can write the functions $A(\zeta)$ and $B(\zeta)$ in the form [10]

$$A(\zeta) = \frac{\alpha_1}{d_{FH}d_{FH}^*}\cdot\frac{\alpha_2}{d_{F_0R}}\cdot\frac{\alpha_2^*}{d_{F_0R}^*} \tag{10}$$

$$B(\zeta) = \frac{\beta\zeta^{\sigma_1-\gamma}}{d_{FH}d_{Q_1R}^*}\cdot\frac{\alpha_2}{d_{F_0R}} \tag{11}$$

where $\alpha_2(\zeta)$ is a polynomial, while $\alpha_1(\zeta)$ and $\beta(\zeta)$ are quasipolynomials. Moreover, since $A_0(s)$ is Hermitian self-conjugated, we have $\alpha_1(\zeta) = \alpha_1(\zeta^{-1})$.

Let us find a stable polynomial $g(\zeta)$ (up to the sign) as a result of the factorization

$$gg^* = \alpha_1\alpha_2\alpha_2^* \,. \tag{12}$$

Let also ν be the minimal nonnegative integer such that $\beta^*\alpha_2^*\zeta^{\gamma-\sigma_1+\nu}$ and $g^*\zeta^\nu$ are polynomials in ζ.

Theorem 1: Let assumptions *A1-A3* hold. Then,

i) the transfer function of the optimal digital filter ensuring the minimum of the criterion (9) is given by

$$C(\zeta) = \frac{n_c}{d_c} = \frac{d_{FHF_0}P}{gd_{Q_1}} \tag{13}$$

where the polynomials $P(\zeta)$ and $\pi(\zeta)$ satisfy the polynomial equation

$$g^*\zeta^\delta P + d_{Q_1R}\pi = \beta^*\alpha_2^*\zeta^{\gamma-\sigma_1+\nu} \tag{14}$$

with π of minimal degree, i.e., deg $\pi < \nu$;

ii) the optimal value of the cost function is

$$J_{opt} = \frac{1}{2\pi j}\oint_\Gamma \left(\frac{\pi\pi^*}{gg^*} + E - \frac{B_1B_1^*}{A_1}\right)\frac{d\zeta}{\zeta} \tag{15}$$

where

$$A_1(\zeta) = D_{A_0G_hG_h^*}(\zeta,0)\,, \quad B_1(\zeta) = D_{B_0G_h}(\zeta,\theta_1)$$

iii) the degrees of the polynomials d_c and n_c are determined by the inequalities

$$\begin{aligned} \deg n_c &\le \chi + \max(0,\gamma-\gamma_1) && (16)\\ \deg d_c &\le \chi\,. && (17)\end{aligned}$$

with $\chi = \delta_{FHF_0R} + \delta_1 - 1$.

The proof of Theorem 1 can be given in analogy to the proof in [16].

VI. Properties of the function $J_{opt}(\tau)$

A. Properties of $J_{opt}(\tau)$ for $\psi = const$

Lemma 1: The following relation holds for any integer $k > 0$:

$$J_{opt}(\tau + kT) \le J_{opt}(\tau)\,.$$

Proof: Let $C_0(\zeta)$ be the transfer function of the optimal digital filter for some fixed τ. Then, as follows from (9), the filter with transfer function ζ^kC_0 provides for the same value of the cost function for the preview interval $\tau + kT$. Using the decomposition

$$\tau = \gamma T + \psi$$

we find that for $\psi =$ const the function $J_{opt}(\gamma T + \psi)$ decreases monotonically as γ increases. ■

B. Properties of $J_{opt}(\tau)$ for $\gamma = const$

Consider relations between the optimal solutions for different preview intervals τ' and τ'' such that

$$\tau' = \gamma T + \psi_1\,, \quad \tau'' = \gamma T + \psi_2\,. \tag{18}$$

Assume that

$$D_{F_0R}(\zeta,\psi_1) = \frac{n_{\psi1}}{d_{F_0R}}\,, \quad D_{F_0R}(\zeta,\psi_2) = \frac{n_{\psi2}}{d_{F_0R}}$$

where $n_{\psi1}(\zeta)$ and $n_{\psi2}(\zeta)$ are polynomials.

Lemma 2: Let the intervals τ' and τ'' satisfy (18) and the polynomial $n_{\psi2}$ be stable. Then,

$$J_{opt}(\gamma T + \psi_2) \le J_{opt}(\gamma T + \psi_1)\,. \tag{19}$$

If the polynomial $n_{\psi1}$ is also stable, equality holds in (19).

Corollary 1: If the numerator of the function $D_{F_0R}(\zeta,\psi)$ is a stable polynomial for all $\psi's$, the function $J_{opt}(\tau)$ is a non-increasing piecewise constant function with possible breaks at the points $\tau_\gamma = \gamma T$.

Proof: Expressions for the coefficients of the functional (9) show that for varying ψ the cost functional depends only on the product $CD_{F_0R}(\zeta,\psi)$.

Let $C_1(\zeta)$ be the optimal controller for the preview interval τ'. Then, it can be easily checked that the controller

$$C_2(\zeta) = \frac{n_{\psi1}}{n_{\psi2}}\,C_1$$

yields

$$C_2D_{F_0R}(\zeta,\psi_2) = C_1D_{F_0R}(\zeta,\psi_1)$$

and, therefore, the same value of the cost function. Moreover, the controller C_1 is admissible, i.e., stable and physically realizable.

If the polynomial $n_{\psi1}$ is also stable, we can prove in a similar way the equality which is inverse to (19).

If the numerator of the function $D_{F_0R}(\zeta,\psi)$ is a stable polynomial for all $\psi's$, then the equality

$$J_{opt}(\gamma T + \psi_1) = J_{opt}(\gamma T + \psi_2)$$

holds for all ψ_1 and ψ_2. Therefore, the function $J(\gamma T + \psi)$ is constant for any fixed γ and $0 \le \psi < T$. ■

C. Breaks of $J_{opt}(\tau)$

As follows from the fomulae for the coefficients A and B in the functional (9), these functions are continuous with respect to ψ for $0 \le \psi < T$. Due to the continuity of the solution to the optimization problem, we find that the function $J_{opt}(\tau) = J_{opt}(\gamma T + \psi)$ is continuous for $\gamma =$ const and $0 \le \psi < T$, i.e., it may have breaks only at the points where $\psi = 0$.

If the pole excess of F_0R is 1, then $D_{F_0R}(\zeta,\psi)$ can have breaks at $\psi = 0$ [10]. Physically, this means that a discontinuous signal acts upon the sampling unit. In this case, the coefficients A and B in the integrand of (9) may change with a jump. Such being the case, the problem becomes non-robust in the sense that the optimal controller and the minimal value of the cost function depend on the fact whether the sampling unit fixes the signal at the moments $kT + 0$ or $kT - 0$.

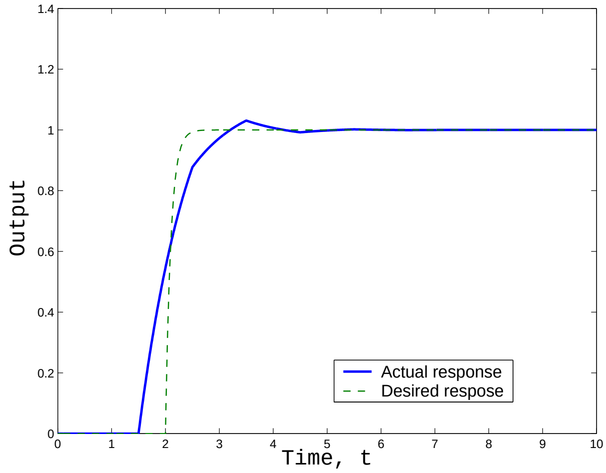

Fig. 3. Transients for Example 1.

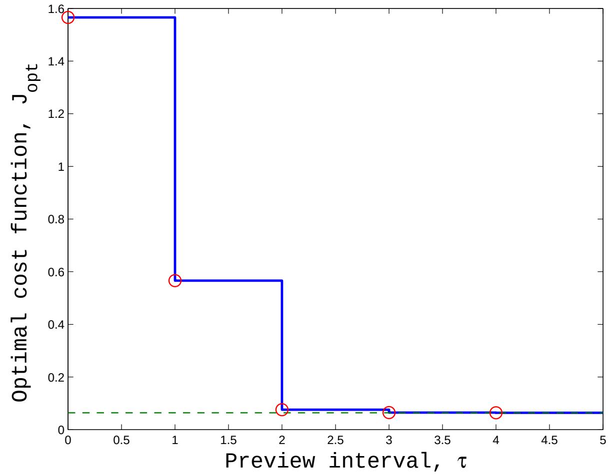

Fig. 4. Curve $J_{opt}(\tau)$ for Example 1.

D. Asymptotic properties of solution as $\tau \to \infty$

Theorem 2: Let $J_{opt}(\gamma, \psi)$ be the minimal value of the cost function (9) for known γ and ψ. Then, there exists the limit $J_\infty = \lim_{\gamma\to\infty} J_{opt}(\gamma, \psi)$, which is independent of ψ. Moreover,

$$J_\infty = \frac{1}{2\pi j} \oint_\Gamma \left(E - \frac{B_1 B_1^*}{A_1} \right) \frac{d\zeta}{\zeta} . \quad (20)$$

The proof can be given in analogy to [16].

A special feature of the problem under consideration consists in the fact that the input signal passes through the sampling unit of the digital filter. Thus, the output of the sampled-data system is influenced by the input values that are fixed by the sampling unit at the moments $t_k = kT$, where k is an integer. Quantization leads to a loss of information, therefore it appears to be impossible to attain zero cost functions in the limit as $\tau \to \infty$ (i.e., when the whole trajectory is known in advance) even when no restrictions are imposed on the control power.

VII. Numerical examples

Example 1. Consider the system shown in Fig. 1 with

$$F(s) = \frac{1}{s+1}, \quad H(s) = F_0(s) = 1$$
$$R(s) = \frac{1}{s}, \quad Q(s) = \frac{1}{0.1s+1}, \quad \tau_1 = 1.5$$
$$T = 1, \quad \tau = 2, \quad \varrho^2 = 0 .$$

Using the proposed technique, we obtain with the help of the *DirectSD* toolbox for Matlab [17]

$$C(\zeta) = \frac{1.388 + 0.08392\zeta - 0.2185\zeta^2 - 7.866 \cdot 10^{-5}\zeta^3}{1 + 0.2534\zeta - 1.151 \cdot 10^{-5}\zeta^2} .$$

This controller ensures $J = 0.0761$. Desired and actual transients are shown in Fig. 3. The curve $J_{opt}(\tau)$ for the system under investigation is shown in Fig. 4. In this case, we have

$$D_{F_0R}(\zeta, -0) = \frac{-\zeta}{\zeta - 1}$$

$$D_{F_0R}(\zeta, \psi) = \frac{-1}{\zeta - 1}, \quad +0 \le \psi < T .$$

Therefore, the conditions of Corollary 1 hold, and $J_{opt} =$ const for $0 \le \psi < T$. Since the pole excess of F_0R is 1, the function $D_{F_0R}(T, \zeta, \psi)$ has a finite break for $\psi = 0$. Hence, the function $J_{opt}(\tau)$ also has finite breaks at the points $\tau = kT$.

As the preview interval τ increases, the curve $J_{opt}(\tau)$ asymptotically approaches the value $J_\infty = 0.06419$ calculated by (20) (dashed line in Fig. 4). Circles denote the values of $J_{opt}(\tau)$ for integer τ (which are multiples of the sampling period).

Notice that for all τ the cost function is nonzero. This is specific for sampled-data systems as distinct from stationary ones and caused by the properties of the sampling and hold process.

Example 2. Consider the system investigated in Example 1 with the input

$$R(s) = \frac{1}{s^2 + s} .$$

The optimal digital filter

$$C(\zeta) = \frac{2.089 - 0.8748\zeta + 0.03911\zeta^2 - 7.501 \cdot 10^{-6}\zeta^3}{1 + 0.2534\zeta - 1.151 \cdot 10^{-5}\zeta^2}$$

gives $J = 0.03915$. Desired and actual transients are shown in Fig. 5. The curve $J_{opt}(\tau)$ for this system is shown in Fig. 6. Now for $\psi < 0.38$ the function $D_{F_0R}(\zeta, \psi)$ has an unstable numerator, for instance,

$$D_{F_0R}(\zeta, 0) = \frac{1.7183\zeta}{\zeta^2 - 3.718\zeta + 2.718}$$

while for $\psi > 0.38$ the zero of the function moves to the stability region. Therefore, as follows from Lemma 2, the function $J_{opt}(\tau)$ decreases for $\psi < 0.38$ and remains constant for $\psi > 0.38$. Since the pole excess of F_0R is 2, the curve $J_{opt}(\tau)$ is continuous.

Circles denote the values of $J_{opt}(\tau)$ for integer τ. As was expected, for a constant ψ the function $J_{opt}(\gamma T + \psi)$

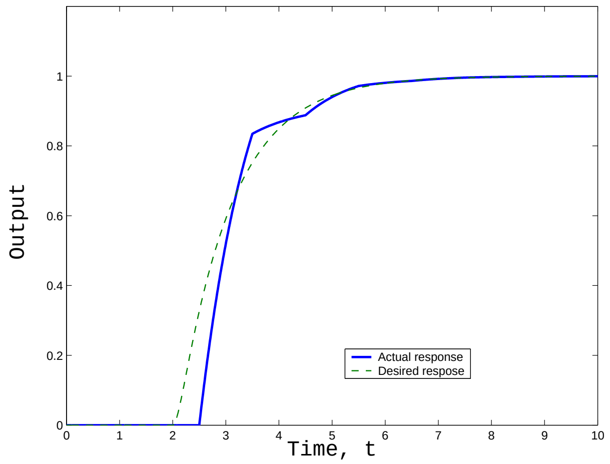

Fig. 5. Transients for Example 2.

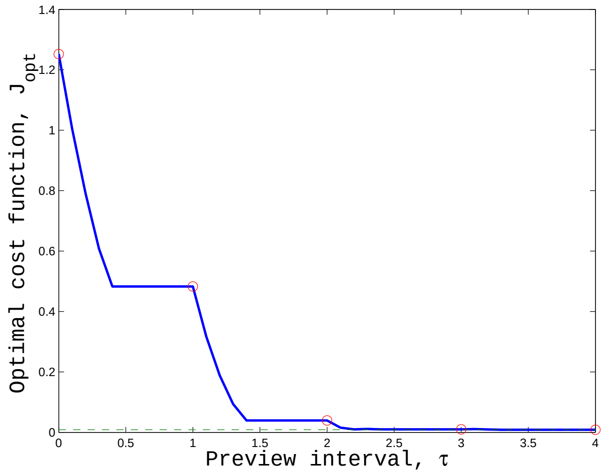

Fig. 6. Curve $J_{opt}(\tau)$ for Example 2.

decreases monotonoulsy and, for any ψ, approaches $J_\infty = 0.008339$ computed by (20) (dashed line in Fig. 6).

VIII. Conclusions

The paper deals with the problem of optimal tracking using SISO sampled-data open-loop filters with preview. A rigorous frequency-domain solution is proposed on basis of the Laplace transform in continuous time. Expressions are given for the direct determination of the degrees of the numerator and denominator of the optimal digital filter using initial data.

It is shown that the properties of the function $J_{opt}(\tau)$ depend on the properties of the real rational function $F_0 R$. In some cases, the curve $J_{opt}(\tau)$ can have finite discontinuities at the points $\tau = kT$.

For $\tau \to \infty$, the lower bound of the optimal cost is obtained. It is demonstrated that, as distinct from the LTI-case, it is impossible to attain zero cost in the limit as $\tau \to \infty$, even if no penalties are imposed on the control power.

References

[1] M. Tomizuka, "Optimal continuous finite preview problem," *IEEE Trans. Autom. Contr*, vol. AC-20, pp. 362–365, 1975.

[2] M.J. Grimble, "LQG-predictive optimal control for adaptive applications," *Automatica*, vol. 26, no. 6, pp. 949–961, 1990.

[3] M.E. Halpern, "Optimal tracking with previewed commands," *IEE Proc. pt D*, vol. 138, no. 3, pp. 237–241, 1991.

[4] M.E. Halpern, "Preview tracking for discrete-time SISO systems," *IEEE Trans. Autom. Contr*, vol. AC-39, no. 3, pp. 589–592, 1994.

[5] M.J. Grimble, *Robust Industrial Control: Optimal Design Approach for Polynomial Systems*, International Series in Systems and Control Engineering. Prentice Hall International (UK) Ltd, Hemel Hempstead, Hertfordshire, 1994.

[6] P.P. Khargonekar and Y. Yamamoto, "Delayed signal reconstruction using sampled-data control," in *Proc. 35th IEEE Conf. Decision Contr*, Kyoto, 1996, pp. 1259–1263.

[7] Y. Yamamoto and S. Hara, "Performance lower bound for a sampled-data signal reconstruction," in *Open Problems in Mathematical Systems and Control Theory*, V. Blondel, E. Sontag, M. Vidyasagar, and J. Willems, Eds., pp. 277–279. Springer, London, 1998.

[8] T. Chen and B.A. Francis, *Optimal sampled-data control systems*, Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[9] T. Hagiwara and M. Araki, "FR-operator approach to the $\mathcal{H}_2$-analysis and synthesis of sampled-data systems," *IEEE Trans. Autom. Contr*, vol. AC-40, no. 8, pp. 1411–1421, 1995.

[10] E.N. Rosenwasser and B.P. Lampe, *Computer Controlled Systems - Analysis and Design with Process-orientated models*, Springer-Verlag, London Berlin Heidelberg, 2000.

[11] E.N. Rosenwasser, K.Yu. Polyakov, and B.P. Lampe, "Frequency domain method for $\mathcal{H}_2-$optimization of time-delayed sampled-data systems," *Automatica*, vol. 33, no. 7, pp. 1387–1392, 1997.

[12] E.N. Rosenwasser, K.Yu. Polyakov, and B.P. Lampe, "Application of Laplace transformation for digital redesign of continuous control systems," *IEEE Trans. Automat. Contr*, vol. 4, no. 4, pp. 883–886, April 1999.

[13] T. Chen and B.A. Francis, "Design of multirate filter banks by H_∞ optimization," *IEEE Trans. Signal Processing*, vol. SP-43, pp. 2822–2830, 1995.

[14] K.Yu. Polyakov, "Polynomial design of optimal sampled-data tracking systems. I: Quadratic optimization," *Automation and Remote Control*, vol. 62, no. 2, pp. 304–316, Feb 2001.

[15] K.Yu. Polyakov, "Polynomial design of optimal sampled-data tracking systems. II: Robust optimization," *Automation and Remote Control*, vol. 62, no. 3, pp. 430–442, Mar 2001.

[16] K.Yu. Polyakov, E.N. Rosenwasser, and B.P. Lampe, "Optimal reconstruction of stochastic signals with preview using sampled-data control," *ASME Journal of Dynamic Systems, Measurement and Control*, June 2003, (acc.).

[17] K.Yu. Polyakov, E.N. Rosenwasser, and B.P. Lampe, "DirectSD - a toolbox for direct design of sampled-data systems," in *Proc. IEEE Intern. Symp. CACSD'99*, Kohala Coast, Island of Hawai'i, Hawai'i, USA, 1999, pp. 357–362.