LQG tracking problem and its tracking robustness

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***Abstract*— This paper deals with the design of LQG optimal control system for tracking problem guaranteed tracking robustness. Presented approach is based on alternative control theory and using non additive optimality criteria. The approach leads through augmenting given controlled subsystem by a model of classifier and evaluating its state in optimality criterion. The definition of correspondent model of classifier is discussed in the paper. Then the robustness control system is obtained using standard LQG design method. Presented method allows to design robust control system for all types of measurement in systematic way. Shown method is illustrated by example.**

I. Introduction

The tracking problem is one of the most common and important issues in designing a control system. Therefore the problem is discussed in literature frequently [1], [2], [3], [4], [5], for example.

When we study alternative control theory [6], which accepts each control system as autonomous system composed from given controlled subsystem and a controller mutually connected via information relation only and they are not influenced by the environment, then the controlled subsystem is considered as a plant comprising all surroundings relevant to the given problem.

Those facts together with application of Kalman's theorems [7], [8] make possible to represent each controlled subsystem in structure with obviously required parts properties [9]. Then the "observer–based" approach [2] enable to use for formulated tracking problem LQG optimization. It is known that optimal LQG controller does not guarantee robustness of tracking in the case when some states of the controlled system are available to the controller through by measurement. Conception of such robust controller design can be found in [10], for example. Thereat is shown approach using a subsystem whose state is evaluated in optimality criterion. This approach is possible to interpret in context of the alternative control theory as using non additive optimality criteria [11], [6].

In this paper the idea is generalized and applied to linear tracking problem. Given controlled system is augmented by a model of classifier. The model is defined as the controllable system with dynamic equals the dynamic of set of assumed exo-disturbance. It is in correspondence of internal model principle [12], [13].

Here presented method is systematic approach to the problem and covers all altogether heuristic methods presented in literature. Further much part of paper deals with discussion about non additive optimality criteria.

The second section introduces the theory of non additive optimality criteria with context of standard tools of LQG optimization. Then the third section show application of it to the design of robust optimal controller for linear tracking problem and the fourth section illustrates obtained result by simple examples.

II. LQG and non additive optimality criteria

Solved optimization problems are obviously focussed to additive optimality criteria only; for example LQ or LQG optimization. Formulation of the problem with additive criterion has obviously defined

$$\dot{s} = f(s,u,t); \qquad s(t_0) = s_0 \tag{1}$$

$$J = \int_{t_0}^{t_e} l\left(s(t), u(t), t\right)\ \mathrm{d}t \tag{2}$$

where s is controlled subsystem state, u is its control input and $l(s(t), u(t), t) \in \mathrm{R}^+$ is additive lost function. When we leave field of additive criteria, it is possible to generalize optimality criterion to the form

$$J = \int_{t_0}^{t_e} l\left(s_{[t_0,t_e]}, u_{[t_0,t_e]}, t\right)\ \mathrm{d}t \tag{3}$$

where $s[t_0, t_e]$ and $u[t_0, t_e]$ are segments of state and control trajectory at time control interval $\langle t_0, t_e\rangle$. Lost function $l\left(s_{[t_0,t_e]}, u_{[t_0,t_e]}, t\right)$ can be assumed as model of classifier of whole trajectory of control system at time t. Further the lost function can be depend by time t. This class of optimality criterion is very few discussed in literature.

Methods of LQG optimization is elaborated in detail and effective design methods exist. Hence it would be profitable to extent the theory by non additive optimality criteria. Here used method follows form papers [11], [6]. It is based on finding suitable transformation to correspondent problem with additive optimality criterion.

In the state space the transformation is based by definition of *model of classifier*. Then the problem can be formulated in the form

$$\begin{aligned} \dot{s} &= f(s,u,t); & s(t_0) &= s_0 \quad (4)\\ \dot{s}_H &= f_H(s_H, s, u, t); & s_H(t_0) &= s_{H0} \quad (5) \end{aligned}$$

$$J_H = \int_{t_0}^{t_e} l_H\left(s_H(t), s(t), u(t), t\right)\ \mathrm{d}t. \tag{6}$$

where s is state of controlled subsystem, s_H is state of classifier and $l_H(s_H(t), s(t), u(t), t)$ is additive lost function evaluated state of classifier, as well.

It can be proofed [11], [6] that if for each $s(t_0)$ exists such $s_H(t_0)$ for which it holds

$$\int_{t_0}^{t_e} l\left(s_{[t_0,t_e]}, u_{[t_0,t_e]}, t\right) \mathrm{d}t = \int_{t_0}^{t_e} l_H\left(s_H(t), s(t), u(t), t\right) \mathrm{d}t \quad (7)$$

for arbitrary $u_{[t_0,t_e]}$ then, it is possible to call s_H as state of model of classifier and to consider J_H as expression of non additive optimality criterion in state space form.

According to alternative control theory [14] it suitable to consider model of classifier as a part of controller. Then the classifier represents given surrounding of given plant and therefore it is included to the controlled subsystem description. Thus augment optimization problem can be solved by standard tools of LQG optimization and obtained feedback controller produces control action u based on state of controlled subsystem s and state of model of classifier s_H. Controller with this dynamic compensates exo-disturbances with such dynamic. This fact allows by appropriate choosing of model of classifier (5) to design LQG optimal controller guaranteed robustness of tracking toward set of given disturbance signal respectively in case of state feedback controller.

Using previous discussion, the problem of design of control system guaranteed tracking robustness can be reformulated to the problem of model of classifier definition and then the LQG optimization is standard problem.

The definition of model of classifier proceeds on internal model principle [12] and its generalization for multi-input and multi-output systems [13]. The principles show that the robustness is guaranteed by control action generated by dynamic of disturbances that have to be compensated. Hence the feedback controller have to include dynamic of all relevant surrounding of given controlled subsystem, and that is why the model of classifier have to copy dynamic of set of relevant disturbance signals.

In the next section we going to apply this theoretic discussion to the problem of linear tracking.

III. Linear Tracking Problem

We consider linear tracking problem defined for controlled subsystem given in the form

$$\dot{s} = \mathcal{A} \cdot s + \mathcal{B} \cdot u + \mathbf{\Gamma} \cdot \xi \quad (8)$$
$$z = \mathcal{G} \cdot s \quad (9)$$

where s is state of controlled subsystem, u is input and ξ is absolutely randomness process with zero mean value and known variance function $\delta(t)\mathbf{Q}$.

As shown in paper [9], using possible transformation, the controlled system (8)–(9) can be described in the form

$$\dot{x} = \mathbf{A} \cdot x + \mathbf{B} \cdot u + \mathbf{G} \cdot w + \mathbf{\Gamma} \cdot \xi \quad (10)$$
$$\dot{w} = \mathbf{F} \cdot w + \mathbf{\Delta} \cdot \xi \quad (11)$$
$$e = \mathbf{C} \cdot x + \mathbf{H} \cdot w \quad (12)$$
$$y = \mathbf{C}_m \cdot x + \mathbf{H}_m \cdot w \quad (13)$$

where $x \in \mathrm{R}^n$ is state of plant, $w \in \mathrm{R}^m$ is state of exo-generator, $e \in \mathrm{R}^p$ is control error and $y \in \mathrm{R}^p$ is measured output. Further the representation (10)–(13) has properly properties as full rank of tuple $(\mathbf{A}, \mathbf{C})$ and full rank of tuple $(\mathbf{A}, \mathbf{B})$.

The solution of optimization problem can be obtained using a plant deviation model defined in [2], [15]. State feedback controller stabilizing augmented plant (10)–(12) exists if and only if the augmented plant can be represented in the form

$$\dot{x}_e = \mathbf{A} \cdot x_e + \mathbf{B} \cdot (u - \mathbf{L} \cdot w) + \mathbf{\Gamma}_E \cdot \xi \quad (14)$$
$$\dot{w} = \mathbf{F} \cdot w + \mathbf{\Delta} \cdot \xi \quad (15)$$
$$e = \mathbf{C} \cdot x_e \quad (16)$$

where $x_e = x - \mathbf{T}w$ and matrices $\mathbf{T}(n \times m)$ and $\mathbf{L}(r \times m)$ exist as the solution of matrix equations system

$$\mathbf{T} \cdot \mathbf{F} - \mathbf{A} \cdot \mathbf{T} - \mathbf{B} \cdot \mathbf{L} = \mathbf{G} \quad (17)$$
$$\mathbf{C} \cdot \mathbf{T} = -\mathbf{H}. \quad (18)$$

Then the influence of exo-disturbance w onto control error e are fully compensated by compensation control

$$u_N = \mathbf{L} \cdot w \quad (19)$$

and plant is stabilized by deviation control

$$u_e = u - u_N = -\mathbf{K} \cdot x_e \quad (20)$$

where $\mathbf{K}$ is optimal state feedback gain minimized optimality criterion

$$J = \mathrm{E}\left\{\int_0^\infty x_e'(t) \cdot \mathbf{Q} \cdot x_e(t) + u_e'(t) \cdot \mathbf{R} \cdot u_e \mathrm{d}t\right\} \quad (21)$$

with weighting matrices $\mathbf{Q}$ and $\mathbf{R}$.

Feedback controller in case of unmeasuring state of the controlled subsystem is designed by standard way by using separation theorem. Appropriate estimation model and conditions of its existence are specified in paper [9].

Now, specification of model of classifier definition will be discussed. From previous text it is known that robustness of tracking is guaranteed by one useful rearranging of exo-generator dynamic exited by control error. Here we use simplest way of definition of model of classifier, we choose the model in the form

$$\dot{s}_{H,i} = \mathbf{F}_H \cdot s_{H,i} + \mathbf{B}_{K,i} \cdot e_i; \quad \text{for } i = 1, 2, \ldots, p, \quad (22)$$

where $s_H \in \mathrm{R}^x$ is state of model of classifier. Matrix $\mathbf{F}_H$ is sorted block of dynamic matrix of exo-generator and matrix $\mathbf{B}_H$ is chosen so that the tuple $(\mathbf{F}_H, \mathbf{B}_H)$ has full rank.

Now we can add the model of classifier (22) into description of the augmented plant (10)–(12)

$$\dot{x}_e = \mathbf{A} \cdot x_e + \mathbf{B} \cdot u_e + \mathbf{\Gamma}_E \cdot \xi \tag{23}$$
$$\dot{s}_{H,i} = \mathbf{F}_H \cdot s_{H,i} + \mathbf{B}_{K,i} \cdot e_i, \quad i = 1, 2, \ldots, p \tag{24}$$
$$e = \mathbf{C} \cdot x_e \tag{25}$$

and this state (x_e, s_n) together with deviation control u_e are evaluated by optimality criterion

$$J = \mathrm{E}\left\{ \int_0^\infty l(s_H, x_e, u_e)\mathrm{d}t \right\} \tag{26}$$

with quadratic lost function

$$l(s_H, x_e, u_e) = s_H' \mathbf{Q}_H s_H + x_e'(t)\mathbf{Q}x_e(t) + u_e'(t)\mathbf{R}u_e \tag{27}$$

Then using methods of LQG design we obtain optimal controller which can be described in the form

$$\dot{s}_r = \mathcal{A}_r \cdot s_r + \mathcal{B}_r \cdot y \tag{28}$$
$$u = \mathcal{C}_r \cdot s_r + \mathcal{D}_r \cdot y. \tag{29}$$

The controller (28)–(29) contains as a part model of classifier (the model of chosen exo-generator dynamic). It is result of adding its state into the set of evaluated variables. The controller guarantees robustness with context of given set of exo-disturbance according to internal model principle.

Exo-generator dynamic is included into controller by estimator of unmeasured exo-disturbances, as well. Then the dynamic is duplicate, but it is not problem, because we can define the controller using its irreducible representation defined as part observable through measurement y and control u. Used techniques based on Kalman's theorems [7], [8] is discussed in [9] in detail.

IV. Example

In this section we apply previous discussion on simple examples. We suppose the controlled subsystem with structure shown on Fig. 1 which is described by equations

$$\dot{z} = -z + d + u \tag{30}$$
$$\dot{d} = 0 \tag{31}$$
$$\dot{z}_{ref} = 0 \tag{32}$$
$$e = z - z_{ref}, \tag{33}$$

where $z \in \mathrm{R}$ is controlled output of plant, $d \in \mathrm{R}$ is additive disturbance and $z_{ref} \in \mathrm{R}$ is nominal controlled output.

Given controlled subsystem (30)–(33) can be equivalently represent in the form

$$\dot{x} = -x - w + u \tag{34}$$
$$\dot{w} = 0 \tag{35}$$
$$e = x - w \tag{36}$$
$$\dot{v} = -x - w + u \tag{37}$$
$$z = -2 \cdot w + v \tag{38}$$
$$d = -x - 3 \cdot w + v \tag{39}$$
$$z_{ref} = -x - w + w \tag{40}$$

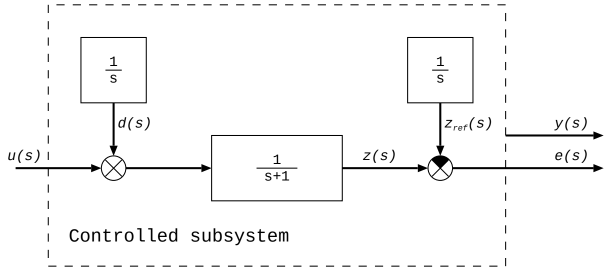


Fig. 1. Block diagram of controlled subsystem

by using regular transformation

$$\begin{bmatrix} x \\ w \\ v \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ 0 & -0.5 & 0.5 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} z \\ d \\ z_{ref} \end{bmatrix} \tag{41}$$

where the (x, w) is the state of augmented plant (34)–(36) and v is state of information system (37)–(40).

Deviation model of given controlled subsystem exists and it has form

$$\dot{x}_e = -x_e + u - L \cdot w \tag{42}$$
$$e = x \tag{43}$$

where $x_e = x + w$ is deviation state and $L = -2$ is gain of nominal control input.

For given controlled subsystem we analyze three types of available measurement; state of controlled subsystem $[z, d, z_{ref}]$, control error e and output z with its nominal signal z_{zef}. It is easy to establish that the state of augmented plant is observable through defined control error in all assumed cases of measurement.

In the first step we define a model of classifier. According to discussion in the previous sections, one will be represented by integrator with control error input due to disturbance dynamic (35). Therefore we suppose model of classifier in the form

$$\dot{s}_H = e. \tag{44}$$

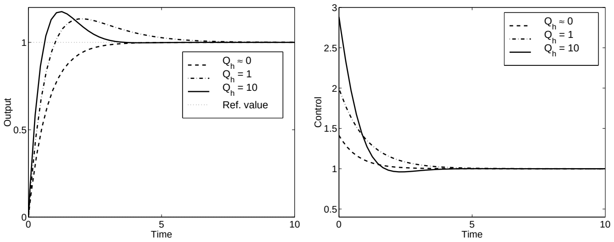
In the next step we add the state of model of classifier to the set of evaluated variables and the optimality criterion has the form

$$J = \lim_{t_F \to +\infty} \frac{1}{t_F} \mathrm{E}\left\{ \int_0^{t_F} \boldsymbol{Q} \cdot x_E^2 + Q_H \cdot s_H + R \cdot u_E^2 \ \mathrm{d}t \right\}. \tag{45}$$

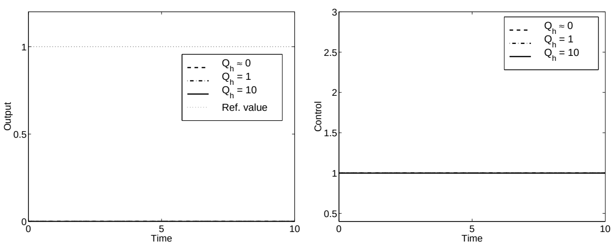
Using standard LQG optimization methods and methods presented in [9] we obtain controller minimized optimality criterion (45). Description of design process is not presented here in detail. Here we analyze only results of specified measurement:

a) $y = [z, d, z_{ref}]$ – State feedback controller

$$\dot{s}_H = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \cdot y \tag{46}$$
$$u^* = -k_H \cdot s_H - \boldsymbol{K} \cdot y; \tag{47}$$

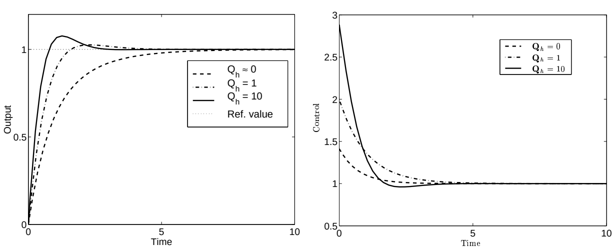


(a) Unit step responses of nominal regulated output z_{ref}

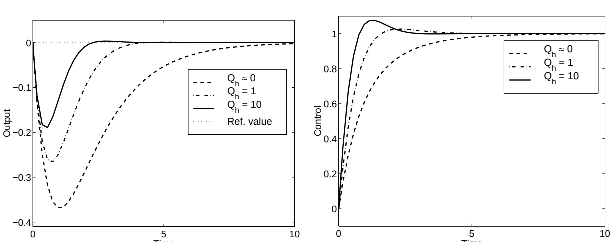


(b) Unit step responses of additive disturbance d

Fig. 2. Characteristics of control system with robustness state feedback controller for differently choosing weighting matrix $\mathbf{Q}_H$



(a) Unit step responses of nominal regulated output z_{ref}



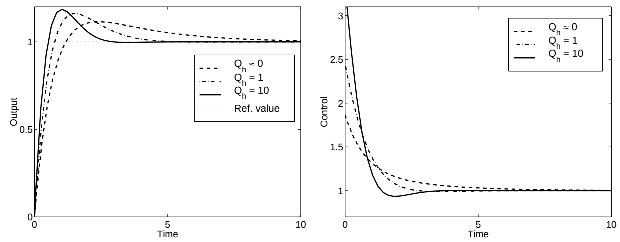
(b) Unit step responses of additive disturbance d

Fig. 3. Characteristics of control system with robustness error feedback controller for differently choosing weighting matrix $\mathbf{Q}_H$

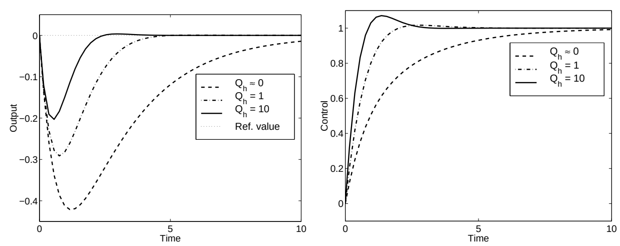
b) $y = e$ – Error feedback controller

$$\dot{s}_r = \boldsymbol{B}_r \cdot y \tag{48}$$

$$\dot{s}_H = y \tag{49}$$



(a) Unit step responses of nominal regulated output z_{ref}



(b) Unit step responses of additive disturbance d

Fig. 4. Characteristics of control system with robustness 2-DOF feedback controller for differently choosing weighting matrix $\mathbf{Q}_H$

$$u^* = \boldsymbol{C}_r \cdot s_r - k_H \cdot s_H + \boldsymbol{D}_r \cdot y \tag{50}$$

c) $y = [z, z_{ref}]$ – 2-DOF controller

$$\dot{s}_r = \boldsymbol{B}_r \cdot y \tag{51}$$

$$\dot{s}_H = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot y \tag{52}$$

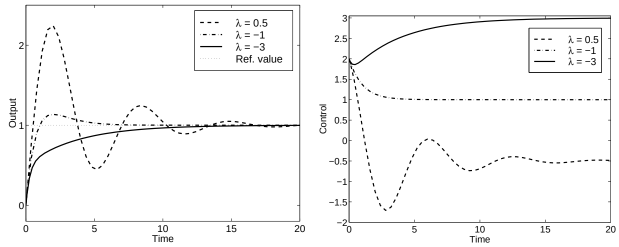
$$u^* = \boldsymbol{C}_r \cdot s_r - k_H \cdot s_H + \boldsymbol{D}_r \cdot y, \tag{53}$$

where matrices k_H, $\boldsymbol{K}$, $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{D}_r$ are dependent by choosing of weighting matrices $\boldsymbol{Q}$, Q_H and R.

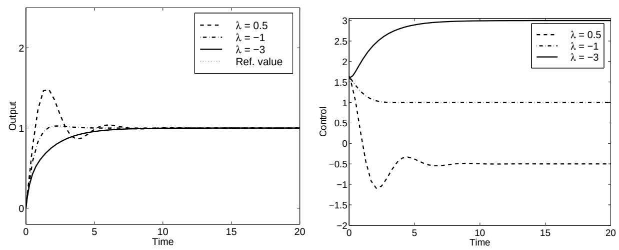
Controller with state feedback contains as a part integration feedback, which guarantees robustness of tracking for given set of disturbance (constant signals). Same result we can found in papers dealing with this problem, but there presented methods are based on authors knowledge and heuristics. Here obtained structure of state feedback controller is implicated by problem formulation and offers right way to generalization.

In the second two cases of measurement, the dynamic of exo-disturbances (integrator) is included to the controller by state estimator designing, as well, because state w of exo-generator is not available through measurement. Then the including by definition of model of classifier is duplicate. This redundancy is removed by definition of suitable irreducible representation of controller which always exists [9].

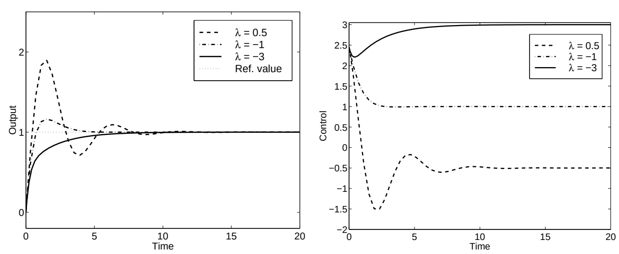
Obtained result are illustrated on several figures. Fig. 2, Fig. 3 and Fig. 4 show control system responses on nominal value step depend choosing weighting matrix of state of model of classifier for all tree assumed cases of measurement. Further Fig. 5 represents same responses, but for the control system with changed parameters in range of system stability.



(a) State feedback controller



(b) Error feedback controller



(c) 2-DOF controller

Fig. 5. Unit step responses of nominal regulated output z_{ref} of control system with changed parameter (eigenvalue of controlled plant λ) in range of control system stability

They are demonstrated tracking robustness towards to additive disturbance signal d.

V. Conclusions

This paper deals with design of LQG optimal controller for linear tracking problem which guarantees robustness of tracking towards given set of disturbance signals. In literature, for example [10], [3], can be found idea of such controller design by including dynamic of exo-disturbance into the state feedback controller. Here presented idea is inspired by [10] and it is generalized for all kind of measurement.

The linear tracking problem formulation is based on alternative control theory [14] using non additive optimality criterion. Because this area is not much discussed in literature, the first part of the paper deals with one. The result is description of transformation of non additive optimality criteria to the standard problem of LQG optimization through a model of classifier definition. Then the second part of the paper studies the model of classifier in linear tracking problem and tracking robustness point of view.

In linear tracking problem, the model of classifier is defined according to internal model principle as subsystem with dynamic copying exo-disturbance dynamic. Such model of classifier includes exo-disturbance dynamic to the optimal controller also in the case of state feedback controller design. In the case when the dynamic is included by the estimator then the redundancy of the dynamic is removed by specification of the controller as its irreducible representation.

Further research in discussed area can be focused to interpretation of weighting matrix of classifier state and to implication of presented methods to the applications, [16] for example. Also it is possible to study the problem for other types (non-quadratic) of optimality criteria.

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