Suboptimal Control of a One-Dimensional Nonlinear Heat Equation Using POD and q - D Techniques

Radhakant Padhi[1], Ming Xin[2] and S. N. Balakrishnan[3]

Department of Mechanical and Aerospace Engineering and Engineering Mechanics
University of Missouri - Rolla, MO 65409, USA

[1]padhi@umr.edu, [2]xin@umr.edu, [3]bala@umr.edu

Abstract

A new technique is presented in this paper for the suboptimal control design of distributed parameter systems in general. This technique is used to synthesize the controller for a nonlinear heat diffusion problem. The method of proper orthogonal decomposition is used for model reduction of the distributed parameter systems. A suboptimal control is then designed using the recently emerging $q-D$ technique for lumped parameter systems. This control for the reduced order system is then mapped back to the distributed domain using the same basis functions, leading to distributed controls. Simulation results indicate that the method holds promise as a control design technique for nonlinear distributed parameter systems.

1. Introduction

Distributed Parameter Systems (DPS) are governed by a set of Partial Differential Equations (PDEs). There exist theoretical methods for the control of distributed parameter systems in the infinite dimensional operator theory framework [Curtain]. While there are many advantages, these approaches are mainly confined to the linear systems, besides having the usual difficulties in control implementation through an infinite dimensional operator.

An engineering approach to deal with the infinite dimensional systems is to have a finite-dimensional approximation of the system using a set of orthogonal basis functions in a Galerkin projection [Ravindran]. However, if arbitrary basis functions are used, it leads to a higher order system of ordinary differential equations (ODEs), which is difficult to handle. In recent literature attention has beeen increasingly focused on the technique of *Proper Orthogonal Decomposition (POD)* [Burns, Ravindran]. This technique essentially helps design a set of problem dependent basis functions, which could lead to a low-order ODE representation of the DPS with sufficient accuracy. This model can then be used for control design using the available and evolving techniques of lumped parameter systems.

Many difficult real-life optimal control problems in the ODE domain can be formulated in the framework of dynamic programming [Bryson]. It attempts to solve for a feedback form of optimal control by producing a family of optimal paths, or what is known as the "*field of extremals*". However, in the process it leads to the Hamilton-Bellman-Jacobi (HJB) equation, which requires a prohibitive amount of computation and storage requirements. Hence even though the formulation is nice, it is not feasible to obtain such a solution for complex problems.

Towards designing an alternate computational tool for finding a feedback form of the optimal control solution, we have attempted to propose a nonlinear suboptimal control technique, namely the $q-D$ method [Xin], which finds an approximate solution to the HJB equation. By introducing an intermediate variable q, the optimal cost is expanded as a power series in terms of q. The HJB equation is then reduced to a set of recursive algebraic Lyapunov equations, which are easier to solve. By tuning the parameters in the perturbation terms of the formulation, one can modulate the system performance. In this paper we combine the ideas of proper orthogonal decomposition and $q-D$ method to come up with an optimal controller of a nonlinear heat conduction problem.

In this paper, first we propose to carry out the proper orthogonal decomposition of the distributed parameter system to design a set of problem oriented basis functions that leads to a low order finite dimensional representation. Since our aim is to design feedback control, we essentially use the same basis functions to decompose the associated control, under the justifiable assumption that the control function can be spanned by the basis functions of the states. We also derive a compatible performance index once again using the same basis functions. In the process we essentially formulate an analogous finite dimensional optimal control problem in the time domain only. After synthesizing the control in the time domain using $q-D$ technique, we generate the control function in the spatial domain by using the same basis functions. We have presented numerical simulation results for one-dimensional linear and nonlinear heat equation problems, with an infinite time optimal control formulation.

2. Proper Orthogonal Decomposition: A Review

In this section we briefly summarize the process of proper orthogonal decomposition. An interested reader can refer to [Burns, Ravindran] for further readings.

Let $\{U_i(y) : 1 \le i \le N,\ \ y \in \Omega\}$ be a set of N *snapshot* solutions (observations) of some physical process over the domain Ω at arbitrary instants of time. The goal of the POD technique is to design a basis function Φ that has the largest mean square projection on the snapshots. As a standard notation the L^2 inner product is defined as $\langle \Phi, \Psi \rangle = \int_\Omega \Phi \Psi \, dy$. We seek $\Phi = \sum_{i=1}^{N} w_i U_i$ where the coefficients w_i are to be determined such that Φ maximizes $(1/N)\left(\sum_{i=1}^{N} \left| \langle U_i, \Phi \rangle \right|^2 / \langle \Phi, \Phi \rangle \right)$. After some algebra it can be shown [Ravindran] that this leads to

$$
\begin{aligned}
& CW = s\,W \\
& C = \left[c_{ij} \right],\ c_{ij} = \frac{1}{N} \int_\Omega U_i(y)\, U_j(y)\, dy
\end{aligned}
\tag{1}
$$

In Eq.(1) $s \in \mathbb{R}$ and $W = [w_1 \ \ w_2 \ \ \cdots \ \ w_N]^T$. So, we have a standard matrix eigenvalue and eigenvector problem to find W.

Matrix C has N *non-negative real* eigenvalues and N orthogonal eigenvectors. Sorting the eigenvectors in descending order, we can write $s_1 \ge s_2 \ge \cdots \ge s_N \ge 0$. Let the corresponding eigenvectors be $W^1 = \left[w_1^1 \ \cdots \ w_N^1\right]^T$, $W^2 = \left[w_1^2 \ \cdots \ w_N^2\right]^T \cdots$ $W^N = \left[w_1^N \ \cdots \ w_N^N\right]^T$. It can be noted that the eigenspectrum can be truncated judiciously such that $\sum_{j=1}^{\tilde{N}} l_j \approx \sum_{j=1}^{N} l_j$. In that case we obtain $\tilde{N}$ orthonormal eigenfunctions as

$$\Phi_1 = \sum_{i=1}^{N} w_i^1 U_i(y), \ \cdots, \ \Phi_{\tilde{N}} = \sum_{i=1}^{N} w_i^{\tilde{N}} U_i(y) \tag{2}$$

The $\|\Phi\| = 1$ condition is met when we normalize W^i s by forcing

$$\left\langle W^j, W^j \right\rangle = 1/\left(N l_j\right) \tag{3}$$

3. Finite Dimensional Approximations

We consider a nonlinear distributed parameter system given by

$$\frac{\partial x}{\partial t} = f\left(x, \ \partial x / \partial y, \ \ \partial x / \partial y^2, \ ..., u\right) \tag{4}$$

with appropriate boundary conditions. The problem is to find the controller that minimizes the performance index

$$J = \int_0^\infty \int_0^L L(x, \ u)\, dy dt \tag{5}$$

where the state x and control u are functions of time t and y is the spatial variable such that $0 \le y \le L$. The process of getting the snapshot solutions for our example problem will be discussed later in Section 5.3. With the snapshot solutions we design the POD basis functions following the idea from Section 2. After obtaining the basis functions, we propose to write

$$x = \sum_{j=1}^{\tilde{N}} \hat{x}_j(t) \cdot \Phi_j(y), \ \ u = \sum_{j=1}^{\tilde{N}} \hat{u}_j(t) \cdot \Phi_j(y) \tag{6}$$

One may notice that we have assumed the same basis functions for x and u. In other words, we assume that the basis functions for the state are capable of representing the control as well. This is because our final aim is to design a *state feedback controller*. Substituting Eq.(6) in Eq.(4) and taking the inner product of this equation on a specific basis function Φ_i we can write

$$\dot{\hat{x}}_i = \hat{F}_i\left(\hat{x}_j, \hat{u}_j, \ \ j = 1, 2, \ldots, \tilde{N}\right) \tag{7}$$

By the definition of inner product all functionality dependence on y is now absorbed in the integrals. Collecting all equations for $i = 1, 2, \cdots, \tilde{N}$ we can write a $\tilde{N}$ dimensional lumped model for the system as

$$\dot{\hat{X}} = \hat{F}\left(\hat{X}, \hat{U}\right) \tag{8}$$

Similarly, we can substitute for x and u from Eq.(6) in the expression for the performance index in Eq.(5) to obtain

$$J = \int_0^\infty \hat{L}\left(\hat{X}, \ \hat{U}\right) dt \tag{9}$$

Eq.(8-9) formulate an analogous optimal control problem in the time domain. We point out that the boundary conditions of the PDE are absorbed in Eq.(8).

4. q - D Suboptimal Control Technique

In this paper we restrict ourselves to the state feedback control problem for the class of nonlinear time-invariant systems described by

$$\dot{\hat{X}} = f(\hat{X}) + B\hat{U} \tag{10}$$

where $\hat{X} \in R^n$, $\hat{U} \in R^m$, $f(\hat{X})$ is continuously differentiable in $\hat{X}$ and B is a constant matrix; The condition $f(0) = 0$ is assumed in order to have the system at equilibrium when it is at the origin. The objective is to find a controller that minimizes the quadratic cost function, J given by

$$J = \frac{1}{2}\int_0^\infty [\hat{X}^T Q\hat{X} + \hat{U}^T R\hat{U}]dt \tag{11}$$

where $Q \in R^{n\times n}, R \in R^{m\times m}$ are assumed to be positive semi-definite and positive definite matrices respectively. To ensure that the control problem is well-posed, we assume that a solution to the optimal control problem in Eq.(10-11) exists. The optimal solution of the infinite-horizon nonlinear regulator problem can be obtained by solving the following HJB equation [Bryson] given by

$$\frac{\partial V^T}{\partial \hat{X}} f(\hat{X}) - \frac{1}{2}\frac{\partial V^T}{\partial \hat{X}} BR^{-1}B^T \frac{\partial V}{\partial \hat{X}} + \frac{1}{2}\hat{X}^T Q\hat{X} = 0 \tag{12}$$

where $$V(\hat{X}) = \min_{U} \frac{1}{2}\int_0^\infty (\hat{X}^T Q\hat{X} + \hat{U}^T R\hat{U})dt \tag{13}$$

It is assumed that $V(\hat{X})$ is continuously differentiable and $V(\hat{X}) > 0$ and $V(0) = 0$. The necessary condition for optimality leads to

$$\hat{U} = -R^{-1}B^T\left(\partial V / \partial \hat{X}\right) \tag{14}$$

However, it is well known that Eq.(12) is extremely difficult to solve in general, rendering optimal control techniques of limited use for nonlinear systems. In this paper, we propose a suboptimal control design technique to solve it approximately. Consider perturbations added to the cost function:

$$J = \frac{1}{2}\int_0^\infty [\hat{X}^T (Q + \sum_{n=0}^{\infty} D_n q^n)\hat{X} + \hat{U}^T R\hat{U}]dt \tag{15}$$

where q and D_i are chosen such that $\left\|\sum_{i=1}^{\infty} D_i q^i\right\|_2$ is small compared to $\|Q\|_2$. We can write Eq.(10) as

$$\dot{\hat{X}} = f(\hat{X}) + B\hat{U} = A_0\hat{X} + q(\frac{A(\hat{X})}{q})\hat{X} + B\hat{U} \tag{16}$$

where A_0 is a constant coefficient matrix such that (A_0, B) is a stabilizable pair and $[A_0 + A(\hat{X}), B]$ is point-wise controllable. Defining $l \triangleq \left(\partial V / \partial \hat{X}\right)$ and using it in Eq.(12), we have

$$l^T f(\hat{X}) - \frac{1}{2} l^T BR^{-1}B^T l + \frac{1}{2}\hat{X}^T (Q + \sum_{n=0}^{\infty} D_n q^n)\hat{X} = 0 \quad (17)$$

Next, we assume a power series expansion of l as

$$l = \sum_{i=0}^{\infty} T_i(\hat{X}) q^i \hat{X} \quad (18)$$

where T_i matrices are assumed to be symmetric. Substituting Eq.(18) in Eq.(17) and equating the coefficients of powers of q to zero, we get:

$$T_0 A_0 + A_0^T T_0 - T_0 BR^{-1}B^T T_0 + Q = 0 \quad (19)$$

$$T_1(A_0 - BR^{-1}B^T T_0) + (A_0^T - T_0 BR^{-1}B^T)T_1 = -\frac{T_0 A(\hat{X})}{q} - \frac{A^T(\hat{X})T_0}{q} - D_1 \quad (20)$$

$$\vdots$$

$$\begin{aligned} &T_n(A_0 - BR^{-1}B^T T_0) + (A_0^T - T_0 BR^{-1}B^T)T_n \\ &\quad = -\frac{T_{n-1}A(\hat{X})}{q} - \frac{A^T(\hat{X})T_{n-1}}{q} + \sum_{j=1}^{n-1} T_j BR^{-1}B^T T_{n-j} - D_n \end{aligned} \quad (21)$$

Then the expression for control can be obtained in terms of a power series for l as

$$\hat{U} = -R^{-1}B^T l = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(\hat{X}) q^i \hat{X} \quad (22)$$

It is easy to find that Eq.(19) is an algebraic Riccati equation. The rest are Lyapunov equations which are linear in terms of T_i. The algorithm without D_i term is called the q approximation. The algorithm in [Wernli] would result in the q approximation. One of the problems with q approximation is that large state values may give rise to large control. In order to deal with this problem, we construct the following expressions:

$$D_1 = k_1 e^{-l_1 t}[-\frac{T_0 A(\hat{X})}{q} - \frac{A^T(\hat{X})T_0}{q}] \quad (23)$$

$$D_2 = k_2 e^{-l_2 t}[-\frac{T_1 A(\hat{X})}{q} - \frac{A^T(\hat{X})T_1}{q} + T_1 BR^{-1}B^T T_1] \quad (24)$$

$$\vdots$$

$$D_n = k_n e^{-l_n t}[-\frac{T_{n-1}A(\hat{X})}{q} - \frac{A^T(\hat{X})T_{n-1}}{q} + \sum_{j=1}^{n-1} T_j BR^{-1}B^T T_{n-j}] \quad (25)$$

where k_i and $l_i > 0,\ i = 1, \cdots n$ are adjustable design parameters. The idea in constructing D_i in this manner is because otherwise large control results from the term $A(\hat{X})$ on the right hand side of Eq.(20-21), if $\hat{X}$ is large. So we choose D_i such that

$$\begin{aligned} &-\frac{T_{i-1}A(\hat{X})}{q} - \frac{A^T(\hat{X})T_{i-1}}{q} + \sum_{j=1}^{i-1} T_j BR^{-1}B^T T_{i-j} - D_i \\ &\quad = e_i(t)\,[-\frac{T_{i-1}A(\hat{X})}{q} - \frac{A^T(\hat{X})T_{i-1}}{q} + \sum_{j=1}^{i-1} T_j BR^{-1}B^T T_{i-j}] \end{aligned} \quad (26)$$

where $e_i = 1 - k_i e^{-l_i t}$ is a small number chosen to suppress the large control. It also leads to $D_i \to 0$ as t increases, meaning that the steady state cost function goes back to the original cost function as we do not want to change the original cost function too much.

Remark-1: q is just an intermediate variable. It turns out to be canceled in the final control expression Eq.(22) [Xin].

Remark-2: Solution of Eq.(19-21) is carried out offline from top to bottom. Once $T_i (i = 0, \cdots n)$ is known, T_{i+1} can be solved by substituting T_i into the equation for T_{i+1}. Eq.(19) is a standard algebraic Riccati equation, where as Eq.(20-21) are linear equations in terms of $T_1, \cdots, T_n$ with constant coefficients $(A_0 - BR^{-1}B^T T_0)$ and $(A_0^T - T_0 BR^{-1}B^T)$. After some algebra, these equations can be rearranged as: $\hat{A}_0 T_i = Q_i(\hat{X}, q, t)$ and $T_i = \hat{A}_0^{-1} Q_i(\hat{X}, q, t)$ where $\hat{A}_0$ is a constant matrix. So essentially closed-form solutions for $T_2, \cdots, T_n$ are obtained with just one matrix inverse operation. Therefore if we take a finite number of terms in Eq.(22), we would get a closed-form expression for the optimal controller.

5. A Nonlinear One-Dimensional Heat Equation

In this section we consider a one-dimensional nonlinear heat diffusion problem. Essentially we formulate the problem as a regulator problem and solve it using the techniques discussed in Sections 2-4.

5.1 Problem Description

The dynamics of the heat conduction problem given by:

$$\partial x / \partial t = \partial^2 x / \partial y^2 - x^3 + u \quad (27)$$

where $x(t, y)$ represents the temperature profile at time $t \in [0, \infty)$ and spatial location $y \in [0, L]$. $u(t, y)$ is the associated control. We consider the infinite time quadratic regulator problem, for which the goal is to drive x and u to zero in the spatial domain considered by minimizing the performance index

$$J = \frac{1}{2} \int_0^{t_f \to \infty} \int_0^L \left(q x^2 + r u^2\right) dy dt \quad (28)$$

where $q, r \in \mathbb{R}^+$ are the weighting factors.

Boundary and Initial Conditions:

We assume the boundary conditions to be $\partial x / \partial y(t, 0) = \partial x / \partial y(t, L) = 0$ (*i.e.* insulation at both ends) and the initial condition can be *any* profile from the *domain of interest* which is described next.

5.2 Domain of interest and state profile generation

We assume an envelope profile

$$f_{env}(y) = a + A Cos\left(-p + (2p y / L)\right) \quad (29)$$

and $S_i \triangleq \{x : \|x\| \le \| f_{env} \|,\ \|x''\| \le \| f''_{env} \|\}$, with $x'(t, 0) = x'(t, L) = 0$ as the domain of interest. The conditions on x ensure that the profiles are smooth and they satisfy the boundary conditions. For our numerical experiments, we choose $a = A = 0.25$. For the envelope profile we get

$$\| f_{env} \|^2 = \left(a^2 + A^2 / 2\right) L, \quad \| f''_{env} \|^2 = A^2 p^4 (2 / L)^3 \quad (30)$$

After fixing $0 \le C_i \le 1$, we assume

$$\|x\|_{\max}^2 = C_i \|f_{env}\|^2, \ \|x''\|_{\max}^2 = \|f''_{env}\|^2 \tag{31}$$

and consider a Fourier cosine series expansion for $x(y)$ as

$$x = a_0 + \sum_{n=1}^{N_f} a_n Cos(np\, y/L) \tag{32}$$

where N_f is a large number. After some algebra, we observe that

$$\begin{aligned} &\frac{L}{2}\left(2a_0^2 + \sum_{n=1}^{N_f} a_n^2\right) \le \quad C_i\left(a^2 + A^2/2\right)L \\ &\frac{L}{2}\left(\sum_{n=1}^{N_f} n^4 a_n^2\right)\left(\frac{p}{L}\right)^4 \le A^2 p^4 \left(2/L\right)^3 \end{aligned} \tag{33}$$

So we select *random* coefficients $a_n, \quad n = 0,1,\ldots,N_f$ to satisfy the inequalities of Eq.(33) and generate a state profile using Eq.(32).

5.3 Snapshot solution generation

To generate the snapshot solutions, we first fix C_i, $0 \le C_i \le 1$ and generate a random initial state profile $x(0, y)$. Then we generate a random control profile as well, similar to the state profile generation. This is done under the assumption that the controller will satisfy $\|u\| \le \|x(0,y)\|$ and $\|u''\| \le \|x''(0,y)\|$. Holding the control as constant, we then simulate the original system in Eq.(27) and randomly collect some profiles at arbitrary instants of time to form the snapshot solutions. We repeat the steps a number of times and to collect some snapshot solutions each time, till a large number of snapshots is collected to properly design the basis functions.

5.4 Finite dimension approximations

First the snapshot solutions are generated and POD basis functions are designed. Substituting Eq.(6) in Eq.(27), taking the inner product with Φ_i we get:

$$\dot{\hat{x}}_i = \sum_{j=1}^{\tilde{N}} \left\langle \Phi_j'', \Phi_i \right\rangle \hat{x}_i - \int_0^L \left(\sum_{j=1}^{\tilde{N}} \hat{x}_j \Phi_j\right)^3 \Phi_i \, dy + \hat{u}_i \tag{34}$$

Using the boundary conditions after some algebra, it leads to:

$$\dot{\hat{X}} = A\hat{X} + f^{nl}\left(\hat{X}\right) + B\hat{U} \tag{35}$$

where,

$$\begin{aligned} &A \equiv \left[a_{ij}\right], \ a_{ij} = -\left\langle \Phi_i', \Phi_j' \right\rangle, \quad B \equiv I \\ &f_i^{nl}\left(\hat{X}\right) \equiv -\int_0^L \left(\sum_{j=1}^{\tilde{N}} \hat{x}_j \Phi_j\right)^3 \Phi_i \, dy \end{aligned} \tag{36}$$

$f^{nl}\left(\hat{X}\right)$ is a nonlinear function that comes from the nonlinear term in Eq.(27). For the performance index, we observe:

$$\begin{aligned} &q\left\langle x, x\right\rangle = \hat{X}^T Q \hat{X}, \ r\left\langle u, u\right\rangle = \hat{U}^T R \hat{U} \\ &\text{where } Q = qI, \ R = rI \end{aligned} \tag{37}$$

Using Eq.(37), the performance index in Eq.(28) can be written as

$$J = \frac{1}{2}\int_0^\infty \left(\hat{X}^T Q \hat{X} + \hat{U}^T R \hat{U}\right) dt \tag{38}$$

5.5 q - D Suboptimal control solution

Eq.(35) and Eq.(38) pose a standard optimal control problem that we can solve using the $q - D$ method. First we write the nonlinear term in Eq. (35) in the form

$$\dot{\hat{X}} = A_0 \hat{X} + \left[q\left(\frac{A(\hat{X})}{q}\right)\right]\hat{X} + B\hat{U} \tag{39}$$

The suboptimal feedback controller becomes

$$\hat{U} = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(\hat{X}, q) q^i \hat{X} \tag{40}$$

where T_i is obtained by solving Eq.(19-21) recursively. We only pick the first three terms in Eq.(22), which has been found to be good enough approximations in this problem.

5.6 Numerical results

For our numerical experiments we chose $q = r = 1$, $L = 4$. After some tuning process, we have chosen $k_1 = k_2 = 0.96$ and $l_1 = l_2 = 40$. The weighting matrices Q, R relate to q, r respectively and they are chosen as $Q = R = I_5$. For implementing the control we assumed a control update scheme with $\Delta t = 0.1$. In the finite difference scheme for generating the snapshot solutions we assumed $\Delta t = 0.002, \quad \Delta y = 0.1$. However for simulating the system after control synthesis, we used $\Delta t = 0.001, \quad \Delta y = 0.05$.

The choice of two different sets of values for $\Delta t, \Delta y$ was to emphasize the point that the control synthesis methodology presented is independent of the grid size. This was also to verify that that the results are not bad because of the spillover effects, by assuming a particular grid size for generating the snapshot solutions and hence the basis functions. However to compute the values of the basis functions at a location other than where it was constructed, we opted for an interpolation scheme based on the Fourier cosine series having the same number of terms as the number of points for which the function values exist.

Both the state and control over the entire spatial domain should proceed towards zero as time progresses for this regulator problem. Even though we have presented simulation results only for 8 Sec, essentially it can be continued as long as one wishes. We have chosen some test case initial profiles for the states (from the domain of interest described in Subsection 5.2) and let the simulations proceed by applying the designed control.

The first initial test profile chosen was of parabolic nature given by $x_0(y) = 0.2(1 - 2y/L)^2$. Figures 1 and 2 correspond to the state and control histories for this case. It is clear from the figure that both state and control are driven to zero with time, which was the design objective. Moreover, it is also clear that both state and control vary smoothly, which is an additional goal of any control related to distributed parameter systems. Figures 3 and 4 correspond to similar results for state and control histories for another initial state profile (a sinusoidal function) given by $x_0(y) = 0.2 + 0.2\cos(-p - 2p\, y/L)$. Figures 5 and 6 correspond to similar results for state and control histories for another initial state profile (a constant function) given by $x_0(y) = 0.2$.

Since there are an infinite number of possibilities of such functions describing possible initial conditions. However, we have run the program for a large number of test cases, each time generating a different random initial profile as described in Subsection 5.2 and have observed similar good results each time. Figures 7 and 8 correspond to the state and control histories from such a random initial state. It is clear the controller was successful in driving the state to zero while its own value goes to zero simultaneously.

6. Conclusions

In this paper a systematic computational tool for the optimal control synthesis of distributed parameter systems is presented. Using the concept of POD a low-order lumped model representation of the infinite dimensional system was developed. This low dimensional ODE model was used to synthesize a suboptimal control following the philosophy of $q-D$ technique, which leads to a control solution in a state feedback sense. We have synthesized the optimal control for a one-dimensional nonlinear conduction problem. Simulations show promising results for all test cases, which validate the technique presented. Further more, since control calculations are not computationally intensive, this methodology can be implemented on-line.

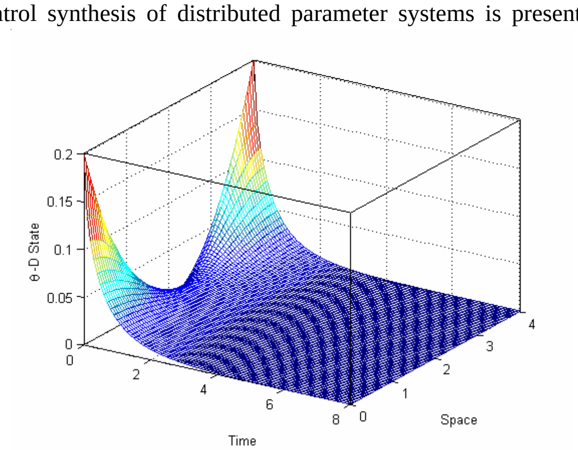

Figure 1: State history from a parabolic initial condition

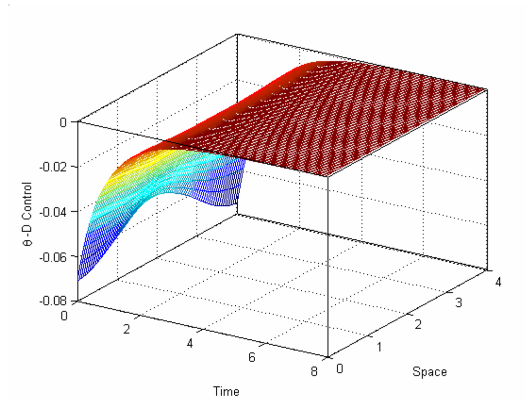

Figure 2: Associated control history for the state with parabolic initial condition

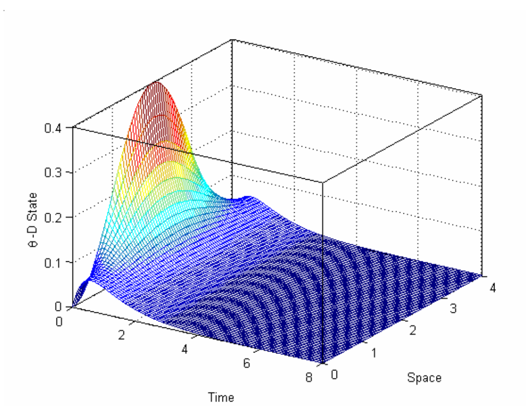

Figure 3: State history from a sinusoidal initial condition

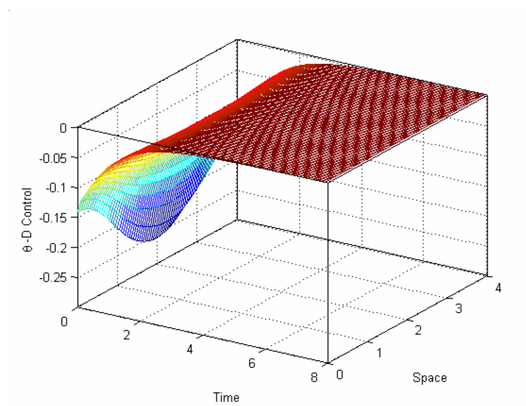

Figure 4: Associated control history for the state with sinusoidal initial condition

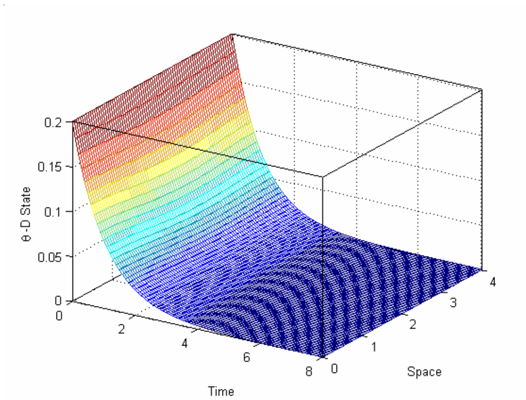

Figure 5: State history from a constant initial condition

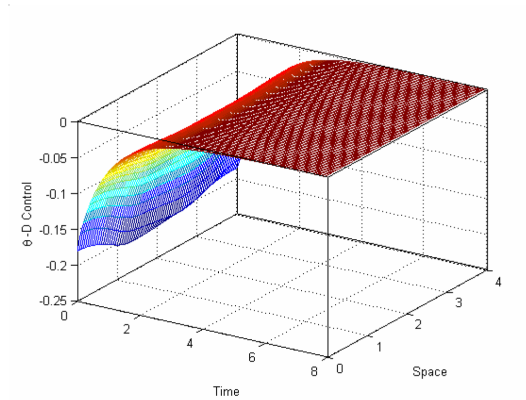

Figure 6: Associated control history for the state with constant initial condition

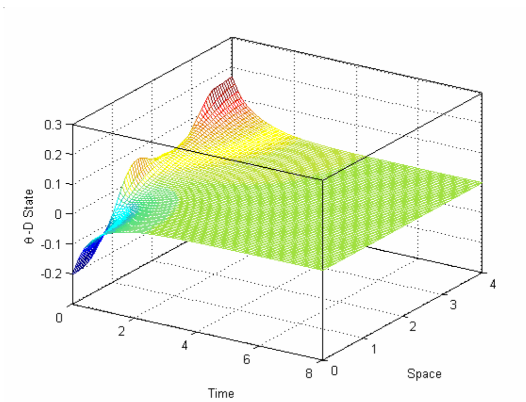

Figure 7: State history from a random initial condition

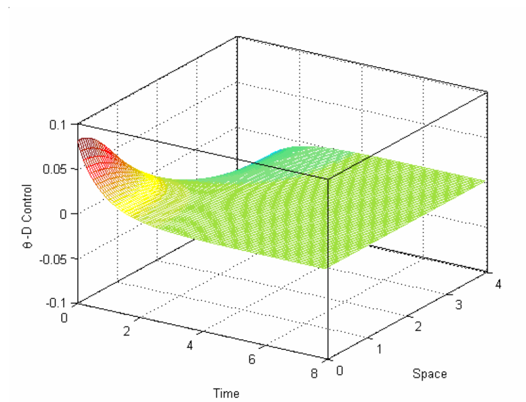

Figure 8: Associated control history for the state with random initial condition

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